ZERMELO-FRAENKEL SET THEORY AND CUMULATIVE TYPE THEORY (*)

John L. Pollock

1. *Introduction.* It is well known that Zermelo-Fraenkel set theory (ZF) is closely connected with cumulative type theory (CT). The sets of ZF (when ZF is taken to include the axiom of regularity and the axiom scheme of replacement) can be arranged into a transfinite hierarchy of cumulative types. The purpose of this paper is to construct a natural axiomatization of CT and compare it with ZF. Ordinal numbers are taken as primitive, along with the ordering relation "$<$", and there is a primitive relation symbol $T(\alpha,X)$ ("X is of type α"). A few obvious axioms are adopted for ordinals, together with axioms characterizing types which make types cumulative. The axiom of extensionality is adopted for sets, together with the axiom scheme of comprehension for each type level. It follows trivially from the above that CT is contained in ZF. But the converse only holds when the axiom of choice is added. In particular, the axiom scheme of replacement is not a theorem of CT. This suggests the construction of a sligthly weakened version of ZF that is equivalent to CT. In this connection it is urged that, in the context of CT, the axiom scheme of replacement is unacceptable. However, it can be replaced in ZF by another intuitively acceptable axiom — the axiom scheme of ordinal replacement, which results from restriction the axiom scheme of replacement to sets of ordinals (the functional image of a set of ordinals is a set). This axiom scheme, although weaker than the principle of replacement, still suffices for all the uses to which the principle of replacement is customarily put. Furthermore, the principle of replacement becomes a theorem when the axiom of choice is added. This version of ZF is equivalent to CT, and it seems to me that it is intuitively preferable to the

(*) This paper was presented at the meeting of the Association for Symbolic Logic in Cleveland, May, 1969.

customary version of ZF. It yields the customary version when the axiom of choice is added.

2. *The axiom of ZF.* There are several versions of ZF, resulting from slightly different choices of axioms. There are also slightly different versions depending upon whether sets of individuals are allowed, or, as has become common contemporary practice, only sets of sets. To me, it seems ridiculous to disallow sets of individuals. Thus versions of ZF and CT which countenance sets of individuals will be developed here. However, this makes no difference to the main result of this discussion, which is the connection between ZF and CT. This connection still holds if sets of individuals are disallowed in both ZF and CT.

As the variables of our theory range over both individuals and sets, we must have a primitive predicate "C" meaning "is a set". We will follow the common practice of using lower case variables as general variables, and upper case variables as shorthand for the relativization of quantifiers to sets. Given this, the axioms are the following:

(Z1) EXTENSIONALITY: $(\forall X)(\forall Y)[X = Y \equiv (\forall z)(z\varepsilon X \equiv z\varepsilon Y)]$.

(Z2) SEPARATON: If φz is a formula in which z is free but X and Y are not, then the following is an axiom:
$(\forall X)(\exists Y)(\forall z)[z\varepsilon Y \equiv (\varphi z \,\&\, z\varepsilon X)]$.

(Z3) UNION: $(\forall X)(\exists Y)(\forall z)[z\varepsilon Y \equiv (\exists W)(W\varepsilon X \,\&\, z\varepsilon W)]$.

(Z4) POWER SET: $(\forall X)(\exists Y)(\forall z)(z\varepsilon Y \equiv z\subset X)$.

(Z5) PAIR: $(\forall x)(\forall y)(\exists Z)(\forall w)[w\varepsilon Z \equiv (w = x \vee w = y)]$.

(Z6) INFINITY: $(\exists X)[(\exists Y)(Y\varepsilon X \,\&\, (\forall z)z\not\varepsilon Y) \,\&\, (\forall Y)(Y\varepsilon X \supset YU\{Y\}\varepsilon X)]$.

(Z7) REGULARITY: $(\forall X)[X \neq \emptyset \supset (\exists y)(y\varepsilon X \,\&\, (\forall z)(z\varepsilon y \supset z\not\varepsilon X))]$.

(Z8) INDIVIDUALS: $(\exists X)(\forall y)(y\varepsilon X \equiv \sim C(y)) \,\&\, (\forall x)(\forall y)(x\varepsilon y \supset \sim C(y))$.

