GRAPHS, GEOMETRIC REPRESENTATIONS AND BINARY RELATIONS

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1. *Introduction.* It is well known that binary relations may be represented by graphs [1]. Now, just as there are geometric representations of finite graphs, there are geometric representations of binary relations. In this paper we explore the relationships between binary relations, their graphs and their geometric representations.

2. *Graph-Theoretical Concepts.* Let $\mathscr{V} = \{v_1, v_2, \ldots, v_n\}$ be a set of elements called "*vertices*". The subscripts 1, 2, ..., n are simply labels to distinguish the vertices.

Let $\mathscr{E} = \{\langle v_i, v_j\rangle : v_i \in \mathscr{V}$ and $v_j \in \mathscr{V}\}$ be the set of ordered pairs of elements of $\mathscr{V}$, called "edges". We denote $\langle v_i, v_j\rangle$ by e_{ij}.

Since v_i and v_j are not necessarily distinct, $\mathscr{E}$ contains n^2 edges.

DEFINITION 1. If $\langle v_i, v_j\rangle$ is an edge, then v_i and v_j are the *end-points* of the edge. v_i is the *initial,* end-point, and v_j is the *terminal* end-point.

DEFINITION 2. If $\langle v_i, v_j\rangle$ is an edge and $v_k = v_i$ or $v_k = v_j$, then

(i) v_k is *incident* to $\langle v_i, v_j\rangle$, and

(ii) $\langle v_i, v_j\rangle$ is *incident* to v_k.

DEFINITION 3. If $T \subseteq \mathscr{E}$, then $(e_{i,j} \in T) \leftrightarrow (\langle v_i, v_j\rangle \in T$ and $\langle v_j, v_i\rangle \in T)$. If $e_{i,j} \in T$, we say that $e_{i,j}$ is a *bi-oriented* edge in T.

DEFINITION 4. If $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, $\mathscr{G}$ is a *graph.*

DEFINITION 5. G is a *subgraph* (also called a *graph*) of $\mathscr{G}$ if and only if $G = \{V, E\}$, where $\emptyset \neq V \subseteq \mathscr{V}$ and $E \subseteq \mathscr{E}$.

DEFINITION 6. A graph G is a *model* of the binary relation R on the set S if and only if

(i) there is a 1 - 1 map f of the elements of S onto the vertices of G such that $\langle s_i, s_j \rangle \in R$ iff $\langle f(s_i), f(s_j) \rangle \in E$,

(ii) there is a 1 - 1 map between the predicates defined for R and the predicates defined for G, and

(iii) every true statement about R translates into a true statement about G.

3. *Geometric Representation of Graphs.* In this section we construct geometric representations of graphs. A geometric representation is a plane figure. The plane may be Euclidean or non-Euclidean.

EXAMPLE 7. Figures 1 and 2, below, are geometric representations of the graph $G = \{\{v_1, v_2, v_3\}, \{\langle v_1, v_1 \rangle, \langle v_2, v_1 \rangle, \langle v_3, v_1 \rangle\}\}$. That is, $V = \{v_1, v_2, v_3\}$ and $E = \{e_{11}, e_{21}, e_{31}\}$. To construct a geometric representation of G, we first select any three points in the plane. To each point we correlate one and only one vertex. In Figure 1, we use the map g such that $g(v_1) = p_1$, $g(v_2) = p_2$ and $g(v_3) = p_3$. In Figure 2, we use the map g′ such that $g'(v_1) = p'_1$, $g'(v_2) = p'_2$ and $g'(v_3) = p'_3$. To show that $\langle v_1, v_1, \rangle$ is an edge in G, we draw a circle at the image of v_1, i.e., p_1 in Figure 1 and p'_1 in Figure 2. We show that $\langle v_2, v_1 \rangle$ and $\langle v_3, v_1 \rangle$ are edges in G by drawing the directed segments from $g(v_2)$ to $g(v_1)$ and $g(v_3)$ to $g(v_1)$ in Figure 1. In Figure 2, the edges are shown to be in G by the directed segments from $g'(v_2)$ to $g'(v_1)$ and $g'(v_3)$ to $g'(v_1)$. The arrows on the segments show the direction of the segments.

