A FITCH-STYLE FORMULATION OF CONDITIONAL LOGIC (1)

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The system **CS** of *sentential conditional logic* is generated by axiom-schemes A1-A6 of [5] together with the rules of *modus ponens* and necessitation. It is given an intuitive justification in [3] and its connection with conditional probability is demonstrated in [4]; in [5] itself this system (or rather, a quantificational extension of it) is shown to be sound and complete with respect to its intended interpretation.

In this paper we will further describe the inferential structure of **CS** by formulating it as a system of *natural deduction*, using the format developed by Fitch in [2] (2). After showing that this system **FCS** is equivalent to **CS**, we will use it to indicate how the rules of **FCS** may be related to the way we reason with conditionals in everyday situations. The natural deduction format has the further advantage of yielding a framework within which theories of the conditional may be classified. In particular, all reasonable theories of the conditional, including **FCS**, may be displayed as results of adding rules to a minimal system that embodies properties common to all such theories.

1. The system **FCS**.

This system has $\supset$, $\sim$, and the conditional connective $>$ as

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(2) This paper presupposes familiarity with this format, or some other one similar to it.

its primitive connectives. We will regard disjunction, conjunction, and material equivalence as defined in the usual way. Necessity and possibility are defined as follows.

$$'\Box A' =_{df} '\sim A > A'$$
$$'\Diamond A' =_{df} '\sim \Box \sim A'$$

FCS has the usual rules for implication and negation, and for reiteration into ordinary derivations (see [6], Chapters III and IV). But besides ordinary derivations in **FCS**, we may also form *strict derivations*, which have the following form.

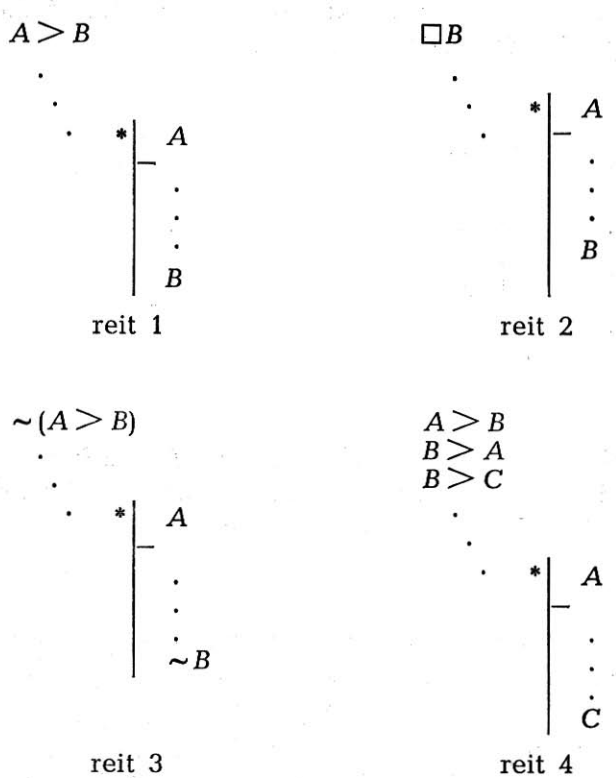
$$
\begin{array}{r|l}
* & A \\
\hline
 & \cdot \\
 & \cdot \\
 & \cdot
\end{array}
$$

The intuitive difference between such a derivation (which we will call a *strict derivation in A*) and an ordinary derivation with hypothesis A is that in the latter, we suppose that it *is* the case that something-or-other; in a strict derivation we suppose that it *were* the case that something-or-other. This means that in a strict derivation we may "bracket" or hold in abeyance certain portions of our knowledge about our actual situation, and envisage another situation in which something is supposed to hold. In an ordinary derivation we merely make a hypothesis about the actual situation.

All our knowledge about the actual situation will not in general obtain in other situations we may envisage. Our formal theory therefore must not allow unrestricted reiteration into strict derivations, so that arguments having the following pattern are not sanctioned as such in **FCS**.

