RESTRICTED INFERENCE

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1. *The Concept of Restricted Proof and Restricted Consequence*

Given a system L of deductive logic, it seems natural that one should be interested not only in the question of what can be demonstrated but also in that of what can be demonstrated *simply*. The idea of a simple proof or demonstration as such is very vague and imprecise. We shall introduce some formal machinery for its precise articulation.

Let L be a system of natural deduction based on the rules $R_1, R_2, \ldots, R_n$. We may now introduce the following definitions:

1. Let $S = \langle m_1, m_2, \ldots, m_n \rangle$ be a sequence of non-negative integers, each correlated with one of the R_i and let k be any integer greater than or equal to 2. Then an (S, k)-*restricted proof* of the proposition β from the premisses $\alpha_1, \ldots, \alpha_n$ is one that (i) applies any rule R_i no more than m_i times, and (ii) is no longer than k lines in length (so $2 \leqslant k \leqslant n + m_1 + m_2 + \ldots + m_n$). We shall write

   $$\alpha_1, \ldots, \alpha_n \ [S, k\} \ \beta$$

   to indicate that β is (S, k)-restrictedly provable from the premisses $\alpha_1, \ldots, \alpha_n$ [1].

2. β is an (S, k)-*restricted consequence* of the premisses $\alpha_1, \ldots, \alpha_n$ if there exists a restricted proof that moves to β as conclusion from $\alpha_1, \ldots, \alpha_n$ as premisses in no more than k steps, i.e., when

$$\alpha_1, \ldots, \alpha_n \ [S, k + n\} \ \beta$$

In the special case that k is so fixed that no restriction over

[1] Note that the inference-claim is automatically false when $k \leqslant n$.

and above that inherent in S itself is imposed — i.e., when $k = m_1 + m_2 + \ldots + m_n$ — we shall speak simply of S-restricted proofs and consequences. In the special where the set $S = \langle m_1, \ldots, m_n \rangle$ is such that all the m_i are equal, $j = m_1 = \ldots = m_n$, we shall speak simply of (j, k)-*restricted* proofs and consequences.

When the above two special cases are combined — i.e., when the proof of β from $\alpha_1 \ldots \alpha_n$ is no longer than k lines *in toto* or when the inference to β from the α_i proceeds in no more than k steps — we may speak simply of a k-*restricted proof* or *consequence*, respectively.

The relationships inherent in the above may be summarized as follows:

(1) $\alpha_1, \ldots, \alpha_n\ [S, k\}\ \beta$ This is basic
(1′) $\alpha_1, \ldots, \alpha_n\ (S\ k\}\ \beta =_{DF} \alpha_1, \ldots, \alpha_n\ [S, n + k\}\ \beta$
(2) $\alpha_1, \ldots, \alpha_n\ [S\}\ \beta =_{DF} \alpha_1, \ldots, \alpha_n\ [S, m_1 + \ldots + m_n\}\ \beta$
(2′) $\alpha_1, \ldots, \alpha_n\ (S\}\ \beta =_{DF} \alpha_1, \ldots, \alpha_n\ [S, n + m_1 + \ldots + m_n\}\ \beta$
(3) $\alpha_1, \ldots, \alpha_n\ [j, k\}\ \beta =_{DF} \alpha_1, \ldots, \alpha_n\ [\langle j, j, \ldots, j\rangle, k\}\ \beta$
(3′) $\alpha_1, \ldots, \alpha_n\ (j, k\}\ \beta =_{DF} \alpha_1, \ldots, \alpha_n\ [\langle k, k, \ldots, k\rangle, k\}\ \beta$
(4) $\alpha_1, \ldots, \alpha_n\ [k\}\ \beta =_{DF} \alpha_1, \ldots, \alpha_n\ [\langle k, k, \ldots, k\rangle, k\}\ \beta$
(4′) $\alpha_1, \ldots, \alpha_n\ (k\}\ \beta =_{DF} \alpha_1, \ldots, \alpha_n\ [\langle k, k, \ldots, k\rangle, n + k\}\ \beta$