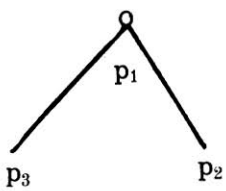


Figure 1

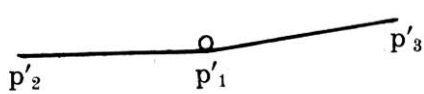


Figure 2

If there were a bi-oriented edge in G, this would be shown by a bi-directed line segment in the geometric representation, i.e., a line segment without an arrow.

Since there are six distinct maps of the vertices of G onto the points in Figure 1, the selection of a set of three points in the plane is not sufficient to determine a unique geometric representation of G. Furthermore, since we can select any one of an infinite number of triples of points in the plane, there are infinitely many possible geometric representations of G. Since the remarks are general, we conclude there is no unique geometric representation of a graph G.

DEFINITION 8. Two graphs, G and H, are *isomorphic* if and only if there is a 1 - 1 map f of the vertices of G onto the vertices of H such that $\langle v_i, v_j \rangle \in G$ if and only if $\langle f(v_i), f(v_j) \rangle \in H$.

DEFINITION 9. Two geometric representations, L and M, are *isomorphic* if and only if there is a 1 - 1 map of the vertex points of L onto the vertex points of M such that

(a) $\overrightarrow{p_i p_j}$ is a directed line segment in L if and only if $\overrightarrow{f(p_i)f(p_j)}$ is a directed line segment in M,

(b) $\overline{p_i p_j}$ is a bi-directed segment in L if and only if $\overline{f(p_i)f(p_j)}$ is a bi-directed segment in M, and

(c) there is a circle at p_i in L if and only if there is a circle at $f(p_i)$ in M.

THEOREM 10. If R is the set of all geometric representations of a graph G, then, if r_i and r_j are any two representations (i.e., elements of R), r_i is isomorphic to r_j.

Proof. Since r_i and r_j are geometric representations of G, there are 1 - 1 maps f_i and f_j of the vertices of G onto the vertex points in r_i and r_j. Since f_i is 1 - 1 onto, f_i^{-1} is a 1 - 1 map of the vertex points of r_i onto the vertices of G. Now $\overline{p_i p_j}$ is a directed segment in r_i if and only if $\langle f_i^{-1}(p_i), f_i^{-1}(p_j) \rangle$ is in G, but $\langle f_i^{-1}(p_i), f_i^{-1}(p_j) \rangle$ is in G if and only if

$\overrightarrow{f_j f_i^{-1}(p_i)\, f_j f_i^{-1}(p_j)}$ is a directed line segment in r_j. Similar considerations for circles and bi-directed segments in r_i show that $f_j f_i^{-1}$ is the desired map or r_i onto r_j, i.e., that r_i is isomorphic to r_j. It can easily be shown that isomorphism of graphs and hence of geometric representations are an equivalence relation.

4. *Binary Relations.*

DEFINITION 11. If A and B are sets, then $A \times B = \{\langle x, y\rangle : x \in A \text{ and } y \in B\}$. $A \times B$ is called the *Cartesian product* of A and B.

DEFINITION 12. R is a binary relation over a set S and only if there exist sets A and B such that

$R \subseteq \{\langle x,y\rangle : x \in A \text{ and } y \in B \text{ and } A \subseteq S \text{ and } B \subseteq S\}$.

DEFINITION 13. If R is a binary relation, then

$\mathscr{D}(R) = \{x : \langle x,y\rangle \in R\}$.

$\mathscr{D}(R)$ is called the *domain* of R.

DEFINITION 14. If R is a binary relation, then

$\mathscr{R}(R) = \{y : \langle x,y\rangle \in R\}$.

$\mathscr{R}(R)$ is called the *range* of R.

DEFINITION 15. If R is a binary relation, then

$\mathscr{F}(R) = \{x : \langle x,y\rangle \in R \vee (y,x) \in R\}$.

$\mathscr{F}(R)$ is called the *field* of R. The field of R is the union of the range and domain of R.

