ON TRUTH-TABLES FOR M, B, S4 and S5

Arnould Bayart

Part I

0. *Introduction*

0.0. In this paper use is made of results and ideas to be found in Gödel's [1] for the formulation of modal propositional calculi, in Leonard's [2], in Anderson's [3] and [4] and in Hanson's [5] for the construction of truthtables, in Kripke's [6] and in Hughes's and Cresswell's [7] for the theory of models and in Church's [8] for plain proposition logic. The paper is however self-contained and demands no previous acquaintance with these results and ideas, except of course for [8].

0.1. The systems treated in this paper are modal *propositional calculi*. They are the Feys- van Wright system M, the "Brouwerian" system B, and Lewis's systems S 4 and S 5.

0.2. *Truth-tables* are constructed as in [3]. Notions of "*control*" and "*acceptability*" will be introduced. The tables combined with these two notions will yield for each system under consideration a suitable notion of "*tautology*". It will be obvious that our definition of "tautology" is a decision procedure for tautologies. The four systems will be formulated in Gödel style. It will be proved for each system that a formula is a theorem iff (if and only if) it is a tautology in the suitable sense. The theory of models yields for each system a suitable notion of "*validity*". It will be proved for each system that a formula is valid iff it is a tautology in the suitable sense.

0.3. We take negation, material implication and necessity as primitive, and we refer to object-language formulas by way of schemata involving "$\sim$", "$\rightarrow$" and "$\Box$" as metalogical signs for the primitive constants, and small Greek letters, eventually followed by suitable suffixes, as metavariables ranging over the class of formulas.

0.4. We use schemata like "$(\alpha \wedge \beta)$" and "$(\alpha \vee \beta)$" as abbreviations for "$\sim(\alpha \rightarrow \sim \beta)$" and $(\sim \alpha \rightarrow \beta)$" respectively. We use also "$(\alpha \equiv \beta)$" as an abbreviation for "$((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$". In schemata beginning with a left parenthesis and ending with a right parenthesis we drop the outmost parentheses. In our metalanguage we use continuous conjunctions and disjunctions in the usual way. We allow ourselves to consider any formula as a degenerate continuous conjunction or disjunction with a single member.

1. *Truth-tables.*

1.0. *Definitions.* A well formed part of a formula α is said to be a *subformula of* α. Every formula is considered as being a subformula of itself.

The propositional variables and the formulas of the form $\Box\alpha$ are said to be *constituents*. The formulas of the form $\Box\alpha$ are said to be *modal constituents*.

A constituent which is a subformula of α is said to be a *constituent of* α. If a constituent α is a proper part of a constituent β then α is said to be a *subconstituent of* β.

1.1. The *degree* of a formula α is defined as follows by induction on the construction of α.

If α is a variable, then the degree of α is 0.

If α is of the form $\sim \beta$ and if the degree of β is n, then the degree of α is n.

If α is of the form $\beta \rightarrow \gamma$ and if the degrees of β and γ are n' and n" respectively, then the degree of α is the highest number n such that n = n' or n = n".

If α is of the form $\Box\beta$ and if the degree of β is n, then the degree of α is n + 1.

1.2. We refer to sets of formulas by way of the letter "$\mathscr{S}$" followed by a suitable index.

Definition. A set $\mathscr{S}_a$ of constituents is said to be *adequate* iff (1) $\mathscr{S}_a$ is not empty, (2) $\mathscr{S}_a$ is finite and (3) every constituent of a member of $\mathscr{S}_a$ is a member of $\mathscr{S}_a$. If an asc (an adequate set of constituents) $\mathscr{S}_a$ contains all the constituents which occur in a formula α [which occur in any formula of a set $\mathscr{S}_b$ of formulas] $\mathscr{S}_a$ is said to be *an asc for* α [*for* $\mathscr{S}_b$].

If an asc $\mathscr{S}_a$ contains just all the constituents which occur in a formula α [which occur in any formula of a set $\mathscr{S}_b$ of formulas] $\mathscr{S}_a$ is said to be *the minimum asc for* α [*for* $\mathscr{S}_b$].

If an asc $\mathscr{S}_b$ contains all the constituents which occur in an asc $\mathscr{S}_a$, then $\mathscr{S}_a$ is said to be *a subasc of* $\mathscr{S}_b$.