The following remarks are in order:

1. There is a uniform modification (viz., the addition of n) in going from an unprimed relationship to its primed counterpart.
2. This relationship is such that in the primed version the number of premisses is immaterial: it is only the number of inferential steps beyond the premisses is here at issue.
3. In cases (4)-(4′) it is only over-all length of the proof or consequence-deduction that matters. All other cases contemplate a more restrictive limitation on the number of times a given rule can be used.
4. Cases (1)-(1′) and (2)-(2′) are prepared to treat the rules of inference differentially. The other cases treat all rules alike.
5. Cases (1)-(1′) and (2)-(2′) are prepared to let m_i be 0, thus

in effect proscribing the use of certain rules in drawning inferences.

6. Although none of these notions of inference allows unrestricted application of a rule (the m_i are integers), this case can be covered in either of two ways — (i) by letting the m_i be either integers or $\aleph_0$ or (ii) by taking the existential quantification of the relevant notion of inference with respect to the m_i in question.

The idea underlying this articulation of various modes of restricted proof (or consequence) is a restriction in the number of times that the several rules of inference can be employed in a proof — and correspondingly a restriction in the over-all length of the proof itself. When that restriction is tight enough, we obtain a correspondingly restrictive notion of simplicity of proof.

One obvious application of this machinery is as follows: Given two systems L and L′ such that every rule of L is a rule of L′, but not conversely, any inference relation for L expressible in terms of the preceding concepts will be identical with some inference relation for L′. If $S = \langle m_1, \ldots, m_n \rangle$ and $S' = \langle n_1, \ldots, n_k \rangle$, where each of the m_i is correlated with a rule R_i of L and each of the n_j is correlated with a rule R'_j oφ L′, and $n_j = m_i$ if R'_j is R_i but if R'_j is none of the R_i, $n_j = 0$, then we have

$$\alpha_1, \ldots, \alpha_n\ [S, k] \vdash \beta \text{ iff } \alpha_1, \ldots, \alpha_n\ [S', k] \vdash \beta.$$

Consider the notion of (S′, k)-restricted proof so introduced and those n_i corresponding to R'_i not occurring in L. By setting these $n_i = 0, 1, 2, \ldots$, we get closer and closer approximations to L′ beginning with L.

2. *Some examples*

Let R_1-R_7 be respectively the rules UI, EG, TF, Cd, UG, EI, and CQ of Quine's natural deduction system ([2]). The notion of

([2]) W. V. Quine, *Methods of Logic*, revised edition (New York: Holt, Rinehart, and Winston, 1959).

a 2-restricted proof for *this* system is captured by the following schemata (where α' is like α except for containing free x' where α contains free x):

(i) $(x)\alpha\,[2\}\,\alpha'$
(ii) $\alpha'[2\}(\exists x)\alpha$
(iii) If $\alpha\supset\beta$ is a truth-table tautology, then $\alpha[2\}\beta$
(iv) $\sim(x)\alpha[2\}\;(\exists x)\sim\alpha$
(v) $(\exists x)\sim\alpha[2\}\sim(x)\alpha$
(vi) $\sim(\exists x)\alpha[2\}(x)\sim\alpha$
(vii) $(x)\sim\alpha[2\}\sim(\exists x)\alpha$

The definitive character of the 2-restricted system is represented in the result that:

There is a 2-restricted proof of β from the premiss α iff $\alpha[2\}\beta$ is an instance of one of (i)-(vii).

There are no inference schemata corresponding to UG and EI since no 2-line deduction using either of these is a *finished deduction* (see Quine, *ibid.*, p. 162). (There is no inference schema corresponding to Cd since correct 2-line deductions using Cd have no premisses — a case lying outside the scope of the notions introduced in § 1.) It should be noted that it is by no means common that the restricted proofs or consequences of a given system can be so easily schematized.

