DOMAINLESS SEMANTICS FOR FREE, QUANTIFICATION, AND SIGNIFICANCE LOGICS

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The standard semantics for quantification logics have serious limitations; they are more complicated than they need be and more set theoretical than they should be. In support of this evaluation alternative simpler and less set-theoretical semantics are provided for quantification and free quantification logics both without and with identity, and for second-order significance logic. These semantics, domainless semantics, are defended against objections as to their intelligibility and satisfactoriness, and appropriate consistency and completeness theorems are proved in order to show the comparative adequacy of the semantics. Domainless semantics, by assigning values *en bloc* to atomic wff, eliminates the otiose notion of a domain of interpretation and n-place relations on this domain of entities, and thereby eliminates the associated correspondance theory of truth which is built into the reference selections and truth evaluations of standard semantics. It does this without introducing names in the style of, what is similar, the substitution interpretation of quantifiers, and so it avoids legitimate objections that have been made to substitution semantics [1].

Domainless semantics shows, then, that *quantification requires neither reference nor individuation.* Not surprisingly such semantics have big advantages over standard semantics. The standard semantics not only unnecessarily restricts the application of quantification logic to suitably purified reference domains, and accordingly, as it does not provide for all logical uses of quantifiers, does not really give the meaning of quantifiers; also it generates gratuitous philosophical problems like the problem of treating non-singular terms such as mass terms

[1] For these objections, and for answers to many objections to substitution semantics, see DUNN and BELNAP [3].

quantificationnally and the problem of combining quantifiers with modal and intensional factors. It is a fact that the logic captured by quantification theory outruns the standard semantics. For the formal schemes of quantification logic apply (see e. g. [11]) to reasoning about items which are not eligible for membership of reference domains just as well as to reasoning about entities of reference domains, e. g. to quantified and unquantified assertions about clouds, sounds, shades of colour, and hills, though the items concerned are often neither distinct nor definite and are certainly not always non-arbitrarily individuable and countable; and to assertions about items(beyond the reach of reference) like abracadabras, illusions, angels, future seabattles and square circles which are often incomplete, indeterminate and indistinct and likewise are frequently not individuable or countable. Standard semantics, nailed down by the complementary requirements of the theory of sets and the theory of reference, cannot easily admit any of these assertions. For theories of reference typically demand of referents definiteness, distinctness, singularity, individuality, and, when massed, countability; and the theory of sets has always been understood, from the time of Cantor's inaugural definition of a *set* as 'a collection into one whole of definite distinct objects...', as concerned solely with domains consisting of items which are definite, distinct, individuable and countable. (Independence proofs investigate the domains of distinct non standard theories). Nor is the link with Cantor's definition completely flimsy: for Ackermann, starting with this definition, argued his way to a theory which proved equivalent to Zermelo-Fraenkel set theory (less the dispensible axiom of regularity), and Zermelo-Fraenkel set theory has become the standard theory of sets. Thus to obtain an adequate semantics, filling out the formalism of the standard semantics, reference has to be replaced by a more liberal relationship, aboutness say, and a non standard theory of domains has to be forged. The theory of aboutness would concede that subjects in sentences, such as 'All the hills in this unmapped mountain chain merge into one another', may, in advance of legislation or surveys, be about items which are not individuable or determinate, and not fit items for reference. The theory of domains would have some unusual features also, e. g. it could admit domains, like the bands of colour in a visible rainbow, which are Dedekind finite but which are not countable in that a definite cardinal n cannot be assigned to the domain; thus the theory of domains would reject the axiom of choice. But though limitations of standard semantics can be overcome, while keeping to the bare formalism of standard semantics, by supplanting reference and sets (essentially the liberalisation adopted in [13]), the emancipated semantics too requires neither reference nor individuation. Standard semantics, in insisting on reference and individuation, try to foist off a particular, questionable, philosophical theory as a formal necessity dictated by quantification logic.

In a sequel domainless semantics for quantified weak modal logics will be studied. Use of domainless semantics for these logics cuts through foundational difficulties (²) over the individuation of possible items and the identity of ideas in different possible domains, by dispensing with domains. Though associated domains can be defined once the semantics have been set up, they do not raise comparable interpretational problems.

1. *Domainless semantics for quantification logic* **Q.**

The standard semantics of quantification logic assumes a modest amount of set theory. The domainless semantics offered for **Q** has a background logic an extended predicate logic **HQ**, with the symbols of **Q**, with truth values *t* and *f* introduced, and containing sentence predicates. For definiteness **Q** is taken to be formulated with primitive connections '$\sim$' and '$\supset$' and primitive universal quantifier 'A'. Such standard results for **Q** as the deduction theorem and substitution for predicate variables are taken for granted.

An interpretation $\mathscr{V}$ is a relation $\mathscr{V}$ of **HQ** which associates uniformly with each atomic wff of **Q** exactly one of the truth

(²) Some of these difficulties are mentioned in Hintikka [7], where a solution within the framework of a thinly disguised reference theory is attempted.

values t and f [3]. Where A is atomic, '$\mathscr{V}$ (A, t)' reads 'A holds, or A registers (under $\mathscr{V}$)', and $\mathscr{V}$ (A, f)' reads 'A does not hold, or A does not register (under $\mathscr{V}$)'. In terms of interpretation $\mathscr{V}$, a registering function is defined recursively for all wff of **Q** as follows:

1. Where A is atomic, $\mathscr{V}(A) = t \equiv . \mathscr{V}(A, t)$ and $\mathscr{V}(A) = f \equiv . \mathscr{V}(A, f)$;

Therefore, where A is atomic, $\mathscr{V}(A) = t \equiv . \mathscr{V}(A) \neq f$;

2. $\mathscr{V}(\sim A) = t \equiv . \mathscr{V}(A) \neq t$;
3. $\mathscr{V}(A \supset B) = t \equiv . \mathscr{V}(A) = t \supset \mathscr{V}(B) = t$;
4. $\mathscr{V}((Ax)B) = t \equiv . (Ax) (\mathscr{V}(B) = t)$.

In 4. the first quantifier is a quantifier of **Q** and also of **QH**, the second quantifier is a quantifier just of **HQ**. Likewise for connectives in 2. and 3. Key semantical notions are defined:

A is true under $\mathscr{V}$	$\equiv$. A is a closed wff & $\mathscr{V}(A) = t$;
A is false under $\mathscr{V}$	$\equiv$. A is a closed wff & $\mathscr{V}(A) = f$;
A is valid	$\equiv . (A\mathscr{V}) . \mathscr{V}(A) = t$;
A is satisfiable	$\equiv . (S\mathscr{V}) . \mathscr{V}(A) = t$.

Finally where Γ is a property of wff, $\mathscr{V}(\Gamma) = t \equiv . \mathscr{V}(B) = t$ for every wff B such that $\Gamma(B)$, and Γ is satisfiable $\equiv . (S\mathscr{V}) . \mathscr{V}(\Gamma) = t$.