1.3. *Theorem.* Let $\mathscr{S}_a$ be a subasc of $\mathscr{S}_b$ such that $\mathscr{S}_b$ contains just n ($n > 0$) constituents more than $\mathscr{S}_a$. Then it is possible to construct a series of ascs $\mathscr{S}_0, \mathscr{S}_1, \ldots, \mathscr{S}_n$ such that $\mathscr{S}_0$ is $\mathscr{S}_a$, $\mathscr{S}_n$ is $\mathscr{S}_b$ and for each i such that $0 \leqslant i < n$ we have that $\mathscr{S}_{i+1}$ contains just one constituent more than $\mathscr{S}_i$.

Proof by induction on n. If $n = 1$ then let $\mathscr{S}_0$ be $\mathscr{S}_a$ and let $\mathscr{S}_1$ be $\mathscr{S}_b$ and the theorem holds trivialy.

Let $n = m + 1$ ($0 < m$). The n constituents of $\mathscr{S}_b$ which are not in $\mathscr{S}_a$ can be ordered so as to form a series $\alpha_1, \ldots, \alpha_m, \alpha_n$ such that for every i and j ($1 \leqslant i < j \leqslant n$) we have that α_j is not a constituent of α_i. To obtain such a series it suffices to see to it that in the series every constituent α_i precedes all the constituents of a degree higher than the degree of α_i. We then define the series $\mathscr{S}_0, \ldots, \mathscr{S}_m, \mathscr{S}_n$ as follows. $\mathscr{S}_0$ is $\mathscr{S}_a$. For every i ($0 \leqslant i < n$) $\mathscr{S}_{i+1}$ contains the constituents of $\mathscr{S}_i$ plus α_{i+1}. It is obvious that $\mathscr{S}_n$ will be $\mathscr{S}_b$.

We prove now that $\mathscr{S}_m$ is an asc. Let α be any constituent of $\mathscr{S}_m$ and let β be any subconstituent of α. If α is in $\mathscr{S}_a$ then, as $\mathscr{S}_a$ is an asc, β is in $\mathscr{S}_a$ and so in $\mathscr{S}_m$. If α is in the series $\alpha_1, \ldots, \alpha_m$ then, as $\mathscr{S}_n$ is $\mathscr{S}_b$ and so an asc, β is in $\mathscr{S}_n$ and so either in $\mathscr{S}_a$ or in the series $\alpha_1, \ldots, \alpha_n$. If β is in $\mathscr{S}_a$ then β is in $\mathscr{S}_m$. If β is in the series $\alpha, \ldots, \alpha_n$ then α_n cannot be β as α_n is of a degree not lower than the degree of α, and so β is in the series $\alpha_1, \ldots, \alpha_m$ and so in $\mathscr{S}_m$. It follows that in every case β is in $\mathscr{S}_m$ and that $\mathscr{S}_m$ is an asc.

Then by the hypothesis of the induction on n we have that the theorem holds for the series $\mathcal{S}_0, \ldots, \mathcal{S}_m$. As $\mathcal{S}_n$ is an asc and as $\mathcal{S}_n$ contains just the constituent α_n more than $\mathcal{S}_m$, the theorem holds for the series $\mathcal{S}_0, \ldots, \mathcal{S}_n$.

1.4. *Definitions.* A *value-pair from an asc* $\mathcal{S}_a$ is a set containing just one of the constituents of $\mathcal{S}_a$ and one of the letters "T" or "F".

A *row from an asc* $\mathcal{S}_a$ is a set containing just for every constituent α of $\mathcal{S}_a$ one and only one of the two value-pairs which contain α.

The *table from an asc* $\mathcal{S}_a$ is the set of all the rows from $\mathcal{S}_a$.

We shall refer to rows by way of the letter "$\mathcal{R}$" followed by a suitable index, and to tables by way of the letter "$\mathcal{T}$" followed by a suitable index.

1.5. *Definitions.* If the tables $\mathcal{T}_a$ and $\mathcal{T}_b$ are the tables from the ascs $\mathcal{S}_a$ and $\mathcal{S}_b$ respectively, then $\mathcal{T}_a$ is said to be a [*proper*] *subtable* of $\mathcal{T}_b$ iff $\mathcal{S}_a$ is a [proper] subasc of $\mathcal{S}_b$.

If a table $\mathcal{T}_a$ is the table from the asc $\mathcal{S}_a$, then $\mathcal{T}_a$ is said to be a [*minimum*] *table for a formula* α [*for a set of formulas* $\mathcal{S}_b$] iff $\mathcal{S}_a$ is an asc [a minimum asc] for α [for $\mathcal{S}_b$].

If $\mathcal{T}_a$ is the table from the asc $\mathcal{S}_a$, if the constituent α occurs in $\mathcal{S}_a$ [is in $\mathcal{S}_a$], we say that α *occurs in* $\mathcal{T}_a$ [*is in* $\mathcal{T}_a$].

