COMPLETENESS IN THE LOGIC OF PREDICATE MODIFIERS

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In [5], Malinas and the author introduced some languages designed to handle predicate modification (i.e. basically adverbial constructions) and gave some formal semantical prescriptions for these languages. Here I extend the languages in a couple of directions (changing those of [5] slightly to facilitate extension) and provide sketches of completeness proofs for suitable axiomatizations of the theories involved.

SYSTEM QTM

The system QTM is the weakest of the systems considered. Its syntax is set out thus:

Primitives

S(i) $p, q, r, p_1, q_1, \ldots$
designated as propositional variables,

S(ii) $x, y, z, x_1, y_1, \ldots$
designated as individual variables,

S(iii) $F^n, G^n, H^n, F_1^n, G_1^n, \ldots$ for each $n \geqslant 1$,
designated as n-ary predicate variables,

S(iv) $f^n, g^n, h^n, f_1^n, g_1^n, \ldots$ for each $n \geqslant 0$,
designated as n-ary predicate modifiers

S(v) $\sim, \supset, \forall$,

S(vi) (,).

In the formation rules now to be stated we use an italicized variable as a metavariable for the appropriate class: thus p ranges over propositional variables, f^n over n-ary modifiers and so on.

Formation Rules

W(i) F^n is well-formed n-ary predicate (nPr),
W(ii) If φ is an nPr then $f^0(\varphi)$ and $f^j(\varphi)x_1 \ldots x_j$, $j \geqslant 1$, are nPr,
W(iii) If φ is an nPr then $\varphi(x_1, \ldots, x_n)$ is a well-formed formula of QTM (WQTM),
W(iv) p is a WQTM,
W(v) If α is a WQTM so is $\sim\alpha$,
W(vi) If α, β are WQTM so is $(\alpha \supset \beta)$,
W(vii) If α is a WQTM so is $(\forall x)\alpha$,
and these are all the WQTM.

We set $M(\alpha) =_{df} \max n(\mathrm{nPrV}(\alpha) \neq \{\})$ and $K(\alpha) =_{df} \max n(\mathrm{nPrM}(\alpha) \neq \{\})$, where nPrV($\alpha$) and nPrM($\alpha$) are respectively the set of n-ary predicate variables and the set of n-ary predicate modifiers in α. Then the semantics for QTM begins by specifying that a QTM model for a wff α is an ordered (mk + 2m + 3)-tuple $\lambda = \langle D, \varphi, \psi, \chi^1, \ldots, \chi^m, \zeta_1^0 \ldots, \zeta_m^0, \ldots, \zeta_1^k, \ldots, \zeta_m^k \rangle$, where $m = M(\alpha)$, $k = K(\alpha)$, and

$$D \neq \{\},$$

$$\varphi \in \{1, 0\}^{\mathrm{Pv}(\alpha)},$$

i.e. φ is a function from the propositional variables of α to the set of truth-values $\{1, 0\}$;

$$\psi \in D^{\mathrm{FIV}(\alpha)},$$

i.e. ψ is a function from the set of individual variables with at least one free occurrence in α to the domain D,

$$\chi^n \in (\mathscr{P}(D^n))^{\mathrm{nPrV}(\alpha)}, \qquad 1 \leqslant n \leqslant m,$$

$$\zeta_n^0 \in (\mathscr{P}(D^n)^{\mathscr{P}(D^n)})^{\mathrm{0PrM}(\alpha)},$$

$$\zeta_n^j \in (\mathscr{P}(D^n)^{\mathscr{P}(D^n) \times D^j})^{\mathrm{jPrM}(\alpha)}. \qquad 1 \leqslant j \leqslant k.$$

In order to define the valuation function λVal for the model λ, we need firstly to define the "pre-valuation" functions $\lambda \mathrm{Set}^n$: these are defined on the nPr, for $1 \leqslant n \leqslant m$, thus:

$$\begin{aligned}
\lambda \mathrm{Set}^n(F^n) &= \chi^n(F^n),\\
\lambda \mathrm{Set}^n(f^0(\varphi)) &= (\zeta^0_n(f^0))(\lambda \mathrm{Set}^n(\varphi)),\\
\lambda \mathrm{Set}^n(f^j(\varphi)x_1 \ldots x_j) &= \zeta^0_j(f^j))(\lambda \mathrm{Set}^n(\varphi),\ \psi(x_1), \ldots, \psi(x_j)).
\end{aligned}$$

We then use $\lambda \mathrm{Set}^n$ in the definition of λVal just as χ^n is used in the usual rules. Thus:

$$\begin{aligned}
\lambda \mathrm{Val}(\varphi(x_1, \ldots, x_n)) &= 1 \text{ iff } \langle \psi(x_1), \ldots, \psi(x_n) \rangle\ \varepsilon\ \lambda \mathrm{Set}^n(\varphi),\\
\lambda \mathrm{Val}(p) &= \varphi(p),\\
\lambda \mathrm{Val}(\sim\beta) &= 1 \text{ iff } \lambda \mathrm{Val}(\beta) = 0,\\
\lambda \mathrm{Val}((\beta \supset \gamma)) &= 1 \text{ iff } \lambda \mathrm{Val}(\beta) = 0 \text{ or } \lambda\ \mathrm{Val}(\gamma) = 1,\\
\lambda \mathrm{Val}((\forall x)\beta) &= 1 \text{ iff } \lambda' \mathrm{Val}(\beta) = 1 \text{ for all x-variants}
\end{aligned}$$

λ' of λ, where λ' is an x-variant of λ iff λ' differs from λ only in containing ψ' where λ contains ψ, and ψ' differs from ψ, if at all, only in what it assigns to x from D (or in that it assigns a value from D to x and ψ assigns none).