The semantics can be reexpressed in terms of set ups. An *atomic set up* is a class of atomic wff (4). Consider the class H of atomic wff B such that $\mathscr{V}(B, t)$: then H is the atomic set up defined by $\mathscr{V}$. Conversely, the valuation function $\mathscr{V}$ determined by atomic set up H may be defined as a characteristic function indicating membership of H, *i. e.* $\mathscr{V}(A) = t \equiv . A \varepsilon H$. The details of this, equivalent, semantics are as follows: A **Q**-*model* is an

(3) Interpretations may be relativised to given wff or sets of wff. A relativised interpretation $\mathscr{V}_\Delta$ for class Δ wff of **Q** is a relation which associates with every atomic wff of Δ exactly one of values t and f.

(4) Were the interpretation defined only for atomic sentences, *i. e.* closed atomic wff — as is done for the substitution interpretation discussed in [3] — then use of the associated H would be tantamount to the use of a state description H^s (as defined by Carnap [1]), for given H, $H^s = H \cup \{\sim B : B$ is an atomic sentence & $B \notin H\}$, and given H^s, $H = \{B \in H^s : B$ is an atomic sentence$\}$.

atomic set up H. In terms of the modelling *holding* or being in H is defined recursively for all wff of **Q** thus:

1'. Where A is atomic, A is in H $\equiv$ A ε H
2'. $\sim$ A is in H $\equiv . \sim .$ A is in H
3'. (A $\supset$ B) is in H $\equiv$. A is in H $\supset$ B is in H
4'. (Ax)B is in H $\equiv$. (Ax). B is in H.

A *set up* **H** = {B: B is in H}; a set-up is an extended state description (in Hintikka's sense in [6]). Since it follows that
A is valid $\equiv$. A is in H, for every **Q**-model H,
it suffices to show the adequacy of just one of the semantics.

Before proving the adequacy of the semantics it is worth elaborating the differences between the semantics and marketed semantics. It differs from various standard semantics in being domainless, in that the truth value of subject predicate sentences is not assessed atomistically in terms of the items designated by the subjects having the property designated by the predicate, and in the interpretation of quantifiers. It differs from state description semantics, and the equivalent substitution semantics in that assignments are made to all wff, not just closed wff, and in the rules for quantifiers. It differs from the liberalised substitution semantics (mentioned in [3]) where assignments are made to all atomic wff, and also from Hintikka's model set semantics (as presented, most fully, in [5]) in rules 4 (and 4') for quantifiers. For in these semantics rule 4 is replaced by 4A. $\mathscr{V}((Ax)\ B(x)) = t$ iff $\mathscr{V}(B(a)) = t$ for all names (closed terms, constants a (of the logic), a deficient rule which causes strong incompleteness. (see [3]).

It is however just on the basis of rule 4 (and 4') that domainless semantics is likely to encounter most opposition; for it will be charged that the rule is meaningless, and unintelligible insofar as it differs from 4A. Consider for simplicity the paraphrase of rule 4. into logicians' English, along the levels of language lines recommended by Tarski and others, in the case of a one place predicate constant 'g_0': it is: 'for every x, $g_0(x)$' is true iff for every x '$g_0(x)$' registers. Here however the variable symbol on the right hand side occurs within quotation marks, and accordingly does not function as a variable and is not within the scope of the right hand quantifier 'for every x'. Thus the rule, were it meaningful, would be wrong since according to it, on removing the vacuous quantifier, 'for every x, $g_0(x)$' is true iff '$g_0(x)$' registers. But it is not meaningful. For it is nonsense to say that a sentence concatenating predicate with the 23rd letter of the alphabet registers, or express a truth. Only names or constants with a definite designation can significantly occur in the place 'x' holds.

Both my ways of meeting this common objection go beyond the limitations of narrow levels-of-language "reconstructions" of discourse. One need have no qualms about transgressing these limits; for the narrow levels theory is not compulsory and is a thoroughly rotten fabric. Of course my proposals can be fitted into a more liberal levels theory which admits quotation and statement functions.

First, truth and falsity are primarily properties of statements and assertions, not of linguistic items such as sentences of a given language (5). What Tarski really defines is not, what we need, 'that ... is true', but '('...') expresses a truth'. To make domainless semantics appropriately assertoric we simply read '$\mathscr{V}(...) = t$' as 'that... registers' or, in case '...' is closed, 'that... is true'. The previously objectionable example of rule 4. becomes on this, the intended interpretation, the innocuous:

> that for every x, $g_0(x)$, is true iff for every x, that $g_0(x)$ registers.

That-clauses, unlike quotation work phrases, are open to quantification and because of this transparency, ω-incompleteness induced by quotation marks vanishes (see the discussion in [4]) (6). Moreover it is much less trouble understanding and operating with that-clauses than handling quotation mark names. The behaviour of 'that...registers', symbolised alternatively '§' (after Kneale [8], p. 531), is logically trivial. The sole axiom $\S p \equiv p$ (the two-valued assertoric theory of truth), together

(5) The reasons for this claim are well-known; some of the reasons are marshalled by STRAWSON [14].

(6) Some of the standard defences of the theory are attacked in [4].

with a substitution rule, immediately yields, and thereby provides an argument for, the valuation rules adopted, *viz.*

$$\S \sim p \equiv \sim \S p, \S(p \supset q) \equiv . \S p \supset \S q. \text{ and } \S(Ax)B \equiv . (Ax) \S B.$$

Alternatively, domainless semantics can be given a linguistic interpretation by replacing 'that' by the quotation function 'qu'. [7] The example of rule 4. then becomes:
'qu (for every x, $g_0(x)$)' is true iff for every x qu ($g_0(x)$) registers. Since variables within quotation functions, unlike those within name-forming quotation marks, are accessible to quantification the objections that the right hand quantification is vacuous and that the whole specification is meaningless both fail. The variables within quotation functions are neither further names nor alphabetic constants; they maintain their usual role as variables, and because they do ω-incompleteness difficulties and the Dunn-Belnap objections ([3], p. 180 and p. 183) evaporate. Since 'for every x, $g_0(x)$' is a closed wff, the interpretation sentence reduces to:
'for every x, $g_0(x)$' is true iff for every x qu($g_0(x)$) registers. But in the general case, where the wff B may contain free variables other than 'x' say, such a reduction is impermissible. Provided predicate parameters are taken as simply schematic letters the interpretation scheme:
qu (for every x, B) registers iff, for every x, qu(B) registers suffices. Otherwise a generalised quotation function '$qu_{x1, \ldots xn}$' which as a function of the variables with which it is subscripted and name-forming on remaining variables is needed.

Proof of the adequacy of domainless semantics for **Q** makes only minor, but nonetheless interesting, deviations from standard proofs.