If $\mathcal{R}_i$ and $\mathcal{R}_j$ are rows of the tables $\mathcal{T}_a$ and $\mathcal{T}_b$ respectively, if $\mathcal{T}_a$ is a [proper] subtable of $\mathcal{T}_b$ and if for every constituent α in $\mathcal{T}_a$ the value-pair which contains α and which occurs in $\mathcal{R}_i$ is the same as the value-pair which contains α and which occurs in $\mathcal{R}_j$, we say that $\mathcal{R}_i$ *is a* [*proper*] *subrow of* $\mathcal{R}_j$.

1.6. *Definition.* Let $\mathcal{T}_a$ be a table for α and let $\mathcal{R}_i$ be a row of $\mathcal{T}_a$. We define under what conditions $\mathcal{R}_i$ *verifies* or *falsifies* α.

If α is a constituent, then $\mathcal{R}_i$ verifies α iff $\mathcal{R}_i$ contains the value-pair (α, T) and $\mathcal{R}_i$ falsifies α iff $\mathcal{R}_i$ contains the value-pair (α, F).

If α is of the form $\sim\beta$, then $\mathcal{R}_i$ verifies α iff $\mathcal{R}_i$ falsifies β and $\mathcal{R}_i$ falsifies α iff $\mathcal{R}_i$ verifies β.

If α is of the form $\beta \rightarrow \gamma$, then $\mathcal{R}_i$ verifies α iff $\mathcal{R}_i$ falsifies β or verifies γ, and $\mathcal{R}_i$ falsifies α iff $\mathcal{R}_i$ verifies β and falsifies γ.

We say also that $\mathcal{R}_i$ *gives the value* T (F) *to* α, and that α *has the value* T (F) *for* $\mathcal{R}_i$.

1.7. *Theorem.* Let $\mathcal{T}_a$ be a subtable of $\mathcal{T}_b$, let $\mathcal{R}_i$ and $\mathcal{R}_j$ be rows of $\mathcal{T}_a$ and $\mathcal{T}_b$ respectively, let $\mathcal{R}_i$ be a subrow of $\mathcal{R}_j$, and let $\mathcal{T}_a$ be a table for α [for the set of formulas $\mathcal{S}_i$]. Then we have for α [for every formula β in $\mathcal{S}_i$] that $\mathcal{R}_i$ verifies α [verifies β] iff $\mathcal{R}_j$ verifies α [verifies β], and $\mathcal{R}_i$ falsifies α [falsifies β] iff $\mathcal{R}_j$ falsifies α [falsifies β].
Proof. By an induction on the construction of α [of β] like in 1.6.

1.8. *Definitions.* Let α be a variable, let β be any formula, let $\mathcal{S}_a$ be an asc, and let $\mathcal{S}_{a'}$ be the set of formulas obtained by substituting in all the constituents of $\mathcal{S}_a$ the formula β for the variable α. Then in general $\mathcal{S}_{a'}$ will not be an asc. Let $\mathcal{S}_b$ be the minimum asc for $\mathcal{S}_{a'}$. Then $\mathcal{S}_b$ is said to be a *β/α-asc from* $\mathcal{S}_a$. Let α be a variable, let β be any formula, let $\mathcal{S}_b$ be the β/α-asc from $\mathcal{S}_a$, and let $\mathcal{T}_a$ and $\mathcal{T}_b$ be the tables from $\mathcal{S}_a$ and $\mathcal{S}_b$ respectively, then $\mathcal{T}_b$ is said to be the β/α-table from $\mathcal{T}_a$.

Let α be a variable, let β be any formula, let $\mathcal{T}_b$ be the β/α-table from $\mathcal{T}_a$, let $\mathcal{R}_j$ be a row in $\mathcal{T}_b$ and let $\mathcal{R}_i$ be the row in $\mathcal{T}_a$ such that for every constituent γ in $\mathcal{T}_a$ $\mathcal{R}_i$ verifies or falsifies γ according as $\mathcal{R}_j$ verifies or falsifies δ, where δ is the result of substituting β for α in γ. Then $\mathcal{R}_j$ is said to be a *β/α-row from* $\mathcal{R}_i$. *Remark.* It will often happen that, if $\mathcal{T}_b$ is the β/α-table from $\mathcal{T}_a$ and if $\mathcal{R}_i$ is a row of $\mathcal{T}_a$, then there will be in $\mathcal{T}_b$ more than one row which are β/α-rows from $\mathcal{R}_i$. It will also happen that for some row $\mathcal{R}_i$ in $\mathcal{T}_a$ there will be no β/α-rows from $\mathcal{R}_i$. A reason for this can be the fact that β is a tautology by plain proposition logic and that $\mathcal{R}_i$ falsifies α. Another reason can be the fact that some constituent δ in $\mathcal{T}_b$ can be considered as the result of substituting β for α in different constituents γ and γ' in $\mathcal{T}_a$ and that $\mathcal{R}_i$ assigns different values to γ and γ'. This can be the case if for instance we have as constituents in $\mathcal{T}_a$ $\Box(\alpha \to \beta)$ and $\Box(\beta \to \alpha)$ where α does not occur in β. However for every row $\mathcal{R}_j$ in $\mathcal{T}_b$ there will always be a unique row $\mathcal{R}_i$ in $\mathcal{T}_a$ such that $\mathcal{R}_j$ is a β/α-row from $\mathcal{R}_i$.