Finally, we say that α is QTM-valid iff it is true in all QTM models, i.e.

$$\text{QTM-Valid}(\alpha) =_{\mathrm{df}} \bigwedge_{\lambda} (\lambda \mathrm{Val}(\alpha) = 1).$$

The notion of a model for a set Γ of WQTM easily generalizes upon the notion of a model for a single wff. In place of PV(α) we use $\bigcup_{\alpha \in \Gamma} \mathrm{PV}(\alpha)$, for nPrV($\alpha$) we have $\bigcup_{\alpha \in \Gamma} \mathrm{nPrV}(\alpha)$, and so on. A model λ satisfies a set Γ of WQTM iff each member of Γ is true in λ i.e. iff $(\alpha)(\alpha \in \Gamma . \supset . \ \lambda \mathrm{Val}(\alpha) = 1)$.

To axiomatize QTM we need to have in mind some complete axiomatization of the underlying ordinary quantifier theory QT. For definiteness, we suppose this to be Church's F^{1p} ([2] p. 172): for ease we suppose the completeness theorem for this to be dealt with as in Leblanc's [3] § 2.7. In particular, we take it that every consistent set Γ of closed wffs of QTM can be extended to a suitable saturated (maximal consistent, Lindenbaum) set Γ^+ (= Leblanc's S^{∞}_{∞}) of wffs of QTM^{∞}, i.e. QTM with $\aleph_0$ extra individual constants $c_1, c_2, \ldots$.

We show now that if we add to F^{1p} the extensionality principles for modifiers, viz.

$$(\text{Mext}^0) \quad (\forall x_1) \ldots (\forall x_n)(F^n(x_1, \ldots, x_n) \equiv G^n(x_1, \ldots, x_n)) \supset (\forall x_1) \ldots (\forall x_n)(f^0(F^n)(x_1, \ldots, x_n) \equiv {}^0(G^n)(x_1, \ldots, x_n)),$$

and

$$(\text{Mext}^j) \quad (\forall x_1) \ldots (\forall x_n)(F^n(x_1, \ldots, x_n) \equiv G^n(x_1, \ldots, x_n)) \supset (\forall x_1) \ldots (\forall x_n)(\forall y_1) \ldots (\forall y_j)(f^j(F^n)y_1 \ldots y_j (x_1, \ldots, x_n) \equiv f^j(G^n)y_1 \ldots y_j(x_1, \ldots, x_n)), \quad \text{for } j \geqslant 1,$$

then we can fish suitable functions ζ^0_n ζ^j_n out of the set Γ^+, and hence that these principles completely axiomatize QTM (since all other elements of a model λ for Γ are given by the standard completeness proof).

We notice that by a routine induction, the principles Mext^i allow us to prove the general extensionality principle:

$$(\forall x_1) \ldots (\forall x_n)(\varphi(x_1, \ldots, x_n) \equiv \psi(x_1, \ldots, x_n)) \supset (\forall x_1) \ldots (\forall x_n)(\forall y_1) \ldots (\forall y_j)(f^j(\varphi)y_1 \ldots y_j(x_1, \ldots, x_n) \equiv f^j(\psi)y_1 \ldots y_j(x_1, \ldots, x_n)),$$ for any nPr's φ and ψ.

Now, given Γ^+ let us define the functions $\bar{\zeta}^0_n$ and $\bar{\zeta}^j_n$, $j \geqslant 1$, for the canonical model $\bar{\lambda}$ by the prescription:

$\bar{\zeta}^0_n(f^0)$ assigns $\{\langle c_1, \ldots, c_n\rangle \mid f^0(\varphi)(c_1, \ldots, c_n) \in \Gamma^+\}$ to $\{\langle c_1, \ldots, c_n\rangle \mid \varphi(c_1, \ldots, c_n) \in \Gamma^+\}$

and assigns $\{\}$ to any set $B \in \mathscr{P}(\bar{D}^n)$ for which there is no nPr φ such that $B = \{\langle c_1, \ldots, c_n\rangle \mid \varphi(c_1, \ldots, c_n) \in \Gamma^+\}$, and

$\bar{\zeta}^j_n(f^j)$ assigns $\{\langle c_1, \ldots, c_n\rangle \mid f^j(\varphi)c'_1 \ldots c'_j (c_1, \ldots, c_n) \in \Gamma^+\}$ to $\langle\{\langle c_1, \ldots, c_n\rangle \mid \varphi(c_1, \ldots, c_n) \in \Gamma^+\}, c'_1, \ldots, c'_j\rangle$,

and assigns $\{\}$ to any $\langle B, c'_1, \ldots, c'_j\rangle \in \mathscr{P}(\bar{D}^n) \times \bar{D}^j$ for which there is no nPr φ such that $B = \{\langle c_1, \ldots, c_n\rangle \mid \varphi(c_1, \ldots c_n) \in \Gamma^+\}$. These prescriptions define $\bar{\zeta}^0_n$ and $\bar{\zeta}^j_n$ unambiguously because of the general extensionality principle: if there are two nPr's φ and ψ such that:

$$\{\langle c_1, \ldots, c_n\rangle \mid \varphi(c_1, \ldots, c_n) \in \Gamma^+\} = \{\langle c_1, \ldots, c_n\rangle \mid \psi(c_1, \ldots, c_n) \in \Gamma^+\},$$