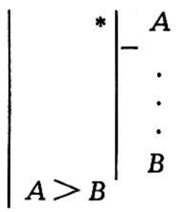
$$
\begin{array}{l}
B \\
\cdot \quad \begin{array}{r|l} * & \\ & A \\ \hline & B \end{array} \\
\cdot \\
\cdot
\end{array}
$$

We obtain a formal theory of conditionality by specifying general circumstances in which information can be obtained inside strict derivations. In other words, such a theory is characterized by a set of *reiteration rules*. The system **FCS** has four such reiteration rules which we will call 'reit 1', 'reit 2', 'reit 3', and 'reit 4'.

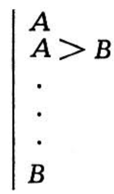


The rule reit 1 permits B to be reiterated into a strict derivation in A which is subordinate to an occurrence of $A > B$. The rule reit 2 permits B to be reiterated into a strict derivation in A which is subordinate to an occurrence of $\Box B$. The rule reit 3 permits $\sim B$ to be reiterated into a strict derivation in A which is subordinate to an occurrence of $\sim(A > B)$. Finally, the rule reit 4 permits C to be reiterated into a strict derivation in A which is subordinate to occurrences of $A > B$, $B > A$, and $B > C$.

These four reiteration rules provide the underlying structure needed for conditional reasoning. In order to describe the logic of the conditional connective we must add to this structure rules for the introduction and elimination of formulas having the form $A > B$. These rules are exact analogues of the corresponding rules for material implication (3).



cond int



cond elim

The rule of cond int allows $A > B$ to be inferred from a strict derivation in A which contains a step B. The rule of cond elim allows B to be inferred from A and $A > B$.

2. Equivalence of **CS** and **FCS**.

As the first step in our proof that **FCS** and **CS** are equivalent we present derivations in **FCS** showing that any instance of the axiom-schemes A1-A6 of **CS** is derivable in **FCS**. These derivations will also serve as illustrations of how the rules of **FCS** can be applied. We will use the notation 'taut' below to justify a step in which the conclusion can be obtained using only rules for negation and material implication.

(3) This holds true for theories of the conditional that do not attempt to embody the principle that the antecedent and conclusion should be relevant, in the sense discussed in [1]. In this case the introduction and elimination rules for $>$ must be adjusted as in [1].

1	$\Box(A\supset B)$	hyp
2	$\Box A$	hyp
3	$\sim B$	hyp
4	A	2, reit 2
5	$A\supset B$	1, reit 2
6	B	4, 5, m p
7	$\Box B$	3-6, cond int
8	$\Box A\supset\Box B$	2-7, imp int
9	$\Box(A\supset B)\supset .\ \Box A\supset\Box B$	1-8, imp int

1	$\Box(A\supset B)$	hyp
2	A	hyp
3	$A\supset B$	1, reit 2
4	B	2, 3, m p
5	$A>B$	2-4, cond int

1	$\Diamond A$	hyp
2	$A>B$	hyp
3	$A>\sim B$	hyp
4	A	hyp
5	$\sim\sim A$	4, taut
6	$A>\sim\sim A$	4-5, cond int
7	$\sim\sim A$	hyp
8	A	7, taut
9	$\sim\sim A>A$	7-8, cond int
10	$\sim\sim A$	hyp
11	$\sim B$	3, 6, 9, reit 4
12	B	2, 6, 9, reit 4
13	$\sim A$	11,12, taut
14	$\Box\sim A$	10-13, cond int
15	$\Diamond A$	1, reit
16	$\sim(A>\sim B)$	2-15, neg int
17	$(A>B)\supset\sim(A>\sim B)$	2-16, imp int
18	$\Diamond A\supset .\ (A>B)\supset\sim(A>\sim B)$	1-17, imp int

1	$A > (B \vee C)$	hyp
2	$\sim(A > B)$	hyp
3	$* \quad A$	hyp
4	$B \vee C$	1, reit 1
5	$\sim B$	2, reit 3
6	C	4, 5, taut
7	$A > C$	3-6, cond int
8	$\sim(A > B) \supset (A > C)$	2-7, imp int
9	$(A > B) \vee (A > C)$	8, taut
10	$(A > (B \vee C)) \supset ((A > B) \vee (A > C))$	1-9, imp int

1	$A > B$	hyp
2	A	hyp
3	$A > B$	1, reit
4	B	2, 3, cond elim
5	$A \supset B$	2-4, imp int
6	$(A > B) \supset . A \supset B$	1-4, imp int