THEOREM 16. If R is a binary relation, then the graph $G = \{\mathscr{F}(R), R\}$ is a model of R.

DEFINITION 17. A graph G is *reflexive* if and only if, for every vertex v_i of G, $\langle v_i, v_i\rangle$ is an edge in G.

DEFINITION 18. A graph G is *symmetric* if and only if, for every vertex v_i and v_j, $\langle v_i, v_j\rangle$ is an edge in G implies $\langle v_j, v_i\rangle$ is an edge in G.

DEFINITION 19. A graph G is *transitive* if and only if whenever $(e_{ij} \in G$ and $e_{jk} \in G)$, then $(e_{ik} \in G)$.

Theorems 20 and 21 follow easily from the definitions.

THEOREM 20. If the graph G is a model of the binary relation R, then

(a) R is reflexive iff G is reflexive.
(b) R is symmetric iff G is symmetric, and
(c) R is transitive iff G is transitive.

THEOREM 21. A graph G is symmetric if and only if all edges in G are bi-oriented.

THEOREM 22. If G is a finite graph, then G has a geometric representation in the plane.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices of G. Let h denote the map of the vertices into the Euclidean plane such that, for every i, $h(v_i) = (i, |\sqrt{(1-i)(i-n)}|)$. The function h maps the vertices onto points on a semicircle in the upper half-plane. The semicircle has radius $(n-1)/2$, and its center at $((n+1)/2, 0)$.

We denote $h(v_i)$ by p_i. We complete the construction of the geometric representation of G as follows:

(a) if e_{ii} is an edge in G, we draw a circle at p_i,

(b) if e_{ij} is in G and e_{ji} is not, we draw the directed segment from p_i to p_j, and

(c) if e_{ij} and e_{ji} are both in G, we draw the bi-directed segment connecting p_i and p_j.

DEFINITION 23. If R is a binary relation and P is a geometric representation of $\{\mathscr{F}(R), R\}$, then P is a *geometric representation* of R.

THEOREM 24. If P is a geometric representation of the binary relation R, then all line segments are bi-directed in P if and only if R is symmetric.

Proof. Let $G = \{\mathscr{F}(R), R\}$. R is symmetric if and only if G is. G is symmetric iff $(e_{ij} \in G) \leftrightarrow (e_{ij} \in G \text{ and } e_{ji} \in G)$. Thus, the geometric representation will contain only bi-directed segments. On the other hand, where f is the appropriate map of $\mathscr{F}(R)$ to the points in the plane, a bi-directed segment connecting p_i and p_j shows that both $\langle f^{-1}(p_i), f^{-1}(p_j) \rangle$ and $\langle f^{-1}(p_j), f^{-1}(p_i) \rangle$ are edges in G, i.e., that G is symmetric.

THEOREM 25. If P is a geometric representation of the binary relation R, then R is reflexive if and only if there is a circle at each vertex point of the representation.

THEOREM 26. If P is a geometric representation of the binary relation R, then R is transitive if and only if

(a) if $\overrightarrow{p_ip_j}$ and $\overrightarrow{p_jp_k}$ are in P, then $\overrightarrow{p_ip_k}$ is in P,

(b) if $\overline{p_ip_j}$ and $\overline{p_jp_k}$ are in P, then $\overline{p_ip_k}$ is in P,

(c) if $\overrightarrow{p_ip_j}$ and $\overline{p_jp_k}$ are in P, then $\overrightarrow{p_ip_k}$ is in P, and

(d) if $\overline{p_ip_j}$ and $\overrightarrow{p_jp_k}$ are in P, then $\overrightarrow{p_ip_k}$ is in P.

THEOREM 27. If R and R′ are two binary relations such that $\mathscr{F}(R) = \mathscr{F}(R')$, P and P′ are the geometric representations of R and R′, and P is isomorphic to P′, then R is isomorphic to R′.

Proof. Let f denote the isomorphism from $\mathscr{F}(R)$ to P, g the isomorphism from $\mathscr{F}(R')$ to P′, and h the isomorphism from P to P′, then $g^{-1}hf$ is an isomorphism from R to R′.