So much for an example based on Quine's system; let us consider a different point of departure. Let R_1-R_{13}(m_1-m_{13}) be respectively the rules (integers) reit (2), imp int (1), imp elim (1), conj int (2), conj elim (2), dis int (2), neg elim (1), neg_2 int (2), neg_2 elim (1), neg conj int (2), neg conj elim (2), neg dis int (2), and neg dis elim (2), of Fitch's system [3]. (So that $S = \langle 2, 1, 1, 2, 2, 2, 1, 2, 1, 2, 2, 2, 2\rangle$.) Rather than attempting to axiomatize some notion of restricted proof or consequence for this system, we simply list some examples of (S, k) and S-restricted consequences for various k and then note some general relations among such (for *this* system). Letting Γ be an arbitrary

[3] Frederick B. FITCH, *Symbolic Logic* (New York: Ronald Press, 1952).

string of premisses (Γ may be empty in the presence of other premisses), we have:

(1) $\Gamma, \alpha\&\beta\ (S, 3)\vdash\beta\&\alpha$.
(2) $\Gamma, \alpha(S, 4)\vdash\sim\alpha\supset\beta$.
(3) $\Gamma, \alpha \vee \beta\ (S, 5)\vdash\beta \vee \alpha$.
(4) $\Gamma, \alpha \vee \beta, \sim\alpha(S, 5)\vdash\beta$.
(5) If $\Gamma(S, k)\vdash\alpha$, then $\Gamma(S)\vdash\alpha$.
(6) If $\Gamma(S, 1)\vdash\gamma$ and $\Gamma, \beta(S, 1)\vdash\gamma$, then $\Gamma, \alpha \vee \beta(S, 5)\vdash\gamma$.

With respect to (6), it might be noted that the more general

(7) If $\Gamma,\alpha(S, k_1)\vdash\gamma$ and $\Gamma, \beta(S, k_2)\vdash\gamma$, then $\Gamma, \alpha \vee \beta(S, k_1 + k_2 + 3)\vdash\gamma$ does *not* obtain since, although we can find a deduction of γ from $\Gamma, \alpha \vee \beta$ given that the antecedent of (7) obtains, we have no guarantee that this (or any) such deduction meets the conditions for $(S, k_1 + k_2 + 3)\vdash$ restricted consequence given S as above. For example, both $\alpha(S, 2)\vdash(\alpha \vee \gamma) \vee (\beta \vee \delta)$ and $\beta(S, 2)\vdash(\alpha \vee \gamma) \vee (\beta \vee \delta)$ obtain, but it is false that $\alpha \vee \beta(S, 7)\vdash(\alpha \vee \gamma)\ \beta$.

It might be noted that for any system L which contains (analogues of) Fitch's rules imp int and imp elim we have the following quasi-deduction-theorem. Let m_i and m_j be the integers in S corresponding to imp int and imp elim respectively, let S′ be like S except for having $m'_i = m_i + 1$ where S has m_i and let S″ be like S except for having $m'_j = m_j + 1$ where S has m_j. Then we have:

(8) If $\Gamma, \alpha[S, k]\vdash\beta$ then $\Gamma[S', k + 1]\vdash\alpha\supset\beta$.
(9) If $\Gamma[S, k]\vdash\alpha\supset\beta$ then $\Gamma, \alpha[S'', k + 2]\vdash\beta$.

3. *Application I: Epistemic Logic*

According to one concept of belief, all of the logical consequences of beliefs are themselves believed. This view of belief is sometimes expressed by saying that the believers at issue in a logical theory of the subject are to be logically omniscient, and finds its formalization in the meta-principle:

(1) If β is a logical consequence of $\alpha_1, \ldots, \alpha_n$, then $Bx\alpha_1\& \ldots \&Bx\alpha_n\supset Bx\beta$ is a theorem.

