ON AXIOMS AND THEIR CORRESPONDING DEDUCTION RULES: A SURVEY

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In the construction of various axiomatizations for systems of mathematical logic, the situation often arises in which, for a certain purpose, we could either add an axiom or else a deduction rule that corresponds to this axiom in some obvious sense. For many purposes, these two processes are equivalent, but this is not always the case. In the course of this paper, we will investigate this situation and determine how the presence of the rule modus ponens and the deduction theorem as rules (or "derived rules") of a given axiomatization affect the situation.

I. *Terminology*. By *modus ponens*, we mean the usual rule: "From $A \supset B$ and A infer B".

By the *deduction theorem*, we mean the following: given an axiomatic system W, let W* be the system that has the same rules of deduction as W and the axioms of W plus the new axiom X; suppose that Y is a theorem of W*; then $X \supset Y$ is a theorem of W. Except where specifically noted, we shall assume that all deduction rules are *finitary*, i.e., that they have a finite number of formulae as hypotheses.

The deduction rule "from hypotheses A_1, ..., A_n, infer B" is said to be a *weak derived rule* of the axiomatic system W if it is the case that whenever the hypotheses of the rule are theorems of W, it happens that the conclusion is also a theorem of W. The same deduction rule is said to be a *derived rule* of the axiomatic system W if it is a weak derived rule of the axiomatic system W" which arises from W by adding A_1, A_2, ..., A_n as additional axioms. The same deduction rule is said to be a *strong derived rule* of the axiomatic system W if it is a weak derived rule of all extensions of W by the addition of new axioms.

Given a deduction rule D of the form "from formula P, infer formula Q", the formula $P \supset Q$ is called *the formula corresponding to D* and D is called *the deduction rule corresponding* to $P \supset Q$. Correspondences for deduction rules with more than one formula as hypotheses will be defined later. We assume that the rule "an axiom is a theorem" is available in all our systems.

II. *The Classical Case.* First we consider the most obvious situation, the complete classical propositional calculus with $\supset$ present as one of the connectives. No matter what formulation of the axioms and rules is chosen for this calculus, modus ponens is always a derived rule in the weak sense. This is simply one way of considering the completeness theorem for the classical propositional calculus, remembering that the rule modus ponens takes tautologies into tautologies.

Trivial examples show that modus ponens is not always a strong derived rule in the case of the complete propositional calculus. Consider a formulation in which all tautologies are axioms and there are no deduction rules (i.e., only the axioms are theorems). If we add to this system the two additional axioms p and $p \supset q$, the rule modus ponens is no longer a weak derived rule of the expanded system, and hence not a strong derived rule of the original system.

The case of the deduction theorem, as defined, is even less satisfactory, since for certain complete propositional calculi, it is not even a weak derived rule. The most obvious example is the case of systems having the rules modus ponens and substitution and any suitable set of axioms. For, in such systems, if we take as additional axiom the propositional variable p and construct a proof in the augmented system with first line p (now an axiom) and second line q (by substitution), the deduction theorem would give us $p \supset q$ as a theorem of the original system, a rather unwelcome development. For a more interesting example, consider the following rule.
R1: "from the formula P, deduce as a new theorem P, if P is a tautology, and deduce — P if P is not a tautology".

Now this rule is a weak derived rule of all of our (complete) propositional calculi, since it always results in a tautology as a theorem if the formula of its hypothesis was a tautology. However, if we add the propositional variable p as an additional axiom to a system with R1 as one of its actual rules, applying the rule once gives us —p as a theorem. Hence if the deduction theorem were true as a weak derived rule, we could conclude that $p \supset -p$ were a theorem of the original system and thus a tautology. A fortiori, the deduction theorem is not always a strong derived rule.

As is well-known, however, for systems that are suitably formulated, e.g., with axiom schemata and with modus ponens as the only rule, the deduction theorem does hold as a weak derived rule. In other cases, restrictions must be made in the formulation of the deduction theorem itself so that a corresponding correct derived rule may be formulated. The following proposition is an example of such a modification of the deduction theorem that is true not only for propositional calculi. For convenience, however, we state it for such calculi. In certain particular cases, stronger theorems can be proven, but in general this proposition is the most that one can expect. It is rather easy to concoct examples with rules similar to R1 showing that any use of the original rules of the system other than modus ponens after the introduction of the new axiom as a theorem may lead to disaster.