Consistency Theorem If $\vdash_Q A$ then A is valid
Proof is, as usual, by induction over the length of the proof of A. Consider to illustrate,
the Instantiation postulate: $\vdash_Q (Ax)\ B \supset S^{v}{}^{x}_{y}\ B|$: the substitu-

([7]) Quotation function are explained and defended against Tarski's objections in [4].

tion notation is that of [13]. If, for arbitrary $\mathscr{V}$, $\mathscr{V}((Ax)B) = t$, by 4., $(Ax) . \mathscr{V}(B) = t$. Hence $S^{vx}_{y}\mathscr{V}(B) = t|$, and so $\mathscr{V}(S^{vx}_{y} B |) = t$, by properties of substitution — properties which follow directly from a recursive definition of substitution. A property Γ of wff (of **Q**, **FQ**, **Q** = , **FQ** =) is *(***L***-) inconsistent* iff for some $A_1, \ldots A_n$ such that $\Gamma(A_1), \ldots \Gamma(A_n)$, and for some wff B, $A_1, \ldots, A_n \vdash_{(L)} B$ and $A_1, \ldots A_n \vdash_{(L)} \sim B$; otherwise Γ is *(***L***-) consistent.* A wff C of **Q** is inconsistent iff the property of being C is inconsistent, i. e. iff $\vdash_Q \sim C$. '$\Gamma \vdash_L B$' abbreviates: for some wff $A_1 \ldots A_n$ such that $\Gamma(A_1)$, $A_1 \ldots A_n \vdash_L B$.

Corollary

If property Γ of wff of **Q** is satisfiable, Γ is consistent.
Proof. Suppose Γ is not consistent. Then for some $A_1, \ldots, A_n$ having Γ, $A_1, \ldots, A_n \vdash B \& \sim B$; hence $\vdash_Q \sim (A_1 \& \ldots A_n)$.
Therefore $\sim (A_1 \& \ldots A_n)$ is valid, $A_1 \& \ldots A_n$ is not satisfiable, and Γ is not satisfiable.
A wff C is *consistent with* property Γ of wff iff the property of having Γ or being C is consistent. A property Γ of wff is *maximal consistent* iff every wff C which is consistent with Γ has Γ.

Lindenbaum Theorem

For every consistent property Γ of wff of **Q** (**FQ**, **Q** =, **FQ** =) there is a maximal consistent property $\overline{\Gamma}$ such that for every wff B if B has Γ B has $\overline{\Gamma}$.
Using the definition, $B \varepsilon \Gamma =_{Df} \Gamma(B)$, standard proofs can be transcribed. '$B \varepsilon \Gamma$' reads 'B has Γ'.

Completeness Theorem

(i) If property Γ of wff of **Q** is consistent then Γ is satisfiable.
(ii) If A is valid then $\vdash_Q A$
Proof: Because when A is valid, $\sim$ A is not satisfiable, and accordingly by (i) not consistent; (ii) follows given (i).
As to (i), assuming Γ is consistent let **K** be the quantification logic got by adding to **Q** the subject constants, $b_1, b_2 \ldots$. Let the wff of K with *a* subject variable free be represented: $A_1(x_{i_1})$, $A_2(x_{i_2})$, ...

where each wff is represented with respect to a given free variable exactly once. Choose distinct constants b_{j_1}, b_{j_2}, ... from the new subject constants of **K** such that b_{j_k} is not contained in $A_1(x_{i_1}), \ldots, A_k(x_{i_k})$.

Let Γ_0 be Γ and let Γ_n be the property of having Γ_{n-1} or being S_n, where S_n is the wff

$$A_n(b_{j_n}) \supset (Ax_{i_n})\, A_n(x_{i_n}),$$

and let $\Gamma\omega$ be the property of having Γ_0 or being S_i for every $i \geqslant 1$. The proof that $\Gamma\omega$ is (**K**-) consistent is only a minor modification of the usual proof. Let J be $\overline{\Gamma}\omega$, i.e. a maximal consistent extension of $\Gamma\omega$. J will have the expected features of maximal consistent properties; in particular every theorem of **Q** has property J, J is closed under material detachment and adjunction, and wff$\sim$B has J iff B does not have J.

Using, J, an interpretation w is specified for each atomic wff C, as follows:

$w\,(C, t) \equiv C\ \varepsilon\ J$, i. e. $J(C)$; $w\,(C, f) \equiv \sim (C\ \varepsilon\ J)$.

By 1. this provides an induction basis for

(*) $w\,(B) = t \equiv .\ B\ \varepsilon\ J$, for every wff B of K.

The induction steps for connectives follow from clauses 2. and 3, properties of J and the induction hypothesis. The remaining case is that where B is (Ax) C for some wff C. Where, first, x is not free in C,

$B\varepsilon J \equiv C\varepsilon J$, since x is not free in C.
$\equiv w\,(C) = t$, by induction hypothesis
$\equiv (Ax)\,.\,w(C) = t$, since x is not free in C
$\equiv w\,((Ax)\ C) = t$, by 4.

Where x is free in C, C is $A_k(x)$ for some $A_k(x_{i_k})$ in the enumeration. The case follows using

(†) $(Ax)\ A_k\ (x)\ \varepsilon\ J \quad \equiv .\ (Ax)\ .\ A_k(x)\ \varepsilon\ J.$
For $(Ax)\ (A_k\ (x)\ \varepsilon\ J) \supset .\ A_k\,(b_{j_k})\ \varepsilon\ J$
$\supset .\ (Ax_{i_k})\ A_k\ (x_{i_k})\ \varepsilon\ J$, by S_k
$\supset .\ (Ax)\ A_k(x)\ \varepsilon\ J$, by variable change.

The converse follows by principles of quantification logic. Then,

$w((Ax)\,A_k(x)) = t \equiv .\ (Ax)\,.\,w(A_k(x)) = t$, by 4.
$\equiv .\ (Ax)\,.\,A_k(x)\ \varepsilon\ J$, by induction hypothesis.
$\equiv\ (Ax)\,A_k(x)\ \varepsilon\ J$, by (†)

Finally, for any $B\ \varepsilon\ \Gamma$, since $B\ \varepsilon\ J$,
$w(B) = t$ by (*). Hence Γ is satisfiable.

A wff A is a *classical consequence* of Γ iff
$(A\mathscr{V})\,(\mathscr{V}(\Gamma) = t \supset .\ \mathscr{V}(A) = t)$.

Strong completeness Theorem

B is a classical consequence of
Γ iff $\Gamma \vdash B$

Proof: By preceding results,
(†) Γ is consistent iff Γ is satisfiable.

B is a classical consequence of Γ
iff $\sim (S\mathscr{V})\,.\,\mathscr{V}(\Gamma) = t\ \&\ \mathscr{V}(A) \neq t$
iff $\sim (S\mathscr{V})\,.\,\mathscr{V}(\Gamma \cup \{\sim A\}) = t$
iff $\Gamma \cup \{\sim A\}$ is not consistent, by (†)
iff $\Gamma \vdash A$.