1	$(A > B) \wedge (B > A)$	hyp
2	$A > C$	hyp
3	$(A > B) \wedge (B > A)$	1, reit
4	$A > B$	3, conj elim
5	$B > A$	3, conj elim
6	$* \quad B$	hyp
7	C	2, 4, 5, reit 4
8	$B > C$	6-7, cond int
9	$(A > C) \supset (B > C)$	2-8, imp int
10	$((A > B) \wedge (B > A)) \supset . (A > C) \supset (B > C)$	1-9, imp int

Besides these six axiom-schemes, **CS** has *modus ponens* and necessitation as its rules of inference. It is plain that *modus ponens* is admissible in **FCS**. To see that necessitation is also admissible in **CS**, suppose there is a derivation of A in **FCS**. Then $\Box A$ can be derived by repeating this derivation inside a strict derivation in $\sim A$ as follows.

```
|  * | ~A
|    |----
|    |  .
|    |  .
|    |  .
|    |  A
| □A
```

From this, it follows that everything provable in **CS** is derivable in **FCS**. To obtain the converse result we argue as in Chapter V of [6] that every derivation of B from A in **FCS** can be transformed into a deduction of B from A in **CS**. To show this, the argument of [6] need only be supplemented as follows. Let

```
| .
| .
| .
|     * | C1
|       |----
|       | C2
|       |  .
|       |  .
|       |  .
|       | Cn
| .
| .
| .
```

be a strict derivation occurring in an *augmented derivation* in **FCS**, i.e. a derivation in which axioms of **CS** may be introduced at any point, and in which the rule of necessitation may be applied to steps established without the help of any hypotheses. We must show that $C_1 > C_n$ can be obtained in this augmented derivation without the use of a strict derivation.

We accomplish this by erasing the line demarcating the strict subproof and prefixing $C_1>$ to each of its steps, thus obtaining the following array.

$$\begin{array}{|l} \cdot \\ \cdot \\ \cdot \\ C_1>C_1 \\ C_1>C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_1>C_n \\ \cdot \\ \cdot \\ \cdot \end{array}$$

It remains to be shown that steps can be inserted in this array in such a way as to make an augmented derivation; this is accomplished by cases according to how C_i was justified in the original augmented derivation. If C_i is the hypothesis C_1 we insert a proof in **CS** of $C_1>C_1$. If C_i came by *modus ponens* from D and $D \supset C_i$, we insert a deduction in **CS** of $C_1>C_i$ from $C_1>D$ and $C_1>(D \supset C_i)$. If C_i is an axiom of **CS** we insert C_i and a deduction in **CS** of $C_1>C_i$. If C_i came by negation elimination, insert a deduction in **CS** of $A>C_i$ from $A>\sim\sim C_i$. If C_i came by reit 1 from $C_1>C_i$, insert nothing; $C_1>C_i$ is justified in the new augmented derivation by ordinary reiteration. If C_i came by reit 2 from $\Box C_i$, insert a deduction in **CS** of C_i from $\Box C_i$. If C_i is $\sim D$ and came by reit 3 from $\sim(C_1>D)$, insert a deduction in **CS** of $C_1>\sim D$ from $\sim(C_1>D)$. If C_i came by reit 4 from $C_1>D$, $D>C_1$, and $D>C_i$, insert a deduction in **CS** of $C_1>C_i$ from $C_1>D$, $D>C_1$, and $D>C_i$. Finally, if C_i came by conditional elimination from $D>C_i$ and D, insert a deduction in **CS** of $C_1>C_i$ from $C_1>(D>C_i)$ and $C_1>D$. This proof depends on the assumption that certain deductions can be carried out in **CS**; e.g. that $\sim(A>B) \vdash_{\mathbf{CS}} A>\sim B$ and that $A>B, A>(B \supset C) \vdash_{\mathbf{CS}} A>C$. These facts can be established easily using the results of [6].

From these considerations it follows that **CS** and **FCS** are equivalent systems. We can therefore regard **FCS** as a reformulation of conditional sentence logic.

3. **FCS** and informal reasoning.