then

$$(\forall x_1) \ldots (\forall x_n)(\varphi(x_1, \ldots, x_n) \equiv \psi(x_1, \ldots, x_n))$$

holds, and hence by extensionality we have:

$$(\forall x_1) \ldots (\forall x_n)(f^0(\varphi)(x_1, \ldots, x_n) \equiv f^0(\psi)(x_1, \ldots, x_n)),$$

and

$$(\forall x_1) \ldots (\forall x_n)(\forall y_1) \ldots (\forall y_j)(f^j(\varphi)y_1 \ldots y_j(x_1, \ldots, x_n) \\ \equiv f^j(\psi)y_1 \ldots y_j(x_1, \ldots, x_n)),$$

and so

$$\{\langle c_1, \ldots, c_n\rangle \mid f^0(\varphi)(c_1, \ldots, c_n) \in \Gamma^+\} = \{\langle c_1, \ldots, c_n\rangle \mid \\ f^0(\psi)(c_1, \ldots, c_n) \in \Gamma^+\},$$

and

$$\{\langle c_1, \ldots, c_n\rangle \mid f^j(\varphi)c'_1 \ldots c'_j (c_1, \ldots, c_n) \in \Gamma^+\} = \\ \{\langle c_1, \ldots, c_n\rangle \mid f^j(\psi)c'_1 \ldots c'_j (c_1, \ldots, c_n) \in \Gamma^+\},$$

making the values of the functions $\bar{\zeta}^0_n$ and $\bar{\zeta}^j_n$ unambiguously defined.

We now claim that $\bar{\lambda}\text{Set}^n(\varphi) = \{\langle c_1, \ldots, c_n\rangle \mid \varphi(c_1, \ldots, c_n) \in \Gamma^+\}$ for any nPr φ, and we back up this claim by a proof by induction on the complexity of the nPr. For the basis clause, $\bar{\lambda}\text{Set}^n(F^n) = \bar{\chi}^n(F^n) = \{\langle c_1, \ldots, c_n\rangle \mid F^n(c_1, \ldots, c_n) \in \Gamma^+\}$ by the usual definition of $\bar{\chi}^n$. For a 0-ary modifier:

$$\bar{\lambda}\text{Set}^n(f^0(\varphi)) = (\bar{\zeta}^0_n(f^0))(\bar{\lambda}\text{Set}(\varphi)) \\ = (\bar{\zeta}^0_n(f^0))\{\langle c_1, \ldots, c_n\rangle \mid \varphi(c_1, \ldots, c_n) \in \Gamma^+\})$$

by the inductive hypothesis,

$$= \{\langle c_1, \ldots, c_n\rangle \mid f^0(\varphi)(c_1, \ldots, c_n) \in \Gamma^+\}$$

by the definition of $\bar{\zeta}^0_n$;

and for a j-ary modifier, $j \geqslant 1$,

$$\bar{\lambda}\text{Set}^n(f^j(\varphi)c'_1 \ldots c'_j) \\ = (\bar{\zeta}^j_n(f^j))(\bar{\lambda}\text{Set}(\varphi), c'_1, \ldots, c'_j) \\ = (\bar{\zeta}^j_n(f^j))(\{\langle c_1, \ldots, c_n\rangle \mid \varphi(c_1, \ldots, c_n) \in \Gamma^+\}, c'_1, \ldots, c'_j)$$

by the inductive hypothesis

$= \{\langle c_1, \ldots, c_n\rangle \mid f^j(\varphi)c'_1 \ldots c'_j\,(c_1, \ldots, c_n) \in \Gamma^+\}$

by the definition of $\bar{\zeta}^j_n$.

Now in the usual claim that $\bar{\lambda}\mathrm{Val}(\alpha) = 1$ iff $\alpha \in \Gamma^+$, so that $\bar{\lambda}$ indeed satisfies $\Gamma \subseteq \Gamma^+$, our basis step for elementary wffs $\varphi(x_1, \ldots, x_n)$ is

$\bar{\lambda}\mathrm{Val}(\varphi(c_1, \ldots, c_n)) = 1$ iff

$\langle \bar{\psi}(c_1), \ldots, \bar{\psi}(c_n)\rangle \in \bar{\lambda}\mathrm{Set}^n(\varphi)$,

i.e. iff $\langle c_1, \ldots, c_n\rangle \in \bar{\lambda}\mathrm{Set}^n(\varphi)$

(since the individual constants denote themselves in the canonical model),

i.e. iff $\langle c_1, \ldots, c_n\rangle \in \{\langle c_1, \ldots, c_n\rangle \mid \varphi(c_1, \ldots, c_n) \in \Gamma^+\}$

(by the previous paragraph),

i.e. iff $\varphi(c_1, \ldots, c_n) \in \Gamma^+$.

The rest of the work for the completeness proof is the same as for the usual work for ordinary QT, and this proves QTM to be completely axiomatized by the principles Mext^0 and Mext^j added to F^{1p}, and incidentally gives the usual Löwenheim-Skolem result for QTM as well.

SYSTEMS QTM(D), QTM(C) and QTM(D, C)

These systems each have the same vocabulary as QTM: semantically they result from the addition respectively of the *inclusion condition*

viz. $\mathbf{f}(A) \subseteq A$, for $\mathbf{f} \in D_1(\zeta^0_n)$

and $\mathbf{f}(A, b_1, \ldots, b_j) \subseteq A$, for $\mathbf{f} \in D_1(\zeta^j_n)$, $j \geqslant 1$,

the *crossover condition*

viz. $\mathbf{g}(\mathbf{f}(A, a_1, \ldots, a_j), b_1, \ldots, b_i) \subseteq \mathbf{f}(\mathbf{g}(A, b_1, \ldots, b_i), a_1, \ldots, a_j)$, for $\mathbf{f} \in D_1(\zeta^j_n$ and $\mathbf{g} \in D_1(\zeta^i_n)$,

(with suitable modification if either $j = 0$ or $i = 0$),
and both of these conditions to the semantics for QTM.