EXAMPLE 28. It is mistake, albeit a natural one, to believe that, under the conditions stipulated in Theorem 27, R and R′ can be shown to be identical. That such a view is mistaken is shown by the following example.

Let $R = \{(a, b), (a, c), (b, c)\}$ and $R' = \{(c, a), c, b), (b, a)\}$. The field of R is the field of R′, and R is the converse of R′.

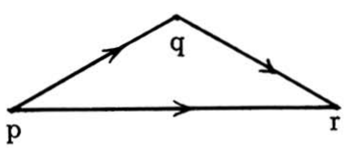


Figure 3

If we use the map f such that $f(a) = p$, $f(b) = q$ and $f(c) = r$, then Figure 3 is a geometric representation of R. If we use the map g such that $g(a) = r$, $g(b) = q$ and $g(c) = p$, then Figure 3 is a geometric representation of R′. Clearly, Figure 3 is isomorphic to itself. Thus, the conditions of Theorem 27 obtain, but $R \neq R'$.

5. *Distance and the Properties of a Binary Relation.*

In this section, we take the concept of distance from one vertex v_i to another vertex v_j in a graph, and use it to identify the properties of a binary relation from its geometric representation.

DEFINITION 29. Suppose v_i and v_j are vertices of a graph G, and Q is a set of edges of G. Q is a *chain of length* n from v_i to v_j if and only if there is a 1 - 1 map q of $\{1, 2, \ldots, n\}$ onto Q such that

(a) $[q(m) = \langle v_s, v_t \rangle$ and $q(m+1) = \langle v_u, v_v \rangle] \rightarrow (v_t = v_u)$,
(b) there is a v_s such that $q(1) = \langle v_i, v_s \rangle$, and
(c) there is a v_t such that $q(n) = \langle v_t, v_j \rangle$.

It is evident that if there is a chain of length n from v_i to v_j in G, then there is a minimal chain from v_i to v_j in G, i.e., a chain that is as short as any other from v_i to v_j. We use such minimal chains to define the *distance* between a vertex v_i and a vertex v_j in G. We use $d(v_iv_j)$ to denote the distance from v_i to v_j.

DEFINITION 30. Suppose v_i and v_j are vertices of the graph G. Then:

(a) if $\langle v_i, v_i \rangle$ is an edge in G, $d(v_iv_i) = 1$,
(b) if $\langle v_i, v_j \rangle$ is an edge in G and $v_i \neq v_j$, $d(v_iv_j) = 1$,
(c) if v_i and v_j are not coincident to an edge in G and there is a minimal chain C of length n from v_i to v_j, then $d(v_iv_j) = n$, and
(d) if v_i and v_j are not coincident to an edge in G, and there is no chain from v_i to v_j in G, then $d(v_iv_j) = \infty$.

The proofs of the following theorems follow easily.

THEOREM 31. v_i and v_j are incident to the bi-oriented edge $e_{i,j}$ (i.e., e_{ij} and e_{ji} are both edges in G) if and only if

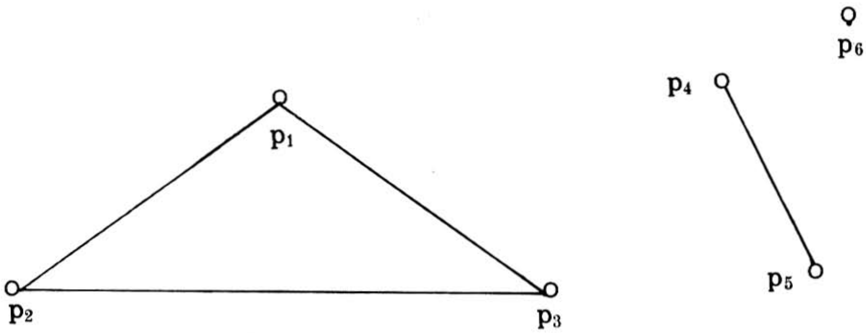
$$d(v_iv_j) + d(v_jv_i) = 2.$$

THEOREM 32. A graph G is symmetric if and only if, for every pair of vertices v_i and v_j in G, $d(v_iv_j) = d(v_jv_i)$.