(Z9) REPLACEMENT: If φxy is a formula in which x and y are free but z, X, and Y are not, then the following is an axiom:

$$(\forall X)[(\forall x)(x \varepsilon X \supset (\exists ! y)\varphi xy) \supset (EY)(\forall y)(y \varepsilon Y \equiv (\exists x)(x \varepsilon X \,\&\, \varphi xy)].$$

(Z10) AXIOM OF CHOICE

The theory resulting from Z1 - Z8 will be called ZF_1. The theory resulting from Z1, Z3, Z4, Z6 - Z9, will be called ZF_2. Z2 and Z5 are theorems of ZF_2. ZF_3 is the theory that results from adding Z10 to ZF_2. ZF_1 is too weak to justify transfinite recursion, and accordingly ZF_2 is the most popular version of ZF without the axiom of choice, or ZF_3 is the most popular version with the axiom of choice.

3. *Cumulative type theory*. Cumulative type theory results from liberalizing the structure of simple type theorem. According to simple type theory, every set has a unique type, and the members of a set must always be of type one less than the type of that set. The rationale often given for this is that it gives a "constructive" picture of sets as being built up by successive steps from objects of type zero (individuals). However, this constructive view of sets does not really justify the structure of simple type theory. Rather, it justifies the less restrictive structure of cumulative type theory. The constructive picture requires that each set be built up out of sets that have already been constructed, but there is no reason why these sets should be only of the next lower type — they can be of *any* lower type. This leads to our defining types by saying that a set is of a given type iff its members are all of lower type. This has the effect of making types cumulative — if a set is of one type, then it is of any higher type. The lowest type of a set is called its *rank*.

Simple type theory generally requires all types to be finite. But again, this is an artificial restriction given the constructive picture of sets. There is no reason why, after having constructed sets of all finite types, these sets cannot be collected together into a new set of type ω. Consequently, a cumulative type theory with transfinite types will be developed here.

Cumulative type theory suffers philosophically from having to presuppose the ordinal numbers rather than constructing them as in ZF. This is undesirable for the simple reason that set theory should ideally provide a foundation for ordinal number theory. In this respect, ZF is preferable to CT. However, this unfortunate aspect of CT is largely rectified once it is seen how ZF and CT compare. The proof of the equivalence of some versions of ZF and CT can be regarded as a kind of completeness theorem for ZF in that it shows that ZF gives us all the structure of CT without presupposing ordinal number theory, and at the same time can be regarded as a kind of soundness theorem for CT in that all the ordinal axioms presupposed by CT are theorems of ZF.

The axioms of CT can be divided into three classes — ordinal axioms, type axioms, and set axioms. The language of CT is different from that of ZF. It no longer contains the primitive predicate "C" (which can be defined in CT), but does contain a binary relation symbol "T" ("$T(\alpha,X)$" means "X is of type α"), a primitive predicate "On" ("is an ordinal"), the constant "0", and the binary relation symbol "$<$". To simplify the notation, we will adopt the abbreviation of using small Greek letters as variables ranging over ordinals. We will also write "$T_\alpha x$" as an abbreviation for "$T(\alpha,x)$". We make the following definitions:

DEFINITION: $\alpha \leqslant \beta \equiv (\alpha < \beta \vee \alpha = \beta)$.
DEFINITION: $\gamma = \alpha+1 \equiv (\forall\beta)(\alpha<\beta \equiv \gamma\leqslant\beta)$.
DEFINITION: $\gamma = \alpha+2 \equiv (\exists\beta)(\beta = \alpha+1\ \&\ \gamma = \beta+1)$.
DEFINITION: $L(\alpha) \equiv (\alpha \neq 0\ \&\ \sim(\exists\beta)\ \alpha = \beta+1)$.