Axiomatically we have the *demodification schema* to match the inclusion condition, and the *commutation schema* to match the crossover condition. These are, respectively:

(DS^0) $(\forall x_1) \dots (\forall x_n)(f^0(\varphi)(x_1, \dots, x_n) \supset \varphi(x_1, \dots, x_n))$,

and

(DS^j) $(\forall x_1) \dots (\forall x_n)(\forall y_1) \dots (\forall y_j)(f^j(\varphi)y_1 \dots y_j(x_1, \dots, x_n) \supset \varphi(x_1, \dots, x_n))$,

and

(CS) $(\forall x_1) \dots (\forall x_n)(\forall y_1) \dots (\forall y_j)(\forall z_1) \dots (\forall z_i)(g^i(f^j(\varphi)y_1 \dots y_j)z_1 \dots z_i(x_1, \dots, x_n) \supset f^j(g^i(\varphi)z_1 \dots z_i)y_1 \dots y_j (x_1, \dots, x_n))$

(where again CS has some degenerate cases if $j = 0$ or $i = 0$).

To show that QTM(D) and QTM(C) are complete, we need to show that if (DS^i) are axioms of the system then the functions $\bar{\zeta}^j_n(f^j)$ satisfy the inclusion condition, and if (CS) are axioms then these functions satisfy the crossover condition.

Well, if (DS^i) are axioms then they and all their instantiations belong to Γ^+. Moreover Γ^+ is closed under detachment, hence if $f^j_1(\varphi)\, c'_1 \dots c'_j\, (c_1, \dots, c_n) \in \Gamma^+$ then:

$$\varphi(c_1, \dots, c_n) \in \Gamma^+, \quad \text{hence:}$$
$$\{\langle c_1, \dots, c_n\rangle \mid f^j(\varphi)c'_1 \dots c'_j(c_1, \dots, c_n) \in \Gamma^+\} \subseteq \{\langle c_1, \dots, c_n\rangle \mid \varphi(c_1, \dots, c_n) \in \Gamma^+\},$$

and this is precisely the inclusion condition. If (CS) are axioms, a similar argument applies: the closure of Γ^+ under detachment translates the $\supset$ in the axiom-schema to the $\subseteq$ in the crossover condition.

In [5], we showed (DS^i) and (CS) to be independent: there is no trouble about combining the conditions, so QTM(D, C) is complete as well as QTM(D) and QTM(C).

POLYADIC MODIFIERS — QTPM

In QTPM, the modifiers of QTM are extended so that they may modify several predicates at once. Examples of such polyadic modifiers are provided by truth-functional conjunction, disjunction etc. of predicates, as in [1] § 28a, and by such adverbs as 'successively' which indeed is a multigrade polyadic modifier since it may take any number of predicates as its arguments to form a new predicate.

For the syntax of QTPM, we replace the S(iv) of QTM by

S(iv)P $f^{r,n}$, $g^{r,n}$, $h^{r,n}$, $f_1^{r,n}$, $g_1^{r,n}$, ...
for each $r \geqslant 1$, $n \geqslant 0$, designated as r-place n-ary modifiers.

The modifiers of QTM correspond to the case where $r=1$, so that our new $f^{1,n}$ corresponds to our previous f^n. We extend W(ii) of QTM to

W(ii)P If $\varphi_1, \ldots, \varphi_r$ are all in nPr,
then $f^{r,0}(\varphi_1, \ldots, \varphi_r)$ and
$f^{r,j}(\varphi_1, \ldots, \varphi_r)x_1 \ldots x_j$ are in nPr.

We let rjPrM(α) be the set of r-place j-ary modifiers in a WQTPM α, and we put $S(\alpha) =_{df} \max r((\exists j)(\text{rjPrM}(\alpha) \neq \{\}))$. Then a QTPM model for a wff α is an ordered (skm + sm + m + 3)-tuple

$$\lambda = \langle D, \varphi, \psi, \chi^1, \ldots, \chi^m, \zeta_1^{1,0}, \zeta_2^{1,0}, \ldots, \zeta_m^{1,0}, \zeta_1^{2,0}, \ldots, \zeta_m^{2,0}, \ldots, \zeta_1^{s,0}, \ldots, \zeta_m^{s,0}, \ldots, \ldots, \zeta_1^{s,k}, \ldots, \zeta_m^{s,k} \rangle$$

where $m = M(\alpha)$, $k = K(\alpha)$, $s = S(\alpha)$, $D, \ldots, \chi^m$ are as for QTM, and where

$$\zeta_n^{r,0} \in \left(\mathscr{P}(D^n)^{(\mathscr{P}(D^n))^r}\right)^{\text{rOPrM}(\alpha)} \qquad \text{for } 1 \leqslant r \leqslant s,\ 1 \leqslant n \leqslant m,$$

and

$$\zeta_n^{r,j} \in \left(\mathscr{P}(D^n)^{((\mathscr{P}(D^n))^r \times D^j)}\right)^{\text{rjPrM}(\alpha)} \qquad \text{for } 1 \leqslant r \leqslant s,\ 1 \leqslant n \leqslant m,\ 1 \leqslant j \leqslant k.$$