In this section we will use our reformulation of conditional logic to refine the account given in [3] of the relationship of conditional logic to reasoning in English. Natural deduction systems such as **FCS** are well adapted to this purpose because of their structure, which allows for the positing and elimination of hypotheses, is much closer to argumentation in natural language than is that of systems such as **CS**. In the following discussion we will take up in turn each of the rules of **FCS** which helps to determine the properties of the connective $>$. Discounting the rules for negation, which characterize the classical, two-valued sort of negation, there are six of these rules: cond int and cond elim, and the four reiteration rules.

The rules for introducing and eliminating formulas of the sort $A > B$ are not peculiar to this connective; for example, they are the same in form as the rules for material and intuitionistic implication. These rules reflect very deep habits of reasoning with conditionals. When required to establish a conditional conclusion, one's natural response is to suppose the antecedent and then try to show that the consequent of the conditional follows. Consider the following illustration.

"Why do you say that if Napoleon had attacked earlier at Waterloo, he would have won?" "Well, suppose he had. As it was, Blücher only arrived in the nick of time, and if Napoleon *had* attacked earlier he would have had only the English to deal with. Besides, only a few days before Waterloo Wellington was miles away from his troops and they were unprepared for a fight. The French would have demoralized them completely with an unexpected attack and achieved an easy victory."

The rule of cond elim can be regarded as either a way of reasoning from conditional statements, or as a way of refuting such statements. The first aspect of this rule constitutes the core of truth in the view of conditionals as "inference tickets"; commitment to a conditional statement entails a conditional commitment to its consequent, should its antecedent be true. If I claim that I will work in the yard if it's sunny outside and then discover that it is sunny outside, my claim entails that I will work in the yard. Turning this around, we can always refute a conditional statement if we know its antecedent is true while its consequent is false. "If you stopped in Detroit then of course you saw Aunt Beatrice." "No, that's not so. I stopped there, but she was visiting a friend in Cleveland and I missed her."

The rule of reit 1 is, I think, an uncontroversial feature of conditional reasoning; if a conditional statement has been posited, the supposition of its antecedent allows its consequent to be asserted. "If the bill were passed, it would be declared unconstitutional." "Well, suppose it *were* passed. Then, according to what you say, it would be declared unconstitutional..."

Together, these three rules determine a *minimal* basis for a logical theory of the conditional. (4) Evidence for this is the fact that most logical theories of conditional statements do satisfy these rules. The only exceptions which need to be taken seriously are, I believe, Anderson and Belnap's systems E and R of entailment and relevant implication. A kind of rock-bottom minimum could be obtained by adding the restrictions of [1] to our three rules. But although minimal systems of this sort may have some formal interest, my immediate concern is to consider enrichments of these basic rules which provide a less impoverished logic. This brings us to the three remaining reiteration rules.

To make clear the import of the rule of reit 2, we should

(4) Call this system **CMS**. It can be axiomatized as a system whose sole rule is *modus ponens* by taking the set S of axioms to be the smallest set containing all instances of the three schemes.

$A > A$

$(A > (B \supset C)) \supset ((A > B) \supset (A > C))$

$(A > (B > C)) \supset ((A > B) \supset (A > C))$

and such that if $E \in S$ then $D > E \in S$.

first say something about the content of $\Box A$, i.e. of $\sim A > A$. It is not easy to find cases in which locutions such as this come naturally to our lips. Perhaps the best way to do it is by taking a limiting use of the locution 'even if'. For instance, a person who says "Even if Hitler had been humane in his policy toward the Jews, he would have been an evil man, even if he had not followed an expansionist foreign policy he would have been an evil man, *even if he hadn't been evil*, he would have been an evil man" is refusing in a picturesque way to acknowledge the possibility that Hitler might not have been evil. He cannot conceive of a situation in which Hitler would not be evil.

Thus, to assert something of the form $\sim A > A$ (or, what is equivalent on principles already established, $\sim A > (B \wedge \sim B)$) is to assert the absurdity of positing the falsity of A. On this understanding of things, the rule of reit 2 says that when $\sim A > A$ has been obtained, A must then be true in any situation whatever, no matter how this situation is posited. To simultaneously assert something of the form $\sim A > A$ and deny something of the form $B > A$ is to say that a situation in which A does not obtain cannot be conceived, and yet that A does not obtain in some situation in which B is posited. It is this kind of paradoxical position that is ruled out by reit 2. The content of the rule, then, is that if $\sim A > A$ is true than A must be true in every situation; if there is any possible situation in which $\sim A$ is true, the one in which it is posited must be such a situation.