We adopt the following ordinal axioms for CT:

(01) On(0).
(02) $(\forall\alpha)(\forall\beta)[\alpha<\beta \supset \sim(\beta<\alpha)]$.
(03) $(\forall\alpha)(\forall\beta)(\forall\gamma)[(\alpha<\beta\ \&\ \beta<\gamma) \supset \alpha<\gamma]$.
(04) $(\forall\alpha)(\forall\beta)(\alpha\leqslant\beta \vee \beta\leqslant\alpha)$.
(05) $(\forall\alpha)(\alpha\neq 0 \supset 0<\alpha)$.
(06) If $\varphi\alpha$ is a formula in which α is free but β is not, then the following is an axiom:
$(\exists\alpha)\varphi\alpha \supset (\exists\alpha)\ [\varphi\alpha\ \&\ (\forall\beta)(\beta<\alpha\supset \sim\varphi\beta)]$.

(8) If $\varphi\beta\gamma$ is a formula in which β and γ are free but α and δ are not, then the following is an axiom:
$(\forall\alpha)[(\forall\beta)(\beta<\alpha \supset (\exists!\gamma)\varphi\beta\gamma) \supset (\exists\delta)(\forall\beta)(\forall\gamma)[(\beta<\alpha \;\&\; \varphi\beta\gamma) \supset \gamma<\delta]]$.

01 - 06 say that the ordinals are well ordered by "$<$", and 0 is the smallest ordinal. 06 is equivalent to the principle of transfinite induction. 07 tells us that there is no limit to the sequence of limit ordinals. 08 tells us that the functional image of a bounded set of ordinals is bounded, or equivalently, a sequence of ordinals of bounded length is bounded. These axioms give us at least a large part of the algebraic structure of the ordinals.

Types are characterized by the following axioms:

(T1) $\alpha\neq 0 \supset (T_\alpha x \equiv (\forall y)(y\varepsilon x \supset (\exists\beta)[\beta<\alpha \;\&\; T_\beta y]))$.

(T2) $(\forall x)(T_0 x \supset (\forall y)y\not\varepsilon x)$.

(T3) $(\forall x)(\exists\alpha)T_\alpha x$.

The individuals are objects of type zero, so we define:

DEFINITION: $C(x) \equiv \sim T_0 x$.

Then once again we use upper case variables as variables ranging only over sets.

Finally, we adopt two set axioms:

(S1) EXTENSIONALITY: $(\forall X)(\forall Y)[X = Y \equiv (\forall z)(z\varepsilon X \equiv z\varepsilon Y)]$.

(S2) COMPREHENSION: If φy is a formula in which y is free but X and α are not, then the following is an axiom:
$(\forall\alpha)(\exists X)(\forall y)[y\varepsilon X \equiv (\varphi y \;\&\; T_\alpha y)]$.

Let us call the theory resulting from the above axioms CT_1. CT_1 is a reasonably strong set theory. With some ingenuity it is sufficient for the development of most of set theory up through ordinal arithmetic.

The form the axiom scheme of comprehension takes in CT_1 is interesting. Russell's paradox shows that the unrestricted axiom scheme of comprehension is inconsistent, and yet it seems intuitively self-evident. In CT_1 we have a "constructive" form of the axiom scheme of comprehension, and it is plausible to suppose that it is really this form of the axiom that is intuitively self-evident — we are misunderstanding the unrestricted axiom when we think that it is self-evident. What is surely self-evident is that given any formula φ, there is a set of all individuals satisfying φ. Furthermore, given any formula φ, at any stage in the process of constructing all sets (i.e., at any type level) we can form the set of all previously constructed sets satisfying φ, and this will be a set of the next type level. In other words, given any previously established domain of objects, we can form the set of all those objects satisfying φ. The problem with the axiom scheme of comprehension comes from supposing that we can talk about the set of *all* sets satisfying φ and that this will be another set of the same group of sets that we began with.

From axioms 06 and T3 it follows that every set has a lowest type — its rank. Let us define:

DEFINITION: $T^*_\alpha x \equiv (\forall\beta)(T_\beta x \equiv \alpha \leqslant \beta)$.