The functions $\zeta_n^{r,j}$ then appear in the definition of λSet^n, as per

$$\lambda\text{Set}^n(F^n) = \chi^n(F^n),$$
$$\lambda\text{Set}^n(f^{r,0}(\varphi_1 \ldots \varphi_r)) = (\zeta_n^{r,0}(f^{r,0}))(\lambda\text{Set}^n(\varphi_1), \ldots, \lambda\text{Set}^n(\varphi_r)),$$

$$\lambda \mathrm{Set}^n(f^{r,j})(\varphi_1 \dots \varphi_r)x_1 \dots x_j)$$
$$= (\zeta_n^{r,j}(f^{r,j}))(\lambda \mathrm{Set}^n(\varphi_1), \dots, \lambda \mathrm{Set}^n(\varphi_r), \psi(x_1), \dots, \psi(x_j)), \qquad j \geqslant 1,$$

and from here on the definition of λVal is as for QTM. Then we have

$$\text{QTPM-Valid}(\alpha) =_{df} \bigwedge_{\lambda}(\lambda \mathrm{Val}(\alpha) = 1),$$

where now the quantifier ranges over all QTPM models for α.

To shorten the axiom-schemata for QTPM, let us write:

$$\mathrm{Coex}(F^n, G^n) =_{df} (\forall x_1) \dots (\forall x_n)(F^n(x_1, \dots, x_n) \equiv G^n(x_1, \dots, x_n)).$$

Then the extensionality principles which serve as axioms for QTPM are:

$$(\mathrm{Mext}_P^0) \quad \bigwedge_{i=1}^{r} (\mathrm{Coex}(F_i^n, G_i^n)) \supset (\forall x_1) \dots (\forall x_n)(f^{r,0}(F_1^n, \dots, F_r^n)$$
$$(x_1, \dots, x_n) \equiv f^{r,0}(G_1^n, \dots, G_r^n)(x_1, \dots, x_n)),$$

and

$$(\mathrm{Mext}_P^j) \quad \bigwedge_{i=1}^{r} (\mathrm{Coex}(F_i^n, G_i^n)) \supset (\forall x_1) \dots (\forall x_n)(\forall y_1) \dots (\forall y_j)$$
$$(f^{r,j}(F_1^n, \dots, F_r^n)y_1 \dots y_j(x_1, \dots, x_n) \equiv f^{r,j}(G_1^n, \dots, G_r^n)$$
$$y_1 \dots, y_j(x_1, \dots, x_n)).$$

For the completeness proof for QTPM we use these extended extensionality principles to justify an unambiguous definition of the functions $\bar{\zeta}_n^{r,j}$ in the canonical model $\bar{\lambda}$. The work entirely parallels that for QTM, and we take the completeness of QTPM as proved.

HETERADIC MODIFIERS — QTHM

We introduce now the notion of *heteradic* modifiers. All of our existing modifiers could be called *homadic,* in that the adinity of the result of the modification is the same as that of the predicate or predicates serving as arguments for the modifier. A heteradic modifier, on the other hand, is a modifier which results in a predicate of adinity different from its argument(s). The simplest kind of heteradic modifier would be one which gave a result of constant adinity regardless of that of its argument(s): a more general kind of heteradic modifier is one such that the adinity of its result is a function of the adinity of its arguments. Thus in general we can countenance μ-resultant r-place j-ary modifiers $f^{\mu,r,j}$, which are like the r-place j-ary modifiers of QTPM except that if $\varphi_1, \ldots, \varphi_r$ have adinities respectively $n_1, \ldots, n_r$ then $f^{\mu,r,j}(\varphi_1 \ldots \varphi_r)x_1 \ldots x_j$ has adinity $\mu(n_1, \ldots, n_r)$, where $\mu \in {}^{(N^r)}$, $N = \{1, 2, 3, \ldots\}$.

To symbolize modifiers of such generality, we have to overcome (or rather ignore) the technical problem that for finite r $N^{(N^r)}$ has cardinality $\aleph = 2^{\aleph_0}$, so that we cannot enumerate the μ-resultant r-place j-ary modifiers. We cannot write a *function* μ into a superscript position in our syntax without going well beyond the usual kinds of syntactic mechanisms, and because we cannot enumerate all the μ's we cannot suppose μ to be an index-number for the function in question. We could suppose that we are restricted to some denumerable subclass of $N^{(N^r)}$), e.g. the general recursive functions; or perhaps we could suppose that we have a variable ordering of the functions in $N^{(N^r)}$ so that in any set of formulae to μ's considered are in the first $\aleph_0$ of the ordering. Whatever we suppose, we will operate as if we had no problem: we will use μ ambiguously as a function and as an index for a function, and will say things like "$f^{\mu,r,0}(\varphi_1 \ldots \varphi_r)$ is a $\mu(n_1, \ldots, n_r)$-adic predicate".

In the syntax for QTHM, S(ii) becomes

S(ii)H $f^{\mu,r,n}$, $g^{\mu,r,n}$, $h^{\mu,r,n}$, $f_1^{\mu,r,n}$, $g_1^{\mu,r,n}$,

for $\mu \geqslant 1$, $r \geqslant 1$, $n \geqslant 0$,

designated as μ-resultant r-place n-ary modifiers, where "$\mu \geqslant 1$" means that μ is an index for some portion of $N^{(N^r)}$, of one of the two kinds we have supposed possible. In defining WQTHM we have:

W(ii)H If $\varphi_1, \ldots, \varphi_r$, are respectively in $n_1Pr, \ldots, n_rPr$, then

$f^{\mu,r,0}(\varphi_1, \ldots, \varphi_r)$ and $f^{\mu,r,j}(\varphi_1, \ldots, \varphi_r)x_1 \ldots x_j$ are in $(\mu(n_1, \ldots, n_r))$Pr.