The rule of reit 3 is equivalent to allowing $\sim(A > B) \equiv (A > \sim B)$ to be derived from $\Diamond A$; this means that for all A which can be true, to deny a conditional of the sort $A > B$ is to *assert* the conditional $A > \sim B$. Inferences of this kind are common in natural language. "If you put your weight on that board, it will break." "That's false: it won't." What is denied here is the conditional statement that if I put my weight on the board, it will break. I then state the denial by conditionally denying the consequent of the first conditional; i.e. I assert that if I put my weight on the board, it *won't* break. Moves of this sort occur constantly in practice.

As a more direct example of the rule reit 3, consider a questionnaire with the following lines.

1. If the U.S. were to adopt the ABM system its military position would be strengthened.	☐	Yes
	☐	No
	☐	Undecided
2. Now, suppose that the U.S. adopts the ABM system. Would its military position be strengthened?	☐	Yes
	☐	No
	☐	Undecided

According to the rule of reit 3, it would be inconsistent to answer "No" to the first question and "Yes" or "Undecided" to the second. This principle seems to be so clearly valid a feature of conditional reasoning that it is surprising that some logicians have chosen to reject it.

The semantic content of reit 3 is that, in principle, *a unique situation* is selected when the antecedent of a conditional is posited. The validity of *conditional excluded middle*, $(A > B) \vee (A > \sim B)$, follows from this for the same reasons that render ordinary excluded middle, $A \vee \sim A$, valid in classical logic.

The rule of reit 4 permits $A > C$ to be inferred from $A > B$, $B > A$, and $B > C$. With this rule, the question whether it permits us to prove enough arises as well as the question whether it is valid. In particular, isn't it also true that $A > C$ should follow from $A > B$ and $B > C$ alone? As is pointed out in [3], the answer to this question is "No"; conditionality is not transitive. Stalnaker's elegant example runs as follows. "If J. Edgar Hoover were today a communist then he would be a traitor. If J. Edgar Hoover had been born a Russian then he would today be a communist. Therefore if J. Edgar Hoover had been born a Russian he would be a traitor."

The reason for this failure of transitivity is that in positing that J. Edgar Hoover were a communist and positing that he were born a Russian, one has two entirely different situations in mind. In the first situation he is an American national and would be a traitor in being a communist; in the second he is a

Soviet national and is a conformist in being a communist. Notice that in this case it is false that if J. Edgar Hoover were a communist he would have been born a Russian, so that we lack the third premiss needed to show by reit 4 that if J. Edgar Hoover had been born a Russian he would be a traitor.

With this in mind, consider the following argument.

1. If my watch cost more it would be accurate.
2. If my watch were accurate it would cost more.
3. If my watch were accurate I wouldn't have been late.
4. Therefore if my watch cost more I wouldn't have been late.

First, notice that, as in Stalnaker's example, step 4 doesn't follow from 1 and 3; nor, for that matter, does it follow from 2 and 3. Steps 1 and 3 may be true because my watch *already is* accurate and I *wasn't* late, while step 4 is false because my lateness would have been caused by my watch costing more. (I have just bought the watch and had only enough money to buy it. If it had cost more I would have had to go to the bank and would have been delayed.) On the other hand, steps 2 and 3 might be true because my watch isn't accurate and any watch that is would cost more. But perhaps any watch I own would be inaccurate; I abuse watches terribly. In that case I would be late even if my watch cost more, and 4 would be false.

As these considerations show, the rule of reit 4 has no redundant premisses; but are its premisses sufficient to yield the conclusion? For instance, does step 4 of the watch example really follow from steps 1 to 3? Certainly, the cases we have considered up to now do not invalidate this inference. In the case in which my watch is accurate and I wasn't late, step 2 is false. Indeed, the point of the first case is that it's false that my watch would cost more if it were accurate. Similarly, the point of the second case, in which my watch isn't accurate but *any* watch I own would be accurate is that even if my watch cost more it would not be accurate.