Clearly there can be no universal set in CT_1, because such a set would contain elements of all types, and so could not itself have a type. However, it follows immediately from the axiom of comprehension that for each ordinal α, there is a universal set U_α consisting of all things of type α:

THEOREM 1: $\vdash_{CT_1} (\forall\alpha)(\exists X)(\forall y)(y \varepsilon X \equiv T_\alpha y)$.

In particular, there is a set U_X of all individuals.

It is a trivial matter to derive all of the axioms of ZF_1 within CT_1. Consequently:

THEOREM 2: All theorems of ZF_1 are theorems of CT_1.

The converse of theorem 2 is not possible, for the simple reason that the language of CT_1 is richer than the language of ZF_1, and therefore there are theorems of CT_1 that are not even sentences of ZF_1. We might, however, inquire whether all theorems of CT_1 that are sentences of ZF_1 are theorems of ZF_1. I don't know whether this is the case, but I conjecture that it is.

The ordinals are presupposed by CT_1. However, they can also be constructed in the usual manner. For example, we can define:

DEFINITION: $\mathrm{Ord}(x) \equiv [(\forall y)(y\varepsilon x \supset y \subset x) \& (\forall y)(\forall z) [(y\varepsilon x \& z\varepsilon x) \supset (y\varepsilon z \vee z\varepsilon y)] \& C(x)]$.

Let us use bold face letters as variables ranging over the constructed ordinals. We can then proceed to develop ordinal number theory just as is customarily done in ZF. Unfortunately, we quickly encounter a difficulty. The principle of transfinite recursion cannot be justified in CT_1. In order to justify definitions by transfinite recursion, we must proceed as follows. First, we define:

DEFINITION: If φxy is a formula in which x and y are free, then

$$\varphi[x] = \begin{cases} \text{the y such that } \varphi xy \text{ if there is a unique} \\ \text{y such that } \varphi xy; \\ \emptyset \text{ otherwise.} \end{cases}$$

Then we must prove the following "recursion principle":

Given any formula φxy and ordinal $\mathbf{x}$, there is a unique function F such that domain (F) $= \mathbf{x} \& (\forall \mathbf{y})(\mathbf{y}<\mathbf{x} \supset F(\mathbf{y}) = \varphi[F|\mathbf{y}])$.

This cannot be proven in CT_1, for precisely the same reason that it cannot be proven in ZF_1. In proving the recursion principle, one constructs the set of functions $F_{\mathbf{y}}$ satisfying the theorem for $\mathbf{y}<\mathbf{x}$, and then defines $F = \cup\{F_{\mathbf{y}}; \mathbf{y}<\mathbf{x}\}$. Unfortunately, in CT_1 and ZF_1 one cannot prove that the set $\{F_{\mathbf{y}}; \mathbf{y}<\mathbf{x}\}$ exists. In ZF the customary remedy is to add the axiom scheme of replacement. Then if we define $\vartheta((\mathbf{y},f)$ to mean "f is a function & domain (f) $= \mathbf{y} \& (\forall \mathbf{z})(\mathbf{z}<\mathbf{y} \supset f(\mathbf{z}) = \varphi[f|\mathbf{z}])$", we have $\vartheta[\mathbf{y}] = F_{\mathbf{y}}$; and by the axiom scheme of replacement, $\{\vartheta[\mathbf{y}];\mathbf{y}<\mathbf{x}\}$ exists. The recursion principle follows immediately from this. However, in CT there is a simpler and more obvious remedy (which will in turn suggest an alternative remedy for ZF).

In CT_1 we have two kinds of ordinals — the constructed ordinals and the presupposed ordinals. CT_1 gives us no way of determining whether these are the same. However, it is natural to add an axiom to the effect that the presupposed ordinals and the constructed ordinals are the same things. This removes some of the air of mystery from presupposing the ordinals in order to construct sets. It has the effect of saying that we are not really presupposing the ordinals at all, but rather constructing them side by side with other sets. All we are presupposing is some of the algebraic properties that will result from this construction. Furthermore, the addition of this axiom has the unexpected side effect of allowing us to prove the recursion principle, and hence justify definition by transfinite recursion. Let us add the axiom:

CORRESPONDANCE AXIOM:
$(\forall x)(\mathrm{Ord}(x) \equiv \mathrm{On}\,(x))\ \&\ (\forall \alpha)(\forall \beta)(\alpha < \beta \equiv \alpha \varepsilon \beta)$.