The logic of this concept of belief has been extensively investigated by Hintikka (4). According to a second and more restricted construction, one believes all and only those things to which one would be prepared to give explicit assent if questioned. The logic of this concept has not (to our knowledge) been investigated and, indeed, it seems doubtful that this concept can have an "interesting" logic. (It is readily conceivable, for example, that a person can believe $\alpha \& \beta$ in this sense, and yet fail to believe $\alpha \vee \gamma$.) With this *overt* sense of belief, we must be prepared to encounter believers who are "logically blind."

It seems desirable to explore the possibility of a middle course between these extremes, to investigate a belief construction such that, to put it metaphorically, believers are neither logically omniscient nor logically blind but rather have a logical vision of limited range. Rescher (5) has proposed a system of epistemic logic for this kind of construction in which (1) is replaces by the weaker:

(2) If β is an *obvious* consequence of $\alpha_1, \ldots, \alpha_n$, then $Bx\alpha_1 \& \ldots \& Bx\alpha_n \supset Bx\beta$ is a theorem,

where the following rough characterization of "obvious consequence" is given: β is said to be an obvious consequent of $\alpha_1, \ldots, \alpha_n$ if β is deducible from $\alpha_1, \ldots, \alpha_n$ in some small number of inferential steps. It is further specified that (2) is not to be used more than once in any proof or deduction. Rescher points out that this notion of obvious consequence has some similarity to Hintikka's notion of a "surface tautology" (6).

It is to be presumed that a rigorous characterization of the notion of an "obvious consequence" can be given in terms of the machinery developed in section 1 above. Given such an explication, the interpretation of (2) becomes well-specified

(4) Cf. Jaakko HINTIKKA, *Knowledge and Belief* (Ithaca: Cornell University Press, 1962).

(5) Nicholas RESCHER, "The Logic of Belief Statements", *Topics in Philosophical Logic* (Dordrecht, Holland: D. Reidel, 1968), pp. 40-53.

(6) Cf. Jaakko HINTIKKA, "'Knowing Oneself' and Other Problems in Epistemic Logic", *Theoria*, vol. 32 (1966), pp. 1-13.

and definite. On the other hand, we are now in a position to make finer distinctions than this construction allows.

Although we are not prepared to give a complete characterization of the notion of obvious consequence, it may be helpful to make some brief remarks on the subject. We assume that the obvious consequence relation is *roughly* a variety of restricted consequence or proof, but one might well want to make additional requirements on the *kind* of deductions that can result in an *obvious* consequence. For example, if the underlying system is one which allows subordinate proofs, one presumably would also want to require that these themselves be (in some yet more stringent sense) "obvious". This might be thought to be especially appropriate where the subordinate proof involves a *reductio* argument. Thus one might be lead to modify the notion of restricted consequence as follows (for simplicity we consider only the case where all subordinate proofs are immediately subordinate to the main proof):

> Let S, the R_i and k be as before. Let $S' = \langle O_1, \ldots, O_n \rangle$ where $O_1 \leqslant m_1, \ldots, O_n \leqslant m_n$, and let $k' < k$. Then an (S, S', k, k')-*restricted proof* of B from $\alpha_1, \ldots, \alpha_n$ is one that satisfies (i) and (ii) as before, and (iii) is such that in any subordinate proof no rule R_i is applied more than 0_i times, and (iv) no subordinate proof is longer than k′ lines in length.

For example, take the system of Fitch considered in § 2. Letting $O_i = m_i - 1$ and $k' = k - 1$, we have as analogues of (1), (3), and (5) of that section:

(1′) $\Gamma, \alpha \,\&\, \beta \;(S, S', 3, 2) \succ \beta \,\&\, \alpha$
(3′) $\Gamma, \alpha \vee \beta \;(S, S', 5, 4) \succ \beta \vee \alpha$
(5′) If $\Gamma \;(S, S', k, k') \succ \alpha$, then $(S, k) \succ \alpha$.