W(ii)H differs from W(ii) and W(ii)P in that it gives a definition by simultaneous recursion of all the classes nPr, rather than applying to each n in turn as do W(ii) and W(ii)P.

If α is a WQTHM, we let μrjPrM(α) be the set of μ resultant r-place j-ary predicate modifiers in α, and we let ωPr(α) be the union over n of the sets of well-formed n-ary predicates in α. We then put:

$$\mathscr{C}_r(\alpha) =_{df} \{ \langle n_1, \ldots, n_r \rangle \mid (\exists\beta)(\beta \in \omega\mathrm{Pr}(\alpha) \\ .\&.\ (\exists\mu)(\exists j)(\beta = f^{\mu,r,j}(\varphi_1 \ldots \varphi_r)x_1 \ldots x_j \\ .\&.\ \bigwedge_{i=1}^{r} (\varphi_i \in n_i\mathrm{Pr})) \}.$$

For any r, $\mathscr{C}_r(\alpha)$ is a finite subset of N^r; let $\mathscr{C}_r(\alpha)$ have t_r members, and index $\mathscr{C}_r(\alpha)$ by $\mathscr{I}_r$, so that $\mathscr{C}_r(\alpha) = \{\mathscr{I}_r(1), \ldots, \mathscr{I}_r(t_r)\}$. For each r-tuple $\mathscr{I}_r(n)$ in $\mathscr{C}_r(\alpha)$ we write $\mathscr{I}_r(n))_i$, for $1 \leqslant i \leqq r$, for the ith element of $\mathscr{I}_r(n)$, i.e. $\mathscr{I}_r(n) = < \mathscr{I}_r(n))_i, \ldots, (\mathscr{I}_r(n))_r >$. Now if we put $S(\alpha) =_{df} \max r(\exists\mu)(\exists j)(\mu rj\mathrm{PrM}(\alpha) \neq \{\ \}))$ and let $s = S(\alpha)$, we can specify the QTHM models.

A QTHM model for α is an ordered:

$$((\prod_{i=1}^{s} (s \times (k+1) \times t_i)) + m + 3)\text{-tuple } \lambda =$$

$$(D, \varphi, \psi, \chi^1, \ldots, \\ \chi^m\ \zeta_1^{1,0}, \zeta_2^{1,0}, \ldots, \zeta_{t_1}^{1,0}, \zeta_{t_2}^{2,0}, \ldots, \zeta_1^{s,0}, \ldots, \zeta_{t_s}^{s,0}, \ldots, \zeta_{t_s}^{s,0}, \ldots, \zeta_1^{s,k}, \ldots, \zeta_{t_s}^{s,k} \rangle$$

where D, ..., χ^m are as in QTPM models and

$$\zeta_n^{r,0} \in \left(\mathscr{P}(D^{\mu(\mathscr{I}_r(n))})^{\overset{r}{\underset{i=1}{\times}} (\mathscr{P}(D^{(\mathscr{I}_r(n))}))} \right)^{\mu r0\mathrm{PrM}(\alpha)} \quad \text{for } 1 \leqslant r \leqslant s,\ 1 \leqslant n \leqslant t_r,$$

and

$$\zeta_n^{r,j} \in \left(\mathscr{P}(D^{\mu(\mathscr{I}_r(n))})^{\overset{r}{\underset{i=1}{\times}} (\mathscr{P}(D^{(\mathscr{I}_r(n))_i}) \times D^j)} \right)^{\mu rj\mathrm{PrM}(\alpha)} \quad \text{for } 1 \leqslant r \leqslant s,\ 1 \leqslant n \leqslant t_r,\ 1 \leqslant j \leqslant k.$$

(If any t_r is 0 then there are no functions $\zeta_n^{r,j}$ in the model.)

We then use the functions $\zeta^{r,j}_{n}$ in the definition of the function λSetω: this differs from the previous functions λSet^{n} in that it is defined over all of $\bigcup_{n<\omega} nPr$, rather than one function for each nPr. The definition is:

$$\lambda Set\omega(F^{n}) = \chi^{n}(F^{n}),$$

$$\lambda Set\omega(f\mu^{,r,0}(\varphi_1 \ldots \varphi_r)) = (\zeta^{r,0}_{\mathscr{I}_r^{-1}(n_1,\ldots,n_r)}(f\mu^{,r,0}))\,(Set\omega(\varphi_1), \ldots, \lambda Set\omega(\varphi_n)),$$

$$\lambda Set\omega(f\mu^{,r,j}(\varphi_1 \ldots \varphi_r)x_1 \ldots x_j) = (\zeta^{r,0}_{\mathscr{I}_r^{-1}(n_1,\ldots,n_r)}(f\mu^{,r,j}))\,(\lambda Set\omega(\varphi\varphi_1), \ldots, \lambda Set\omega(\varphi_r)\,\psi(x_1), \ldots, \psi(\varkappa_j)),$$

$$j \geqslant 1$$

where $\varphi_i \in n_i Pr$ for $1 \leqslant i \leqslant r$. We now use λSetω in place of λSet^{n} and proceed as for QTM or QTPM to gain the definition of λVal and thence the definition of QTHM-Validity.