Given an interpretation $\mathscr{V}$, a domain d associated with $\mathscr{V}$ may be distilled, for example by identifying subjects with equivalent logical behaviour. Thus, where $x = y$ iff, for every wff C, $\mathscr{V}(C(x)) = t$ iff $\mathscr{V}(C(y)) = t$,
equivalence class $\bar{x} = \{y : y = x\}$ is defined for each subject x of **Q**. Then where D is the class of subjects (variables and constants) of **Q**, $d = \{\bar{x} : x\ \varepsilon\ D\}$.
An aboutness function I is defined as a mapping $I: D \rightarrow d$, which maps x to $\bar{x}$. I may be extended to predicates in various ways; for an "extensional" interpretation,
$I(f^n) = \{\langle \bar{x}_1, \dots \bar{x}_n \rangle : \mathscr{V}(f^n(x_1, \dots x_n)) = t\}$.
$\langle I, d \rangle$ is the neutral semantics defined on $\mathscr{V}$. It follows that the defined neutral semantics satisfies desired conditions: thus

1^{11}. $\mathscr{V}(f^n(x_1 \dots x_n)) = t$ iff $\langle I(x_1), \dots I(x_n) \rangle\ \varepsilon\ I(f^n)$.
2^{11}. and 3^{11}. are the same as 2. and 3.
4^{11}. $\mathscr{V}((Ax)B) = t$ iff $\mathscr{V}(B) = t$ for every I(x) in d.

Γ is satisfiable in a domain of type t iff Γ is satisfiable under some interpretation $\mathscr{V}$ and the associated domain *d* of $\mathscr{V}$ is of type t.

Skolem Lowenheim Theorem

If Γ is satisfiable then Γ is satisfiable in an at most denumerable domain.

Proof: If Γ is satisfiable then Γ is consistent (as in Church [2], ** 454), hence Γ is satisfiable under interpretation w defined in the Completeness Theorem. But the domain *d* associated with w is at most denumerable since class D system **K** is denumerable.

§ 2. *Domainless semantics for free quantification logic* **FQ**.

The primitives of **FQ** differ from those of **Q** only in the following respects: the universal quantifier 'A' is replaced by the quantifier '$\forall$', read 'for every existent', and some one-place predicate, say 'd', of **Q** is omitted and (in its stead) the predicate constant 'E', read 'exists', is introduced. **FQ** is characterised by the following schemata (see, e. g. Meyer and Lambert [10]; it was incompletely formulated in [11]):

A1. $\vdash_{FQ}$ A, where A is classical tautology.
A2. $\vdash_{FQ} (\forall x)(A \supset B) \supset . A \supset (\forall x) B$, with x not free in A.
A3. $\vdash_{FQ} (\forall x)(A \supset B) \supset . (\forall x) A \supset (\forall x) B$.
A4. $\vdash_{FQ} (\forall x) A \supset . E(y) \supset S^{v\,x}_{\ y} A$.
A5. $\vdash_{FQ} (\forall x) E(x)$.
R1. Material detachment. R2. Generalisation.

The following results on **FQ** are needed.

Alphabetic change of bound variables principle: Where x does not occur free in B and y does not occur in B and A^1 results from A by replacing one or more occurrences of B in A by $S^x_y B\,|$,

$$\text{if } \vdash_{FQ} A \quad \text{then} \quad \vdash_{FQ} A^1.$$

Proof is given in [10], p. 25.

Principle of substitution for subject variables: Where x is a subject variable and y is a subject term,

$$\text{if } \vdash_{FQ} A \quad \text{then} \quad \vdash_{FQ} S^{v\,x}_{\;\;y} A \,|.$$

Proof is by induction over the proof of A.

Principles of substitution for sentential and predicate variables:

If $\vdash_{FQ} A$ then $\vdash S^{vp}_{B} A \,|$, where p is a sentential variable.

If $\vdash_{FQ} A$ then $\vdash S^{vf(x_1, \ldots, x_n)}_{B} A \,|$, where f is an n-place predicate.

Proof and notation are as in Church [2], p. 193.

Deduction Theorem: If $A_1, \ldots A_n \vdash_{FQ} B$ then $A_1, \ldots A_{n-1} \vdash_{FQ} A_n \supset B$.

Proof and notation are as in Church [2], pp. 196-198, except that the notation '$\ldots \vdash_{FQ} \ldots$' is used in place of '$\ldots \vdash$ ---' to stress that the axioms used in a proof from hypotheses are those of **FQ**.

An *interpretation* μ *of* **FQ** is a pair

$$\mu = \langle v, d \rangle,$$

where v is a binary sentence predicate of **HQ** which associates with every atomic of **FQ** exactly one of t and f, and d is a predicate of **HQ** not in **FQ**. Thus with its first place restricted to atomic wff and its second place to truth-values, v is a mapping. On this basis a related mapping is defined recursively for all wff of **FQ**, as follows:

1a. Where A is atomic and not of the form E(x) for some x, $v(A) = t \equiv . \ v(A, t)$ and $v(A) = f \equiv . \ v(A, f)$.

1b. Where A is of the form E(x), $v(E(x)) = t \equiv . \ d(x)$, i. e. $x \varepsilon d$.

2. $v(\sim A) = t \equiv . \ v(A) \neq t$

3. $v(A \supset B) = t \equiv . \ v(A) \neq t \lor v(B) = t$

4. $v((\forall x) B) = t \equiv . \ (Ax \varepsilon d) . v(B) = t$

B is (**FQ**-) valid $\equiv . \ (A\mu) . v(B) = t$

B is (**FQ**-) satisfiable $\equiv . \ (S\mu) . v(B) = t$

Hence B is **FQ**-valid $\equiv . \sim$ B is not **FQ**-satisfiable.

Γ is **FQ**-satisfiable $\equiv . \ (S\mu) . v(\Gamma) = t$

B is an **FQ**-*consequence* of $\Gamma \equiv . \ (A\mu) . v(\Gamma) = t \supset . v(B) = t$.

Consistency and inconsistency of free quantificational systems is defined analogously to that for **Q**.

Consistency Theorem If $\vdash_{FQ} B$ then B is **FQ**-valid.

Completeness Theorem

(i) If property Γ of wff of **FQ** is consistent then Γ is **FQ**-satisfiable

(ii) If A is (**FQ**-) valid then $\vdash_{FQ}$ A.