> *Proposition 1*: Given the axiomatic system W for the complete classical propositional calculi with finitary rules and give the formula A, suppose that B is a theorem of the axiomatic system W' whose axioms are the axioms of W plus the formula A and whose rules are the rules of W plus the additional rule of modus ponens with the restriction that once the axiom A has been listed as a theorem in the course of a proof, only the rule modus ponens may be applied; further suppose that the connective ⊃ ("implies") is present in W; then $A \supset B$ is a theorem of the original system W.

Proof: This follows more or less as usual, by induction on the number of lines in the proof of B. If B is proved in one line, either it is one of the original axioms or else it is A. In the first case, B is a tautology, so that $A \supset B$ is a tautology and hence a theorem, since our axiomatization is complete. In the second case, the formula in question is $A \supset A$ which is also a tautology. Now suppose that for all formulae B' provable in fewer than n lines in the new system, the formula $A \supset B$ is provable in the old system W. Also suppose that B is probable in n lines in the new system W'. If the new hypothesis A is not used at all in the proof of B, then B is a theorem of the old system (since modus ponens is a weak derived rule) and hence a tautology. Hence, as before, $A \supset B$ is also a tautology and hence a theorem of the old system W. If the last line of the proof of B is the new axiom A, the formula to be proved in W is $A \supset A$, again a tautology. Otherwise, the formula A has been introduced earlier in the proof and the last line of the proof results from two previous lines say C and $C \supset D$, by modus ponens. Thus we know, by the induction hypothesis that $A \supset C$ and $A \supset (C \supset D)$ are theorems of the original system and thus tautologies. Hence, as a check of the truth tables shows, $A \supset D$ is also a tautology and hence a theorem of W as desired. Q.E.D.

III. *The General Case: Deduction Rules with One Formula as Hypothesis.* Here we consider more general calculi, including partial propositional calculi, modal logics, cut logics [cf. 2], first order logics. Any one of these calculi will be called simply an axiomatic system in what follows. For convenience, we shall also assume that the connective $\supset$ is always present. As the section title indicates, we consider here the relationship between deduction rules with one formula as hypothesis and their corresponding formulae (cf. I: Terminology).

Proposition 2: Given axiomatization C, suppose that for every theorem of C of the form $P \supset Q$, the corresponding rule is a weak derived rule of C. Then modus ponens is a weak derived rule of C. Conversely, if modus ponens is a weak derived rule, then, for every theorem of C of the form $P \supset Q$, the corresponding rule is a weak derived rule.

Proof: Given the theorem $P \supset A$ which is to serve as the major premiss for modus ponens, by assumption, the rule "from P, infer A" is available as a weak derived rule. Hence the (weaker) rule "from P, $P \supset Q$, infer Q" is also valid. But this says that modus ponens is available for all the cases that its premisses are available, so that modus ponens is a weak derived rule. Conversely, if modus ponens is available and $P \supset Q$ is a theorem of C, we have the rule "from $P \supset Q$, P, infer Q". But $P \supset Q$ is known to be a theorem. Hence we have the rule "from P, infer Q", as desired. Q.E.D.

Proposition 3: Given an axiomatic system W, if the deduction theorem is a (meta)-theorem for W, then for each derived rule D of W of the form "from P, infer Q", the theorem corresponding to D is a theorem of W. Conversely, if for each such derived rule D, the corresponding formula is a theorem of W, then the deduction theorem is a (meta)-theorem for W.

Proof: This follows immediately from the definitions of derived rule and deduction theorem. Q.E.D.

Thus we see, as a consequence of these propositions that the free interchangeability of axioms and deduction rules, at least in the case when the deduction rules have exactly one formula as hypothesis, is possible precisely when the rules modus ponens and the deduction theorem are both available in the system in question.

It should be noted that these two rules are rather independent; i.e., we may have both of them available or neither, or either singly. The examples of the complete propositional calculus formulated with axiom schemata and modus ponens is an example of the first case, and the corresponding calculus with modus ponens and substitution is an example in which modus ponens holds but the deduction theorem does not. The calculus with $p \supset p$ and $(p \supset p) \supset (p \supset (q \supset p))$ as its only axioms and substitution as its only rule satisfies neither modus ponens nor the deduction theorem. Proposition Four below gives as example of a calculus which satisfies the deduction theorem but not modus ponens. Note that for any such calculus all the instances of the schema $P \supset P$ must be present as theorems, assuming the rule that any axiom is a theorem.