But a mere absence of counterexamples does not show that step 4 really follows from steps 1 to 3: for this we need a general argument. To devise such an argument, we must return to the idea that to posit a condition is to imagine a situation in which this condition is true. Steps 1 to 3 involve two conditions, and hence determine a situation α in which it is posited that my watch costs more and a situation β in which it is posited that my watch is accurate. If steps 1 and 2 are true, my watch is then accurate in α and costs more in β. But then *there is no reason for* α *and* β *to differ.* Having envisaged the situation α in positing one condition I need not choose a different situation in positing the second condition, since this second condition is already true in α; it is true in α that my watch costs more. And if α and β are identical, it follows that in β my watch costs more, and therefore step 4, the conclusion of the argument, is true.

Thus, reit 4 will be valid if we assume that there is economy in the choosing of situations, i.e. if we assume that in positing a condition we imagine a situation β differing from a situation α already imagined only if we are forced to do so in virtue of the fact that the condition is *false* in α. This rule therefore reflects a "law of least effort" in envisaging situations. Equally well, we can regard reit 4 as reflecting an *orderliness* in the choosing of situations; we can suppose that the choice of situations is dependent on a preferential arrangement of situations which can be described independently of the process of imagining. For example, the amount of physical energy required to obtain the situation β as compared with that required to obtain the situation γ may be pertinent to such a description. If this order is strict enough to permit one always to speak of the closest situation in which a condition is true, we can then define the situation selected in positing a condition as the closest one in which the condition is true. The validity of reit 3 follows at once from ths characterization of the choosing of situations.

This argument for the validity of reit 4 may strike some as less persuasive than our arguments in favor of the other rules of **FCS.** Whether or not this is so, I am reluctant to admit that reit 4 represents a superficial and dispensible feature of conditional logic. If it weren't true that our selection of situations in positing conditions were rulelike and orderly, we wouldn't be able to communicate effectively by means of subjunctive conditionals and they would lose their objective and factual character. Ordinarily, we are prepared to take such conditionals as matter-of-fact statements about the world; we are prepared to argue their truth or falsity with complete seriousness. Reasonable people will dispute subjunctive conditionals in debate, and those who are less dispassionate may even come to blows over them. It is this seriousness we are prepared to lavish on conditionals that gives examples such as the one about Bizet and Verdi (5) their paradoxical flavor.

The rule of reit 4 does not reflect any particular system of preferences among worlds, but it does reflect the requirement that these preferences should be systematic. I would claim that this requirement is a precondition of the intersubjective use of subjunctive conditionals. Communication would break down if the Bizet and Verdi phenomenon were the rule rather than the exception. Furthermore, I would maintain that it should be possible to describe the conventions used by a particular community in selecting situations, and I believe that this would be a philosophically useful and illuminating task. However, this enterprise should be distinguished from the more general task of exhibiting logical form and providing a semantic theory of validity.

One unpleasant consequence does follow from our account of situation-choosing as systematic; it appears on this account to be impossible to imagine one situation without somehow imagining them *all*. Every situation must be chosen in the light of a complete system in which all possible situations are arranged in some preferential order.

I do feel that this requirement is too strong, and that it needs to be modified by introducing the more modest notion of a *partial* rule of selecting situations. Such a partial rule would select situations with regard to some conditions, but not necessarily for all. With respect to a partial rule of selection, a subjunctive conditional having an antecedent condi-

(5) One man says "If Bizet and Verdi had been compatriots, Bizet would have been Italian." The other says "No. Verdi would have been French." The example is due to Quine.

tion not falling under the rule would not in general be true or false. For instance, if we have not made a decision about which situation is intended as the one in which Bizet and Verdi are compatriots, then it is neither true nor false that if Bizet and Verdi were compatriots then Bizet would be Italian. In van Fraassen's technique of *supervaluations,* discussed in [7], there exists a semantic technique of providing for truth-value gaps such as these. The application of this technique to the subjunctive conditional is left for a future paper. When modified in this way, the criterion of orderliness in the selection of situations becomes a plausible as well as a necessary feature of the logic of conditionals.

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