Proof: Assuming Γ is consistent, let K be the free logic obtained by adding to **FQ** the constant subjects b_1, b_2 .... Let $A_1(x_{i_1})$, $A_2(x_{i_1})$, ... be an enumeration of wff of **K** such that each wff occurs n + 1 times, where n is the number of distinct free variables in the wff, and each wff is represented with respect to each free variable exactly once, variable x_{i_k} being displayed just in case the wff is represented with respect to that free variable. Choose a sequence b_{j_1}, b_{j_2}... of distinct constants from the adjoined subject constants such that b_{j_k} is not contained in $A_1(x_{i_1})$ ... $A_k(x_{i_k})$.
Let K_0 be Γ and let K_n be the property of having K_{n-1} or being T_n, where T_n is the wff

$$E(b_{j_n}) \supset A_n(b_{j_n}) \supset . (\forall x_{i_n}) A_n(x_{i_n})$$

and Let K_ω be the property union of K_n, for each $n \geqslant 0$. Apart from the induction step, showing that if K_{n-1} is consistent K_n is consistent, the proof that K_ω is consistent is quite standard. Suppose, for the induction step, that K_n is not consistent. Then for some wff B, K_{n-1}, $T_n \vdash_{FQ} B$ & $\sim B$ hence $K_{n-1} \vdash_{FQ} \sim (\forall x_{i_n}) A_n(x_{i_n})$ and $K_{n-1} \vdash_{FQ} E(b_{j_n}) \supset A_n(b_{j_i})$. Choose a new variable 'x_k' not occurring in the last proof sequence: it follows by induction over the sequence, since 'b_{j_n}' does not occur in T_j for $j \leqslant n-1$, $K_{n-1} \vdash_{FQ} E(x_k) \supset A_n(x_k)$. Generalising, and then distributing, $K_{n-1} \vdash_{FQ} (\forall x_k) E(x_k) \supset (\forall x_k) A_n(x_k)$. Hence detaching and changing the bound variables $K_{n-1} \vdash_{FQ} (\forall x_{i_n}) A_n(x_{i_n})$, contradicting the consistency of K_{n-1}. Let J be the maximal consistent extension of K_ω. An interpretation $\mu = \langle \nu, d \rangle$ is defined as follows:

$v(C, t) \equiv C \varepsilon J$; $v(C, f) \equiv \sim C \varepsilon J$;
$d(x) \equiv E(x) \varepsilon J$

As before

(*) $v(B) = t \equiv . J(B)$, for every wff B of K, holds; only the induction step for '$\forall$' is new.

Where B is of the form $(\forall x)$ C, there are these cases:

$$
\begin{aligned}
(\forall x) C \varepsilon J \quad & \supset . E(x) \supset C \varepsilon J. \\
& \supset . d(x) \supset v(C) = t, \text{ by induction hypothesis etc.} \\
& \supset . (Ax)(d(x) \supset v(C) = t), \text{ by HQ.} \\
& \supset . v((\forall x) C) = t \quad , \text{ by 4.}
\end{aligned}
$$

Conversely, since C is $A_k(x)$ for some element of the enumeration of wff, with x the place-holder for the free variable (if any) with respect to which it is represented,

$$
\begin{aligned}
v((\forall x) C) = t \supset & . (Ax) . d(x) \supset v(A_k(x)) = t, \text{ by 4.} \\
& \supset . d(b_{j_k}) \supset v(A_k(b_{j_k})) = t, \quad \text{by HQ} \\
& \supset . E(b_{j_k}) \supset A_k(b_{j_k}) \varepsilon J, \text{ by induction hypothesis.} \\
& \supset . (\forall x_{i_k}) A_k(x_{i_k}) \varepsilon J, \quad \text{by } T_k \\
& \supset . (\forall x) C \varepsilon J, \text{ by change of bound variables.}
\end{aligned}
$$

The translation theorem of Meyer and Lambert [10] is an easy consequence of the preceding theorems. Let 'd_1' be a one-place predicate of **Q** which does not occur in **FQ**. Where A is a wff of **FQ** let A^* be recursively defined in **Q** as follows:

(1) Where A is atomic, if A is of the form E(x), A^* is $d_1(x)$, and otherwise A^* is A.
(2) If A is $\sim B$ ($B \supset C$ for some B and C) then A^* is $\sim(B^*)$ ($B^* \supset C^*$)
(3) If A is $(\forall x)$ B, A^* is $(Ax)(d_1(x) \supset B^*)$.

Translation Theorem

$$\vdash_{FQ} A \text{ iff } \vdash_{Q} A^*$$

Proof applies the following lemma,

($\nrightarrow$) For every interpretation $\mu = \langle v, d \rangle$ of **FQ**, there is an interpretation $\mathscr{V}$ of **Q**, and conversely, such that

$$\nu(A) = t \text{ iff } \mathscr{V}(A^*) = t.$$

For

$\sim \vdash_{FQ} A$ iff A is not **FQ**-valid
iff $(S\mu) . \nu(A) \neq t$
iff $(Sv) . \mathscr{V}(A^*) \neq t$, by $\nrightarrow$
iff A^* is not **Q**-valid iff $\sim \vdash_Q A$.

Lemma ($\nrightarrow$) is proved by induction, starting from these prescriptions:
Given $\mu = \langle \nu, d \rangle$ define $\mathscr{V} = \nu$, and

given $\mathscr{V}$ define $\nu = \mathscr{V}$ and $d(x) \equiv . \mathscr{V}(d_1(x)) = t$.

(1) Where A is atomic, if it is of the form E(x),
$\nu(A) = t$ iff $\nu(E(x)) = t$ iff $\mathscr{V}(d_1(x)) = t$ iff $\mathscr{V}(A^*) = t$
Otherwise since $A = A^*$ $\nu(A) = \mathscr{V}(A^*) = t$.
(2) Where A is $\sim B$ ($B \supset C$) the case follows from the induction hypothesis and the valuation rules for $\sim$ ($\supset$).
(3) Where A is $(\forall x)$ B,

$\nu(A) = t$ iff $(Ax) . d(x) \supset \nu(B) = t$
iff $(Ax) . \mathscr{V}(d_1(x) = t) \supset \mathscr{V}(B^*) = t$
iff $(Ax) . \mathscr{V}(d_1(x) \supset B^*) = t$
iff $\mathscr{V}((Ax)\ (d_1(x) \supset B^*) = t$ iff $\mathscr{V}(A^*) = t$.

Where **Q*** is a neutral quantification logic containing the predicate 'E' (e. g. the system **R*** of [11]), and A + ... the wff obtained by replacing each occurrence of $(\forall x)$ B in a wff of **FQ** by (Ax) (E(x) $\supset$ B). (in order from smallest to largest scope), then

Corollary

$$\vdash_{FQ} A \text{ iff } \vdash_{Q^*} A +$$

Strong Completeness Theorem. B is an **FQ**-consequence of Γ iff $\Gamma \vdash_{FQ} B$.
Proof is as for **Q**.

§ 3. *Domainless semantics for first order logics* **Q** = *and* **FQ** =.