Let CT_2 be the resulting theory.

An immediate consequence of the correspondence axiom is a characterization of the rank of ordinals. It follows by transfinite induction that a finite ordinal n is of rank n+1 and an infinite ordinal α is of rank α:

THEOREM 3: $\vdash_{CT_2} (\forall \alpha)[(\alpha < \omega \supset T^*_{\alpha+1}\alpha)\ \&\ (\omega \leqslant \alpha \supset T^*_{\alpha}\alpha)]$.

From this together with T1 and T3 it follows that every set of ordinals is bounded:

THEOREM 4: $\vdash_{CT} (\forall X)[(\forall x)(x \varepsilon X \supset \mathrm{On}\,(x)) \supset (\exists \alpha)(\forall \beta)(\beta \varepsilon X \supset,$

$\beta < \alpha)]$.

Proof: X is bounded by the rank of X.

We justify the principle of transfinite recursion by proving a special case of the principle of replacement. Notice that in the above proof of the recursion principle we only need the principle of replacement for cases in which the set whose image is being formed is a set of ordinals, i.e., we only need:
If $\varphi\alpha y$ is a formula in which α and y are free but X and Y are not, then

$$(\forall X)[(\forall \alpha)(\alpha \varepsilon X \supset (\exists! y)\varphi\alpha y) \supset (\exists Y)(\forall y)(y \varepsilon Y \equiv (\exists \alpha)(\alpha \varepsilon X\ \&\ \varphi\alpha y))].$$

Although the general principle of replacement is not (apparently) a theorem of CT_2, this restricted principle of "ordinal replacement" is a theorem. This theorem results from the fact that, in ordinal axiom 08, we have in effect an even more restricted case of the principle of replacement — the case in which both the set whose image is being formed and the image are sets of ordinals. Suppose X is a set of ordinals and $\varphi\alpha y$ is a formula assigning a unique object to each element of X. Let $\vartheta\alpha\beta$ be the formula "$(\exists y)(\varphi\alpha y \,\&\, T^*_\beta y)$". Then to each ordinal in X, ϑ assigns a unique ordinal — the rank of the object assigned to that ordinal by φ. By theorem 4, X has an upper bound. Then by 08, the ordinals assigned by ϑ to members of X are all less than some ordinal γ, i.e., the ranks of the objects assigned by φ to elements of X are all less than γ. But then by the axiom of comprehension, these objects can be collected into a set. Thus

THEOREM 5: If $\varphi\alpha y$ is a formula in which α and y are free but X and Y are not, then

$$\vdash_{CT_2} (\forall X)[(\forall \alpha)(\alpha \varepsilon X \supset (\exists! y)\varphi\alpha y) \supset (\exists Y)(\forall y)(y \varepsilon Y \equiv (\exists\alpha)(\alpha\varepsilon X \,\&\, \varphi\alpha y))].$$

The above principle of ordinal replacement suffices for the justification of definition by transfinite recursion. An examination of the use of the general axiom scheme of replacement in ZF_2 reveals that in all of the theorems that are customarily proven with its help, it can always be replaced in CT_2 by the principle of ordinal replacement, together with other principles sanctioned by CT_2. Thus, although CT_2 is weaker than ZF_2 (because it does not contain the general principle of replacement), CT_2 nevertheless contains all the customary theorems of ZF_2.