> *Proposition 4*: The partial propositional calculus with no deduction rules (i.e., only the axioms are theorems) and with the axiom schemata listed below does not have modus ponens as a weak derived rule but does satisfy the deduction theorem.
> A_0: $P \supset P$; A_1: $Q_1 \supset (P \supset P)$; and for $n > 1$ A_n: $Q_n \supset A_{n-1}$; B_0: $(P \supset P) \supset (P \supset (Q \supset P))$; and for $n > 0$, B_n: $R \supset A_{n-1}$; where the P's, Q's, and R's are, as usual arbitrary formula of the calculus.

Proof: If modus ponens were a weak derived rule, from axiom schemata A_0 and B_0 we could conclude that $(p \supset (q \supset p))$ was a theorem of the calculus. But this is impossible since that formula is not an instance of A_0 nor of A_1 and all of the other A's and all of the B's of length greater than the length of $(p \supset (q \supset p))$, using anyone's definition of the length of a formula, so that this formula cannot be a substitution instance of any of those axioms.

On the other hand, the deduction theorem holds. For suppose that the formula T is taken as an additional axiom. If, in the extended system, the theorem proved is T itself, then the theorem to be proven in the original system is $T \supset T$, an instance of A_0. The only other theorems provable in the expanded system are theorems provable in the original system, i.e., instances of some axiom schemata A_n or B_n. Then the corresponding theorem to be proved to verify the deduction theorem is the theorem $T \supset A_n$ or else $T \supset B_n$. But these are respcetively instances of A_{n+1} and B_{n+1}, so that they are theorems of the original system. Q.E.D.

IV. *Deduction Rules with More than One Formula as Hypotheses.*

In the case of rules which have more than one formula as hypotheses, there are several alternatives as choices for the corresponding theorem of the language. We will consider here two of the possibilities. To the rule D "from A_1, A_2, ..., A_n, infer B" might correspond the formula $A_1 \wedge A_2 \wedge \ldots \wedge A_n \supset B$ or else $A_1 \supset (A_2 \supset (\ldots \supset (A_n \supset B))\ldots)$. We call the first of these the *corresponding conjunct formula to D* and the second the *corresponding implicational formula to D*; D will be called the *deduction rule corresponding* to either of these two formulae. Thus a formula may correspond to more than one rule. (The formula $(p \wedge q) \supset r$ corresponds to the rules "from $p \wedge q$, inpfer r" and the rule "from p, q, infer r".)

The conjunct formula seems somehow more natural if the connective $\wedge$ ("and") is present, but even in that case the implicational formula has advantages.

Proposition 5: A. Given a logical system W, if the deduction theorem is valid for that system, then for every derived rule C of the form "from A_1, A_2, ..., A_n, infer B", the corresponding implicational formula to C is a theorem of W; conversely, if for every derived rule of W of the form C, the implicational formula corresponding to C is a theorem of W, then the deduction theorem holds.

B. If modus ponens is a weak derived rule of the logical system W, then given a formula P of W of the form $A_1 \supset (A_2 \supset (\ldots \supset (A_n \supset B)))$, the corresponding deduction rule is a weak derived rule of W; conversely, if for each such P, the corresponding deduction rule is a theorem of W, then modus ponens is a weak derived rule of W.

Proof: A. If the deduction theorem is valid and C is a derived rule, apply the deduction theorem n times to get the corresponding implicational formula. Conversely, the hypothesis is logically stronger than the deduction theorem which is the special case of it for $n=1$.

B. Given modus ponens, to get the rule, as a weak derived rule, use the corresponding theorem as major premiss for modus ponens and A_1 as minor premiss, then use the conclusion as major premiss again with A_2 as minor premiss, etc. Again for the converse, the case $n=1$ suffices as was shown in Proposition 2. Q.E.D.

If one uses the corresponding conjunct formulae, The situation is not quite as satisfying as that in Proposition 5. The converse sections of both parts A and B still hold, again because the assumptions made are formally stronger than the corresponding assumptions in Propositions 3 and 4. However, neither of the direct parts of the theorem holds. In the case of the deduction theorem this is inevitable from the definition of deduction theorem which perforce handles the introduction of one new assumption at a time. In the case of modus ponens, the direct part of the theorem goes through exactly as before if the rule Cn below is assumed, as a weak derived rule, as the reader may easily verify.

Cn: "From $A_1, A_2, \ldots, A_n$ infer $A_1 \wedge A_2 \wedge \ldots \wedge A_n$.

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