The preceding results are extended to quantification and free quantification logics with identity, where (extensional) identity, symbolised '=', satisfies the usual postulates, e. g.
= 1. x = x,
= 2. $x = y \supset . A \supset B$, where B is obtained from A by replacing an occurrence of subject x by subject y, this occurrence of x not being within the scope of quantifiers binding x or y. Call the system obtained by adding '=' and = 1 and = 2 to **Q**, **Q** =, and that obtained by a similar addition to **FQ**, **FQ**=. As is well-known, the predicate 'E' is eliminable in **FQ**= using the definition $E(x) =_{Df} (\exists y)(x = y)$. Results such as the deduction theorem and substitution principles extend to **Q**= and **FQ**=.

Interpretations of **Q**= and **FQ**= are defined precisely as for **Q** and **FQ**; and the extensions made of the valuation functions differ only where A is atomic. In the case of **Q**=, $\mathscr{V}$ is extended as follows:

1a. Where A is atomic and not of the form x = y,

$$\mathscr{V}(A) = t \equiv . \mathscr{V}(A, t) \text{ and } \mathscr{V}(A) = f \equiv . \mathscr{V}(A, f).$$

1c. Where A is of the form x = y,

$\mathscr{V}(x = y) = t \equiv . \mathscr{V}(A(x)) = \mathscr{V}(A(y))$, for every atomic wff of type 1a, i. e. where R is the class of atomic sentence forms which are not identities,

$$\mathscr{V}(x = y) = t \equiv . (Af \in R)\mathscr{V}(f(x)) = \mathscr{V}(f(y)).$$

As before $\mathscr{V}(A) = t \equiv . \mathscr{V}(B) = t$ is abbreviated: $\mathscr{V}(A) = \mathscr{V}(B)$. The extension of v in the case of **FQ**= is amended in a similar fashion, viz.

1a. Where A is atomic and not of the form E(x) or x = y,

$$v(A) = t \equiv . v(A,t) \text{ and } v(A) = f \equiv . v(A, f).$$

1b. Where A is of the form E(x), $v(E(x)) = t \equiv . d(x)$.

1c. Where A is of the form x = y,

$$v(x = y) = t \equiv . (Af \in R)v(f(x)) = v(f(y)).$$

Validity and satisfiability are defined as before. Consistency and completeness theorems are simplified by the following result.

Theorem. Where two place predicate = of **Q** or **FQ** is an equivalence relation which satisfies =2 where A is atomic and not an identity, then = is an identity, i.e. = 2 holds generally. Proof, the same for both **Q** and **FQ**, is by induction over the number of occurrences of connectives and quantifier. The details are like those given in Mendelson [9], Proposition 2.25.

Logic **L** is an *extension of* $\mathbf{L}^1$ iff, for every formula B, B is a wff of **L** iff B is a wff of $\mathbf{L}^1$ and B is a theorem of **L** if B is a theorem of $\mathbf{L}^1$.

Consistency Theorems:
(1) If $\vdash_{Q=}$ B then B is **Q**=-valid.
(2) If $\vdash_{FQ=}$ B then B is **FQ**=-valid
Proof adds to the proofs for **Q** and **FQ** verification that the equivalence postulates and reduced =2 are valid.

Completeness Theorems (1) (i) If set Γ of wff of **Q**= is consistent, then Γ is **Q**=-satisfiable.
(ii) If A is **Q**=-valid, then $\vdash_{Q=}$ A.
(2) (i) If set Γ of wff of **FQ**= is consistent, then Γ is **FQ**=-satisfiable.
(ii) If A is **FQ**=-valid, then $\vdash_{FQ=}$ A.
Proofs vary those given for **Q** and **FQ**, replacing 'Q' by '**Q**=' and '**FQ**' by '**FQ**='. The salient differences are these: For (1) (i) let **L** be the extension of **Q**= by Γ; let $\mathbf{L}_0$ be the extension of **L** obtained by adding the denumerable set of subject constants $\{b_1, b_2 \ldots\}$ and the one-place predicate 'f_0' to$\mathbf{L}_0$; and let **K** be the extension of $\mathbf{L}_0$ got by adding:

I. $f_0(x) \equiv f_0(y) \supset . x = y.$

A routine proof shows that $\mathbf{L}_0$ is consistent. Suppose **K** is not consistent. Then $\vdash_{L_0} \sim (x = y)$ and $\vdash_{L_0} f_0(x) \equiv f_0(y)$. Choose a new one-place predicate variable 'g' not occurring in this last proof sequence; then $\vdash_{L_0} g(x) \equiv g(y)$. Hence by the rule of predicate substitution for **Q**=, since g is not free in the hypotheses of the proof, $\vdash_{L_0} x = x \equiv x = y$. But, since $\vdash_{L_0} x = x$, $\vdash_{L_0} x = y$, contradicting the consistency of $\mathbf{L}_0$. Hence **K** is consistent. Extend it, as before, to a maximal consistent set J, and define interpretation w as before. The further step, $w(x = y) = t \equiv . \vdash_J x = y$ is proved as follows:

$\vdash_J x = y \supset . \vdash_J f(x) \equiv f(y)$, for $f \varepsilon R$ by =2
$\supset . \vdash_J f(x) \equiv \vdash_J f(y)$, for $f \varepsilon R$.
$\supset . w(f(x)) \equiv w(f(y))$, for $f \varepsilon R$, by induction hypothesis.
$\supset . w(x = y)$, by 1c.

Conversely, $w(x = y) \supset . w(f_0(x)) = w(f_0(y))$, since $f_0 \varepsilon R$
$\supset . \vdash_J f_0(x) \equiv f_0(y)$, reversing steps above
$\supset . \vdash_J x = y$, by I.

A similar strategy works in the case of **FQ**=.

The translation theorem may be extended at once; for it follows from clause (1) of the specification of A* in terms of wff (now) of **FQ**=, that if A is of the form x = y then A* is x = y.

Translation Theorem.

$\vdash_{FQ=} A$ iff $\vdash_{Q=} A^*$.

Proof is precisely as before.

Corollary. $\vdash_{FQ=} A$ iff $\vdash_{Q^*=} A+$,
where A+ is defined as before.

Strong Completeness Theorems. (1) B is a **Q**=—consequence of Γ iff $\Gamma \vdash_{Q=} B$. (2) B is a **FQ**=—consequence of Γ iff $\Gamma \vdash_{FQ=} B$.

§ 4. *Domainless semantics for a second-order significance logic*

The second-order significance logic $\mathbf{2QS}_6$ has primitive connective set $\{\neg, \rightharpoonup, \not{D}\}$, primitive quantifier 'A', and satisfies the following postulates (see also [12]):

1. The posulates of its sentential sublogic S_6, viz.

6.1 $B \rightharpoonup . C \rightharpoonup B$

6.2 $A \rightharpoonup (B \rightharpoonup C) \rightharpoonup . A \rightharpoonup B \rightharpoonup . A \rightharpoonup C$

6.3 $\neg B \rightharpoonup \neg C \rightharpoonup . C \rightharpoonup B$

6.4 $S(B \rightharpoonup B)$, where the significance connective 'S' is defined thus: $SB =_{Df} \neg \not{D}B$.