Let CT_3 be the result of adding the axiom of choice to CT_2. Interestingly enough, the general principle of replacement is a theorem of CT_3. The axiom of choice implies the Numeration Theorem according to which every set is equipollent to an ordinal:

THEOREM 6: $\vdash (\forall X)(\exists\alpha)(\exists f)\ f{:}\alpha \xrightarrow[\text{onto}]{1-1} X.$

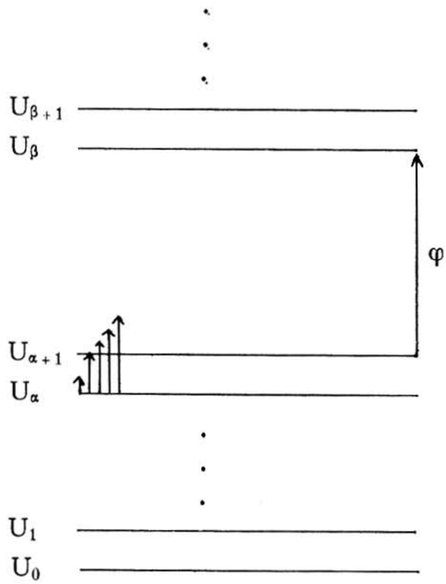
We can now prove the general principle of replacement as follows. Suppose $(\forall x)(x\varepsilon X \supset (\exists! y)\varphi xy)$. By the numeration theorem, for some α and f, $f : \alpha \xrightarrow[\text{onto}]{1-1} X$. Let $\vartheta\beta y$ be "$\varphi(f(\beta),y)$". $(\forall \beta)(\beta\varepsilon\alpha \supset (\exists! y)\vartheta\beta y)$, so by the principle of ordinal replacement, $\{y; (\exists\beta)(\beta\varepsilon\alpha \,\&\, \vartheta\beta y)\}$ exists. But this is just the set $\{y; (\exists x)(x\varepsilon X \,\&\, \varphi xy)\}$. So we have:

THEOREM 7: If φxy is a formula in which x and y are free but X and Y are not, then:

$$\vdash_{CT} (\forall X)[(\forall x)[x\varepsilon X \supset (\exists! y)\varphi xy] \supset (\exists Y)(\forall y)[y\varepsilon Y \equiv (\exists x)(x\varepsilon X \,\&\, \varphi xy)]].$$

This theorem, together with the fact that CT_1 contains all the theorems of ZF_1, shows that CT_3 contains all the theorems of ZF_2, and hence also of ZF_3.

The fact that the general principle of replacement does not seem to be a theorem of CT_2 provides what seems to be a good reason for suspecting that if the axiom of choice is not true, then the replacement principle will not hold. Speaking loosely, the axiom of choice implies that our hierarchy of sets is not as fat as it is tall. In other words, it tells us that if we look at U_α, the set of all sets of some type α, and consider a functional relation φ which may map the elements of U_α onto new sets of successively higher types ("standing U_α on end" so to speak), there will be an upper bound β to the ranks of the sets in the image of U_α. In other words, the situation looks as follows:
The reason for this is that the axiom of choice puts an upper bound on the "fatness" of each U_α by implying that it is equipollent to an ordinal. But without the axiom of choice there remains the possibility of having sets fatter than any ordinal, in which case there might be no upper bound to the ranks of the sets in the φ-image of U_α, and hence those sets in the φ-image cannot themselves be collected together into a single set. This is a reason for being suspicious of the presence of the axiom scheme of replacement in ZF_2.



4. *Zermelo-Fraenkel Set Theory*. It has been seen that CT_1 contains ZF_1, and CT_3 contains ZF_2 and ZF_3. Now let us explore converse inclusions. It is easily shown that ZF_2 contains CT_2. This inclusion cannot be taken in the straightforward sense that all theorems of CT_2 are theorems of ZF_2, for the simple reason that the language of CT_2 contains primitive symbols not contained in the language of ZF_2. But we can *translate* sentences of CT_2 that contain type symbols into sentences of ZF_2, and then show that given any theorem of CT_2, its translation is a theorem of ZF_2. This is familiar ground, but I will go over it for the sake of completeness.