THEOREM 33. A graph G is reflexive if and only if, for each vertex v_i in G, $d(v_iv_i) = 1$.

THEOREM 34. A graph G is transitive if and only if, for every triple of distinct vertices v_i, v_j and v_k, if $d(v_iv_j) = 1$ and $d(v_jv_k) = 1$, then $d(v_iv_k) = 1$.

EXAMPLE 35. Consider the graph whose geometric representation is as shown:



This represents the graph $G = \{\{v_1, v_2, v_3, v_4, v_5, v_6\}, \{\langle v_1, v_1\rangle, \langle v_2, v_2\rangle, \langle v_3, v_3\rangle, \langle v_1, v_2\rangle, \langle v_2, v_3\rangle, \langle v_1, v_3\rangle, \langle v_3, v_1\rangle, \langle v_3, v_2\rangle, \langle v_4, v_4\rangle, \langle v_5, v_5\rangle, \langle v_4, v_5\rangle, \langle v_5, v_4\rangle, \langle v_6, v_6\rangle\}\}$. G is reflexive, symmetric and transitive. That is, if G is a model of the relation R, then R is an equivalence relation.

THEOREM 36. Any complete plane polygon of n bi-directional sides with a circle at each vertex and bi-directional diagonals is a geometric representation of a strongly connected equivalence relation R on a set S of n elements.

Proof. Let $S = \{1, 2, \ldots, n\}$. The desired relation is $S \times S$.

6. *Euler Graphs and Euler Relations.*

DEFINITION 37. A graph G is *connected* if and only if, for every pair of distinct vertices v_i and v_j, there is a chain from v_i to v_j in G, or a chain from v_j to v_i.

DEFINITION 38. The graph $G = \{V, E\}$ is an *Euler graph* if and only if there is a subgraph G' of G such that $G' = \{V, E'\}$, such that

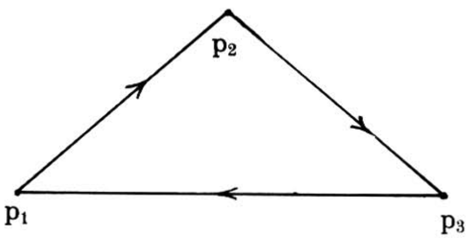
(a) E' is connected,
(b) for each v_i in V, E' is a chain from v_i to v_i, and
(c) E' is asymmetric.

E' is called an *Euler path* in G.

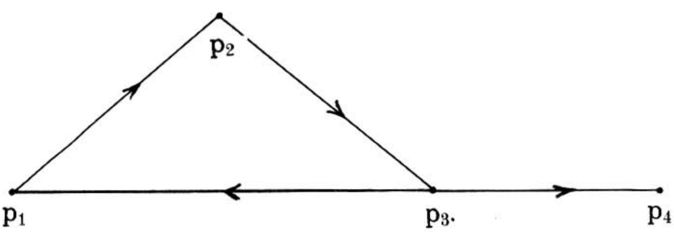
DEFINITION 39. $G = \{V, E\}$ is a *directed Euler graph* if and only if E is an Euler path in G.

DEFINITION 40. Suppose the graph G is a model of the binary relation R. R is an *Euler relation* if and only if G is an Euler graph.

EXAMPLE 41. Let $S = \{s_1, s_2, s_3\}$ and $R = \{\langle s_1, s_2\rangle, \langle s_2, s_3\rangle, (s_3, s_1)\}$. A geometric representation of R is:



This is evidently an Euler graph. If we were to add an element s_5 to S and $\langle s_3, s_4\rangle$ to R, we would obtain the graph:



This figure does not represent an Euler graph, for there is no path from p_4 to p_3. Further, the inclusion of still another element $\langle s_4, s_3\rangle$ would not suffice to make the graph an Euler graph.

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[1] ORE, Oystein, *Theory of Graphs*, American Mathematical Society Colloquium Publications, XXXVIII, Providence, R. I., 1962, pp. 13.