Given a notion of obvious consequence, one could then define the following concepts: α is *patently valid* iff α is an obvious consequence of $\beta \vee \sim\beta$ for some formula β; α is *patently inconsistent* iff for some formula β, $\beta \,\&\, \sim\beta$ is an obvious consequence of α.

Assume that certain beliefs are in some appropriate sense epistemically *basic*. In this case we write: $B_0x\alpha$. We define $B_{n+1}x\beta$ so as to obtain iff β in an obvious consequence of $\alpha_1, \ldots, \alpha_k$ where $B_nx\alpha_1, \ldots, B_nx\alpha_k$. Then, instead of (2), we can adopt the meta-principle:

(3) If β is an obvious consequence of $\alpha_1, \ldots, \alpha_n$, then $B_{m_1}x\alpha_1 \,\&\, \ldots \,\&\, B_{m_n}x\alpha_n \supset B_{\max(m_i)+1}x\beta$ is a theorem.

This embraces the special case:

(4) If β is an obvious consequence of α, then $B_nx\alpha \supset B_{n+1}x\beta$. If the relation of obvious consequence allows repetition, i.e., if for all i such that $1 \leqslant i \leqslant n$, α_i is an obvious consequence of $\alpha_1, \ldots, \alpha_n$, then (3) follows from the definition of $B_nx\alpha$.

4. *Application II: Imperfect Reasoners*

Consider a computer which knows several natural deduction rules (perhaps learned at Turing's knee), but which — being "all too human" — can err. Our computer, call it Mycroft, is known to make one mistaken step in 100 in making deductions and his errors are independent in the sense that mistakes made before the *n*th step in a deduction have no bearing on whether the *n*th step goes wrong. Assume that Mycroft, or Mike for short, knows only *modus ponens* and *modus tollens* and that we want to know the consequences derivable from a set of formulas using these two rules, but are only interested in those formulas that Mike claims as consequences which have at least a .96 chance of being correct. Such a consequence — a "*reliable* consequence" — amounts to a 6-restricted consequence in terms of the machinery introduced in section 1 with L = ⟨*modus ponens, modus tollens*⟩ (7). (For $(.99)^x = .96$, x $\simeq 6$; it would help if Mike were good with logarithms.)

Consider a slightly different situation in which Mike is

(7) It is to be assumed that we can ask Mike to carry through any particular deduction but once.

completely reliable but takes ten times as long to do *modus tollens* as *modus ponens* (say he takes, respectively, ,one and ten seconds for these operations). Call β a t second consequence of $\alpha_1, \ldots, \alpha_n$ (for Mike) iff Mike can derive β from $\alpha_1, \ldots, \alpha_n$ in t seconds. We can define this in terms of the machinery developed in section 1. Let L = ⟨*modus ponens, modus tollens*⟩ and S_i = the ith member (under some appropriate ordering) of the set of all ordered pairs $\langle m_1, m_2 \rangle$ such that $m_1 \geqslant 0$, $m_2 \geqslant 0$, and $m_1 + 10m_2 = p$. Then

> β is a t second consequence of $\alpha_1, \ldots, \alpha_n$ iff β is an S_i-restricted consequence of $\alpha_1, \ldots, \alpha_n$ for some i.

5. *Conclusion*

The idea of a restricted inference seems to offer a rigorous way of introducing into logic a conception that seemingly has no place here: the economists' concept of limited resources, of a finiteness of means, in short, of *scarcity*. In ways we have attempted to define and illustrate, the mechanisms of restricted inference provide a basis for injecting into considerations of logical deduction the operation of the familiar restrictive limitations of human finitude: limitations forced upon us in situations of limited time, accuracy, or logical acumen. The implications of such limitations may well repay further study — they are unquestionably of *practical* importance and may be presumed to prove of *theoretical* interest as well.

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