The generalization of the extensionality principes $Mext^{i}$ and $Mext^{i}_{P}$ for heteradic modifiers is:

$$(Mext^{0}_{H}) \quad \bigwedge_{i=1}^{r}(Coex(F_i^{n_i}, G_i^{n_i})) \supset (\forall x_1) \ldots (\forall x_{\mu(n_1,\ldots,n_r)})$$
$$(f\mu^{r,0}(F^{n_1} \quad F_r^{n_r})\,(x_1, \ldots, x_{\mu(n_1,\ldots,n_r)}) \equiv$$
$$f\mu^{r,0}(G^{n_1} \ldots G^{n_r})\,(x_1, \ldots, x_{\mu(n_1,\ldots,n_r)}))$$

and

$$(Mext^{j}_{H}) \quad \bigwedge_{i=1}^{r}(Coex(F_i^{n_i}, G_i^{n_i})) \supset (\forall x_1) \ldots (\forall x_{\mu(n_1,\ldots,n_r)})$$
$$((\forall y_1) \ldots (\forall y_j)(f\mu^{,r,j}(F_1^{n_1} \ldots F_r^{n_r})\,y_1 \ldots y_j\,(x_1, \ldots,$$
$$x_{\mu(n_1,\ldots,n_r)}) \equiv f\mu^{,r,j}(G_1^{n_i} \ldots G_r^{n_r})\,y_1 \ldots y_j(x_1, \ldots,$$
$$x_{\mu(n_1,\ldots,n_r)})).$$

Again we follow the pattern of QTM for the completeness proof: the extensionality principles allow unambiguous definitions for functions $\bar{\zeta}^{r,j}_n$ in the canonical model $\bar{\lambda}$, and these functions are compatible with $\bar{\lambda}Set\omega$ in the required way to make $\bar{\lambda}$ into a model for the initial set of sentences Γ.

SOME CONSTANT MODIFIERS

We now introduce into QTHM some 0-ary constant modifiers which "correspond" to the usual propositional and quantificational operators. For the constant corresponding to negation, we have the 1-place homadic modifier n^0. Being 1-place and homadic, n^0 could have been introduced into QTM, and indeed in [5] we did introduce it when dealing with contraries and contradictories. The constant n^0 can be characterized by the axiom:

(Ax n) $n^0(\varphi)(x_1, \ldots, x_n) \equiv \sim\varphi(x_1, \ldots, x_n)$.

For the constant corresponding to conjunction, we have the 2-place homadic modifier $k^{2,0}$. Being homadic, $k^{2,0}$ could have been introduced into QTPM, but being polyadic (2-place) it could not have been introduced into QTM. Axiomatically $k^{2,0}$ is characterized by:

(Ax k) $k^{2,0}(\varphi\psi)(x_1, \ldots, x_n) \equiv (\varphi(x_1, \ldots, x_n) \& \psi(x_1, \ldots, x_n))$.

For a modifier corresponding to quantification, it seems natural to consider existential rather than universal quantification since e.g. the predicate of being a parent is formed from the parenthood relation by a suitable existential quantification. There is, however, not just one modifier corresponding to existential quantification, since an n-ary relation can be existentially quantified in each of its n places to form a different $(n-1)$-ary relation. Thus we introduce the 1-place heteradic modifiers $s^{1,0}_u$, $u \geqslant 1$: if φ is in nPr for $n \geqslant 2$ then if $1 \leqslant u \leqslant n$ then $s^{1,0}_u(\varphi)$ is in $(n-1)$Pr (we add formation rules preventing $s^{1,0}_u(\varphi)$ from being well-formed if $u > n$ or if $n = 1$). Thus for each $s^{1,0}_u$, the resultant function μ is simply the predecessor function. The modifiers $s_u^{1,0}$ are characterized axiomatically by:

(Ax s) $s_u^{1,0}(\varphi)(x_1, \ldots, x_{u-1}, x_{u+1}, \ldots, x_n) \equiv (\exists x_u)$
$\varphi(x_1, \ldots, x_{u-1}, x_u, x_{u+1}, \ldots, x_n)$,

where $\varphi \in nPr$, $n > 1$, and $1 \leqslant u \leqslant n$.

The question now arises as to the relation that modifiers in general bear to these constant modifiers: for instance whether the following schema holds in general:

(Cn 1) $f^0(n^0(\varphi))(x_1, \ldots, x_n) \supset n^0(f^0(\varphi))(x_1, \ldots, x_n)$?

The answer to this question is surely no, since if f^0 is some $s_u^{1,0}$ then in effect the schema would license as valid the invalid $(\exists x) \sim Fx \supset \sim (\exists x)Fx$. If we consider the converse of (Cn 1), viz.

(Cn 2) $n^0(f^0(\varphi))(x_1, \ldots, x_n) \supset f^0(n^0(\varphi))(x_1, \ldots, x_n)$,

then this is also not intuitively acceptable as valid, since it would license the inference from

This is a non red-house
to This is a red non-house,

which isn't valid.

Similarly we can investigate the schemata

(Ck 1) $f^0(k^{2,0}(\varphi\psi))(x_1, \ldots, x_n) \supset k^{2,0}(f^0(\varphi)f^0(\psi))(x_1, \ldots, x_n)$,

and

(Ck 2) $k^{2,0}(f^0(\varphi)f^0(\psi))(x_1, \ldots, x_n) \supset f^0(k^{2,0}(\varphi\psi))(x_1, \ldots, x_n)$.

Here, (Ck 1) is invalid if f^0 is n^0, and (Ck 2) is invalid if f^0 is some $s_u^{1,0}$: both schemata will be valid if f^0 takes on only *detachable* functions, in the sense of [5] § 7, but since we have no axiomatic characterization of these we cannot suitably formulate a syntactic system in which (Ck 1) and (Ck 2) are valid.