6.5 $SC \rightharpoonup S(B \rightharpoonup C)$

6.6 $S(B \rightharpoonup C) \rightharpoonup . B \rightharpoonup SC$

6.7 $S \neg B$

6.8 $B \rightharpoonup \not{D}\not{D}B$

6.9 $\not{D}\not{D}B \rightharpoonup B$

R1. Detachment for '$\rightharpoonup$'

2. These quantificational schemes:

6.10 $(Au)(B \rightharpoonup C) \rightharpoonup . B \rightharpoonup (Au)C$, where u is a variable (subject, predicate, or sentence) which is not free in B.

6.11 $(Au)B \rightharpoonup \overset{vu}{S_N} B \mid$, where (i) u is a subject variable and N is a subject term, or (ii) u is an n-phrase predicate variable and N is a predicate term, or (iii) u is a sentential variable and N is a wff.

A term of a given sort (subject or predicate) includes at least variables and constants of that sort. But the completeness of the logic is insensitive as to further extent of terms, especially predicate terms: for it makes no difference to completeness whether predicate terms include all sentence-frames, as in standard formulations of second order logic or some or none. For the present, then, the extent of predicate terms is left open.

6.12 $(Au)SB \rightharpoonup S(Au)B$

6.13 $S(Au)B \rightharpoonup (Au)SB$.

R2. Generalisation for quantifier 'A'.

Several results on $\mathbf{2QS_6}$ are now stated, without proof.

1. Where $TB =_{Df} \neg\neg B$, $t =_{Df} (Ap)p \rightharpoonup p$,
$f =_{Df} \neg t$ $\quad n =_{Df} \not{D}t$,

$\mathbf{2QS_6}$ $\quad S^p_t TA \mid \rightharpoonup . S^p_f TA \mid \rightharpoonup S^p_n TA \mid \rightharpoonup (Ap)TA$, where sentential variable p is not bound in A.

2. The principle of change of bound variable holds. The statement differs from that for **FQ** above only in that the variables concerned are required to be of the same sort.

3. A rule of substitution for variables follows from R2 and 6.11.

4. A deduction theorem — if $A_1, \ldots A_n \vdash B$ then $A_1 \ldots A_{n-1} \vdash A_n \rightharpoonup B$ — holds, where a proof from hypothesis is so defined as to admit derivation by R1, R2 (with the usual restriction), 2. and 3.

Where Γ is a set or property of wff, $\Gamma \vdash B$ iff for some $A_1, \ldots A_n \in \Gamma$, $A_1 \ldots A_n \vdash B$. Γ is $\neg$*-inconsistent* iff for some wff B, $\Gamma \vdash B$ and $\Gamma \vdash \neg B$; otherwise Γ is $\neg$*-consistent*. Maximal $\neg$-consistent sets are defined in the expected way; and Lindenbaum's lemma can be proved in almost standard ways.

An interpretation $\mathscr{V}$ *of* $\mathbf{2QS_6}$ is a mapping $\mathscr{V}$ from atomic wff to value set {t, f, n}, where 'n' symbolises the value *nonsense*. The valuation function $\mathscr{V}$ is extended inductively to all wff as follows:

1. $\mathscr{V}(\neg B) = t \quad \equiv . \quad \mathscr{V}(B) \neq t$
   $\mathscr{V}(\neg B) = f \quad \equiv . \quad \mathscr{V}(B) = t$
2. $\mathscr{V}(B \rightharpoonup C) = t \quad \equiv . \quad \mathscr{V}(B) \neq t \vee \mathscr{V}(C) = t$
   $\mathscr{V}(B \rightharpoonup C) = f \quad \equiv . \quad \mathscr{V}(B) = t \,\&\, \mathscr{V}(C) = f$
   $\mathscr{V}(B \rightharpoonup C) = n \quad \equiv . \quad \mathscr{V}(B) = t \,\&\, \mathscr{V}(C) = n$
3. $\mathscr{V}(\not{D}B) = t \quad \equiv . \quad \mathscr{V}(B) = n$
   $\mathscr{V}(\not{D}B) = f \quad \equiv . \quad \mathscr{V}(B) = f$
   $\mathscr{V}(\not{D}B) = n \quad \equiv . \quad \mathscr{V}(B) = t$
4. $\mathscr{V}((Au)B) = t \quad \equiv . \quad (Au) . \mathscr{V}(B) = t$
   $\mathscr{V}((Au)B) = n \quad \equiv . \quad (Zu) . \mathscr{V}(B) = n,$

where the particular quantifier 'Z' is defined: $(Zu)B =_{Df} \not{D}(Au)\not{D}B$

$$\mathscr{V}((Au)B) = f \equiv . (Zu)(\mathscr{V}(B) = f) \,\&\, (Au)(\mathscr{V}(B) \neq n).$$

It is assumed that certain sentences of the background logic are two-valued, in particular valuation sentences like '$\mathscr{V}(B) = n$' and membership statements like '$B \varepsilon J$' are always significant where well-formed. If C is such a significant sentence, $(Su)C \equiv (Zu)C$, where $(Su)C =_{Df} \sim(Au)\sim C$. The main semantical notions are defined as before:

B is $\mathbf{2QS_6}$-valid $\equiv (A\mathscr{V}) . \mathscr{V}(B) = t$
B is $\mathbf{2QS_6}$-satisfiable $\equiv (Z\mathscr{V}) . \mathscr{V}(B) = t$
Γ is $\mathbf{2QS_6}$-satisfiable $\equiv (Z\mathscr{V}) . \mathscr{V}(\Gamma) = t$

Consistency Theorem. If $\vdash_{\mathbf{2QS}_6}$ A, then A is $\mathbf{2QS}_6$-valid.

Corollary 1. $\mathbf{2QS}_6$ is $\neg$-consistent, i.e. for no wff B are both B and $\neg$B theorems.

Since it is essential that the background logic included $\mathbf{2QS}_6$ this proof is not as decisive as a syntactical proof which maps $\mathbf{2QS}_6$ into extended sentential logic.

Completeness Theorem. (i) If set Γ of wff of $\mathbf{2QS}_6$ is $\neg$-consistent then Γ is $\mathbf{2QS}_6$-satisfiable. (ii) If A is $\mathbf{2QS}_6$-valid then $\vdash_{\mathbf{2QS}_6}$ A.