The ordinal numbers can be constructed in ZF_2, so there is no problem translating the predicate "On", the relation symbol "$<$", or the constant "0" into the language of ZF_2. The only problem is the type relation "T". Letting U_α be the set of all objects of type α, the sets U_α are characterized by the following two principles:

(i) $U_0 = \{x;\ \sim C(x)\}$;
(ii) $\alpha \neq 0 \supset U_\alpha = U\{U_\beta;\ \beta<\alpha\} \cup \mathscr{P}\{U_\beta;\ \beta<\alpha\})$

(where $\mathscr{P}(X)$ is the power set of X). Consequently, these sets can be defined in ZF_2 by tranfinite recursion. Then the relation "T" can be defined by stipulating that $T(\alpha,X) \equiv x\varepsilon U_\alpha$. It is a trivial matter to verify that all the axioms of CT_2 except for T3 are theorems of ZF_2. The proof of T3 is more involved, but can be done as follows. From the axiom of regularity we can obtain what Tarski calls *the principle of set-theoretic induction* (which is a theorem of ZF_1):

THEOREM 8: $\vdash_{ZF_1} (\forall x)[(\forall y)(y\varepsilon x \supset \varphi y) \supset \varphi x] \supset (\forall x)\varphi x.$

The principle of set-theoretic induction is actually equivalent to the axiom of regularity. Next, using the axiom scheme of replacement, we prove:

THEOREM 9: $\vdash_{ZF_2} (\forall y)(y\varepsilon x \supset (\exists\alpha)T_\alpha y) \supset (\exists\alpha)T_\alpha x.$

Proof: Suppose every member of x has a type. Then every member has a unique lowest type (its rank), and consequently by the principle of replacement there is a set X consisting of all ranks of elements of x. Let α be the smallest ordinal greater than all the ordinals in X. Then $T_\alpha x$.

T3 follows immediately from theorems 8 and 9.

Consequently, ZF_2 contains CT_2. It follows that ZF_3 contains CT_3. But we saw that CT_3 also contains ZF_3. Therefore, ZF_3 and CT_3 are equivalent theories. We can diagram the relative strengths of the different versions of ZF and CT thus far considered:

$$ZF_1 \dashv CT_1 \dashv CT_2 \dashv ZF_2 \dashv CT_3 \approxeq ZF_3.$$

Because of the naturalness of CT_3, this can be construed as a justification for the less natural but type-free axioms of ZF_3. On the "constructive" picture of sets, if one accepts the axiom of choice, there seems to be no questions but that ZF_3 is a sound theory. Unfortunately, the situation is not quite so nice with respect to ZF_2. ZF_2 is the most popular set theory without the axiom of choice, but as we have seen, there is reason to be skeptical about the acceptability of the axiom scheme of replacement when one does not have the axiom of choice. It was remarked that CT_2, which does not contain the general principle of replacement, is adequate for the derivation of all the theorems of ZF_2 that people generally want. This gives reason to hope that a somewhat weaker version of ZF can be found which, although still type-free, will be equivalent to CT_2. In effect, such a theory can be obtained by simply replacing the axiom scheme of replacement in ZF_2 by the axiom scheme of ordinal replacement. More precisely, let us define ZF_4 to be the theory that results from adding the axiom scheme of ordinal replacement to ZF_1. We have seen that the principle of ordinal replacement (which is a theorem of CT_2) suffices for the justification of transfinite recursion. Adding this axiom to ZF_1, we can define the type symbols just as we did in ZF_2. Furthermore, we can prove all the axioms of CT_2 other than T_3 just as in ZF_2. The proof of T_3 is rather complicated. We need some additional machinery for its proof.

DEFINITION: If φxy is a formula in which x and y are free, then X is *closed under* φ iff $(\forall x)(\forall y)[(x \subset X \; \& \; \varphi xy) \supset y \varepsilon X]$.

The *closure* of a set A_0 under φ is the smallest set containing A_0 and closed under φ. In naive set theory the closure always exists, but not so in ZF or CT. This is because the closure, if it existed, might contain sets of all ranks, which is impossible. However, although the closure does not necessarily exist as a set, it is still possible to define a *predicate* which characterizes the elements of the closure. In naive set theory, the closure of A_0 under φ can be defined as the union of the range of the function F defined by transfinite recursion by specifying:

(i) $F(0) = A_0$;
(ii) $F(\alpha+1) = F(\alpha) \cup \{y; (\exists Y)[Y \subseteq F(\alpha) \; \& \; \varphi Yy]\}$;
(iii) $L(\alpha) \supset F(\alpha) = \cup \{F(\beta); \beta < \alpha\}$.