We may also consider the following schemata relating modifiers in general to the "existential" modifiers $s_u^{1,0}$:

(Cs 1) $f^0(s_u^{1,0}(\varphi))(x_1, \ldots, x_{n-1}) \supset s_u^{1,0}(f^0(\varphi))(x_1, \ldots, x_{n-1})$,

and its converse

(Cs 2) $s_u^{1,0}(f^0(\varphi))(x_1, \ldots, x_{n-1}) \supset f^0(s_u^{1,0}(\varphi))(x_1, \ldots, x_{n-1})$.

Letting f^0 be n^0, (Cs 2) turns into the previously rejected instance of (Cn 1), so Cs 2) must be rejected. Cs 1), on the other hand, licenses inferences like:

Hamlet killed (someone) quickly
⁂ Hamlet quickly killed (someone),

and such inferences are intuitively valid. Apart from intuitive acceptability, there is an important technical reason why (Cs 1), or something like it, should be required to hold. The reason is that (Cs 1) gives a link between the functioning of a modifier on predicates of different adinities: in the antecedent f^0 modifies a predicate in (n — 1)Pr, and in the consequent it modifies a predicate in nPr. At present in the semantics for QTM (or QTPM or QTHM) the functions ζ_n^0 ζ_m^0 for $n \neq m$ have no connection with each other, and there is no requirement that the modification given by an adverb like "quickly" when applied, say, to dyadic relations should have anything in common with the same adverb's modification when applied to monadic predicates. There should be some kind of connection, and such a connection will be given syntactically by laying down some generalization of (Cs 1) as an axiom. The generalization of (Cs 1) for 1-place homadic modifiers is just:

(Cs 1^j) $f^j(s_u^{1,0}(\varphi))y_1 \ldots y_j(x_1, \ldots, x_{n-1}) \supset s_u^{1,0}(f^j(\varphi)y_1, \ldots, y_j)$
$(x_1 \ldots x_{n-1})$.

In order to state the semantical condition corresponding to (Cs 1^j) we begin by giving the semantics for all the constant modifiers n^0, $k^{2,0}$, $s_u^{1,0}$.

Clearly we require:

$$(\zeta_n^0(n^0))(A) = \mathscr{P}(D^n) - A,$$

and

$$(\zeta_n^{2,0}(k^{2,0}))(AB) = A \cap B.$$

For the $s_u^{1,0}$ we use the cylindrification functions C_u:

$$C_u(A) =_{df} \{\langle a_1, \ldots, a_{u-1}, a_{u+1}, \ldots, a_n\rangle \mid (\exists a_u)(\langle a_1, \ldots, a_{u-1}, a_u, a_{u+1}, \ldots, a_n\rangle \in A),$$

for $A \in \mathscr{P}(D^n)$ and $1 \leqslant u \leqslant n$. Then simply:

$$\zeta_n^{1,0}(s_u^{1,0}))(A) = C_u(A).$$

The semantical condition corresponding to (Cs 1^j) can now be stated.

If $f_{n-1} = \zeta_{n-1}^0(f^0)$ and $f_n = \zeta^0(f_1^0)$ or $f_{n-1} = \zeta_{n-1}^j(f^j)$ and $f^n = \zeta_n^j(f^j)$ then for (Cs 1^j) to be valid we require that:

$$f_{n-1}(C_u(A)) \subseteq C_u(f_n(A)),$$

and

$$f_{n-1}(C_u(A), a_1, \ldots, a_j) \subseteq C_u(f_n(A, a_1, \ldots, a_j)),$$

for $A \in \mathscr{P}(D^n)$ and $1 \leqslant u \leqslant n$. This condition does indeed give a link between the functions ζ_{n-1}^j and ζ_n^j (and thereby any ζ_n^j, ζ_m^j, $n \neq m$), and thus gives at least a minimal requirement for a modifier f^j to modify predicates of different adinity in compatible ways. To illustrate the condition, we show how it can fail for a 0-adic modifier f^0, where the conditions QTM(D, C) are satisfied and in fact all functions are detachable. Let $D = \{a\}$, $f_1 = \zeta_1^0(f^0), f_2 = \zeta_2^0(f^0)$ where $f_1 = \{\langle\{\},\{\}\rangle, \langle\{a\},\{a\}\rangle\}$ and $f_2 = \{\langle\{\},\{\}\rangle, \langle\{\},\{\langle a, a\rangle\}\rangle\}$. Then if $A = \{\langle a, a\rangle\}$ then $C_1(A) = \{a\}$, $f_1(C_1(A)) = f_1(\{a\}) = \{a\}$; but $f_2(A) = f_2(\{\langle a, a\rangle\}) = \{\}$ so $C_1(f_2(A)) = C_1(\{\}) = \{\}$ and not $\{a\} \subseteq \{\}$.

REMARKS

The logic of predicate modification is a useful and indeed vital device for various philosophical purposes. We do not claim that there is just one such logic, any more than there is just one logic for the quantifiers $(\forall x)$ and $(\exists x)$ (see e.g. Leblanc and Thomason [4] for the latter point): we do claim that it is useful to be able to locate accurately the kind of modification occurring in a given sentence or argument, to symbolize such modification and to demarcate the valid formulae in such symbolism. It is perhaps no accident that the three constant modifiers, for negation, conjunction and existential quantification, require successively the system QTM, QTPM and QTHM of increasing generality: the structure and leading characteristics of these and many other modifiers can best be investigated by having available a range of logics of predicate modification.

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