Proof: (ii) follows from (i), since if A is valid, $\neg$ A is not satisfiable. Proof of (i) is like the analogous result for **Q**. Let **K** be the logic obtained by adding to $\mathbf{2QS}_6$ the constant symbols, $b_1^0, b_2^0, \ldots; b_1^1, b_2^1 \ldots; \ldots; b_1^k, b_2^k, \ldots; \ldots$, where $b_1^0 \ldots b_j^0 \ldots$ are denumerably many subject constants, and, for each positive k, $b_1^k, b_2^k \ldots$ are denumerably many k-place predicate constants. Now let the denumerable set of wff of **K** with *a* subject or predicate variable free be represented.

$$A_1(u_{i_1}),\ A_2(u_{i_2})\ \ldots\ A_n(u_{i_n})\ \ldots,$$

where, though a given wff may occur finitely many times, each wff is represented with respect to a given free variable exactly one. Choose a sequence, $b_{j_1} \ldots b_{j_k}, \ldots$, of new constants such that b_{j_k} is n-place iff u_{i_k} is n-place, such that b_{j_k} is not contained in $A_1(u_{i_1}) \ldots A_k(u_{i_k})$, and such that b_{j_k} is distinct from each of $b_{j_1} \ldots b_{j_{k-1}}$. Let ∇_n be the system obtained by adding to ∇_{n-1} the wff S_n:

$$A_n(b_{j_n}) \rightharpoonup (Au_{i_n})\, A_n(k_{i_n})\ ;$$

and let ∇_ω be the system obtained by adding S_n, for every $n \geqslant 1$, to ∇_0, i.e. Γ. By much the usual argument it follows that Λ_w is maximal $\neg$l-consistent; hence it has a maximal $\neg$-consistent extension J say.

A canonical interpretation w of $\mathbf{2QS}_6$ is defined thus: For each atomic wff **C**;

$w(C) = t \equiv . C \varepsilon J$, i.e. $\equiv TC \varepsilon J$, where $TC =_{Df} \neg\neg C$;
$w(C) = f \equiv . FC \varepsilon J$, where $FC =_{Df} \neg(\neg C \rightharpoonup \neg \not{D} C)$;
$w(C) = n \equiv . \not{D} C \varepsilon J$, i.e. $\equiv \neg SC \varepsilon J$.

It follows, by induction on this basis, that
(*) For every wff C of K,
$w(C) = t \equiv . C \varepsilon J$; $w(C) = f \equiv . FC \varepsilon J$;
and $w(C) = n \equiv . \not{D} C \varepsilon J$.

The induction steps for connectives '$\neg$', '$\rightharpoonup$' and '$\not{D}$' follow straightforwardly using theorems of $\mathbf{S}_6$ and features of maximal $\neg$-consistent set J. It suffices to establish the cases for values t and n, since that for f then follows.

Case 4, where C is (Au)B divides into these subcases:

4.1 B is closed. Then
$(Au B \varepsilon J \equiv B \varepsilon J \equiv w(B) = t \equiv (Au)(w(B) = t) \equiv w((Au)B) = t$
$\not{D}(Au) B \varepsilon J \equiv (Zu)\not{D} \varepsilon J \equiv w(B) = n \equiv (Zu)(w(B) = n) \equiv w((Au)B) = n$

4.2 C is (Au)B(u), where u is a subject or predicate variable free in B(u). Then
(α). $(Au)B(u) \varepsilon J \equiv . (Au) . B(u) \varepsilon J$.
For first B(u) is $A_k(u)$ for some member $A_i(u_{i_k})$ of the enumeration. Hence $(Au)B(u) \varepsilon J \equiv (Au_{i_k})A_k(u_{i_k}) \varepsilon J$, by change of variable. Thus

$$(Au)(B(u) \varepsilon J) \supset . A_k(b_{j_k}) \varepsilon J, \text{ by instantiation}$$
$$\supset . (Au_{i_k})A_k(u_{i_k}) \varepsilon J, \text{ by } S_k.$$

Conversely, for every u, $(Au)B(u) \varepsilon J \supset . B(u) \varepsilon J$, whence generalising and distributing, $(Au)B(u) \varepsilon J \supset . (Au) . B(u) \varepsilon J$.

Next, $(Au)B(u) \varepsilon J \equiv . (Au) . B(u) \varepsilon J$, by ($\alpha$)
$\equiv . (Au) . w(B(u)) = t$, by induction hypothesis
$\equiv . w((Au)B(u)) = t$, by 4.
Also, $\not{D}(Au)B(a) \varepsilon J \equiv . S(Au)B(u) \not\varepsilon J$
$\equiv . (Au)SB(u) \not\varepsilon J$, by 6.12, 6.13

$\equiv . \sim(Au) . Sb(u) \varepsilon J$, by ($\alpha$)
$\equiv . (Zu) . \neg SB(u) \varepsilon J$
$\equiv . (Zu) . w(B(u)) = n$, by induction hypothesis
$\equiv . \mathscr{V}((Au)B(u)) = n$, by 4.

4.3 C is (Au)B(u), where u is a sentence variable free in B(u). The desired result follows as in 4.2 given

(β). $(Au)B(u) \varepsilon J \equiv . (Au) . B(u) \varepsilon J$.

That $(Au) . B(u) \varepsilon J \supset (Au) . B(u) \varepsilon J$ follows as in (α). Conversely

$(Au)B(u) \varepsilon J \supset . TB(t) \varepsilon J \,\&\, TB(f) \varepsilon J \,\&\, TB(n) \varepsilon J$
$\supset . TB(t) \,\&\, TB(f) \,\&\, TB(n) \varepsilon J$
$\supset . (Ap)TB(p) \varepsilon J$, with p not bound in B(p)
$\supset . T(Ap)B(p) \varepsilon J$
$\supset . (Au)B(u) \varepsilon J$, by change of bound variable.

Strong Completeness Theorem. $\Gamma \vdash B$ iff B is a classical consequence of Γ, i.e. iff $(A\mathscr{V})(\mathscr{V}(\Gamma) = t \rightharpoonup . \mathscr{V}(A) = t)$. Proof uses the fact that Γ is $\neg$-consistent iff Γ is $\mathbf{2QS}_6$-satisfiable.

Similar results can be similarly established for higher-order predicate logics and for second and higher-order many-valued logics which contain the Rosser-Turquette J-functions. As validity coincides with theoremhood for domainless semantics for second-order logic, domainless semantics takes as (primary) validity what is only *secondary* validity under the standard semantics.

R. ROUTLEY [8]

Added in proof: (1) The no-entity-without-identity thesis is falsified by such items as clouds, hills and colour surfaces; and theoretical notions are not destroyed by lack of sharp identification.

(2) Corollaries of the Skolem paradox do not undermine domainless semantics; for clause 4. of the semantics is not restricted by countability requirements.

[8] Since this paper was written independent and more extensive investigations of these semantics have been published: see H. LEBLANC. Three generalisations of a theorem of Beth's, *Logique et Analyse*, No. 47, (September 1969), pp. 205-220, and references cited there.

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