In ZF and CT no such *function* F exists (where a function is a set), because its domain would be the set of all ordinals, which does not exist. But by the familiar theorems that justify definition by transfinite recursion, we can instead define a *formula* Fxy such that

(i) $F[0] = A_0$;
(ii) $F[\alpha+1] = F[\alpha] \cup \{y; (\exists Y)(Y \subseteq F[\alpha] \ \& \ \varphi Yy]\}$;
(iii) $L(\alpha) \supset F[\alpha] = \cup \{F[\beta]; \beta<\alpha\}$;

provided that for each set X, $\{y; (\exists Y)(Y \subset X \ \& \ \varphi Yy)\}$ exists. Then to say that the union of the range of F is closed under φ means

$$(\forall X)([(\forall x)(x\varepsilon X \supset (\exists\beta)x\varepsilon F[\beta]) \ \& \ \varphi Xy] \supset (\exists\beta)y\varepsilon F[\beta]).$$

That this always holds is *the principle of set-theoretic closure*. It can be proven in ZF_4 as follows:

THEOREM 10: It is a theorem of ZF_4 that if $(\forall X)(\exists Y)(\forall z)[z\varepsilon Y \equiv (\exists Z)(Z \subseteq X \ \& \ \varphi Zz)]$, and F satisfies conditions (i) - (iii) above, then

$$(\forall X)([(\forall x)(x\varepsilon X \supset (\exists\beta)x\varepsilon F[\beta]) \ \& \ \varphi Xy] \supset (\exists\beta)y\varepsilon F[\beta]).$$

Proof: For each α, let $X_\alpha = X \cap (F[\alpha] - \cup\{F[\beta]; \beta<\alpha\})$. Let $K = \{X_\alpha; X_\alpha \neq \emptyset\}$. K exists by the power set and separation axioms. X = UK. K is well ordered by the relation $R = \{\langle X_\alpha, X_\beta\rangle; \alpha<\beta \ \& \ X_\alpha\varepsilon K \ \& \ X_\beta\varepsilon K\}$. Let α be the ordinal of this well ordering. Then there is function $f : \alpha \xrightarrow[\text{onto}]{1-1} K$. Let $\vartheta\delta\beta$ be the formula "$f(\delta) = X_\beta$". By ordinal replacement, $\{\vartheta[\delta]; \delta<\alpha\}$ exists. This is a set of ordinals, so it is bounded by some ordinal β. But this means that for each γ, $X_\gamma \subseteq F[\beta]$. Thus $X \subseteq F[\beta]$. Then if φXy, $y\varepsilon F[\beta+1]$.

From the principle of set-theoretic closure we obtain:

THEOREM 11: $\vdash_{ZF_4} (\forall x)[(\forall y)(y\varepsilon x \supset (\exists\beta)T_\beta y) \supset (\exists\beta)T_\beta x]$.

Proof: Let φxy be "$y \subseteq x$". Then $\{z; (\exists Z)(Z \subseteq X \ \& \ \varphi Zz)\} = \mathscr{P}(X)$, which exists by the power set axiom. Thus F can be defined as above, with the result that $U_\alpha = F[\alpha+1]$, and $L(a) \supset F[\alpha] = \cup\{U_\beta; \beta<\alpha\}$. Thus by the closure principle, letting $y = X$, $[(\forall x)(x\varepsilon X \supset (\exists\beta)x\varepsilon U_\beta) \ \& \ X \subseteq X] \supset (\exists\beta)X\varepsilon U_\beta$.

The principle of set-theoretic induction (theorem 8) is a theorem of ZF_1, and hence of ZF_4, and from this together with theorem 11 we immediately obtain T3. Thus ZF_4 is equivalent to CT_2.

Consequently, we have a type-free theory equivalent to CT_2, and this theory suffices for all the uses to which ZF_2 is customarily put. Furthermore, adding the axiom of choice to this theory gives us ZF_3. To me ZF_4 seems to be intuitively preferable to the more customary ZF_2.

State University of New York at Buffalo John L. Pollock