1.9. *Theorem.* Let α be a variable, let β be any formula, let γ be any formula, let δ be the result of substituting β for α in γ, let $\mathscr{T}_a$ be a table for γ, let $\mathscr{T}_b$ be the β/α-table from $\mathscr{T}_a$, let $\mathscr{R}_i$ and $\mathscr{R}_j$ be rows of $\mathscr{T}_a$ and $\mathscr{T}_b$ respectively, and let $\mathscr{R}_j$ be a β/α-row from $\mathscr{R}_i$. Then we have that (1) $\mathscr{T}_b$ is a table for δ and (2) $\mathscr{R}_j$ verifies (falsifies) δ iff $\mathscr{R}_i$ verifies (falsifies) γ.
Proof of (1). By an induction on the construction of γ like in 1.6 and by the definition of "β/α-table".
Proof of (2). By the same kind of induction and by the definition of "β/α-row".

2. *Control and acceptability.*

2.0. *Definitions.* Let $\mathscr{R}_i$ and $\mathscr{R}_j$ be rows of a stable $\mathscr{T}_a$, the case where $i = j$ not being excluded. We define as follows the *relations* C0 and C1.

$\mathscr{R}_i$ C0 $\mathscr{R}_j$ iff for every constituent $\Box\alpha$ in $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ verifies $\Box\alpha$ then $\mathscr{R}_j$ verifies α.

$\mathscr{R}_i$ C1 $\mathscr{R}_j$ iff for every constituent $\Box\alpha$ in $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ verifies $\Box\alpha$ then $\mathscr{R}_j$ verifies $\Box\alpha$.

2.1. *Definitions.* Let $\mathscr{R}_i$ and $\mathscr{R}_j$ be rows of a table $\mathscr{T}_a$, the case where $i = j$ not being excluded. We shall say that
$\mathscr{R}_i$ M-*controls* $\mathscr{R}_j$ iff $\mathscr{R}_i$ C0 $\mathscr{R}_j$,
$\mathscr{R}_i$ B-*controls* $\mathscr{R}_j$ iff $\mathscr{R}_i$ C0 $\mathscr{R}_j$ and $\mathscr{R}_j$ C0 $\mathscr{R}_i$,
$\mathscr{R}_i$ S4-*controls* $\mathscr{R}_j$ iff $\mathscr{R}_i$ C0 $\mathscr{R}_j$ and $\mathscr{R}_i$ C1 $\mathscr{R}_j$,
$\mathscr{R}_i$ S5-*controls* $\mathscr{R}_j$ iff $\mathscr{R}_i$ C0 $\mathscr{R}_j$ and $\mathscr{R}_j$ C0 $\mathscr{R}_i$, and $\mathscr{R}_i$ C1 $\mathscr{R}_j$ and $\mathscr{R}_j$ C1 $\mathscr{R}_i$.

2.2. *Definition.* For all natural numbers n we define recursively as follows what it is for a row $\mathscr{R}_i$ of a table $\mathscr{T}_a$ to be n-M-*acceptable*, n-B-*acceptable*, n-S4-*acceptable* or n-S5-*acceptable*.
For M. $\mathscr{R}_i$ is 0-M-*acceptable* iff for every constituent $\Box\alpha$ in $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ verifies $\Box\alpha$ then $\mathscr{R}_i$ verifies α.
$\mathscr{R}_i$ is n + 1-M-*acceptable* iff (1) $\mathscr{R}_i$ is n-M-acceptable and (2) for every constituent $\Box\alpha$ in $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ falsifies $\Box\alpha$

there is in $\mathscr{T}_a$ an n-M-acceptable row $\mathscr{R}_j$ such that $\mathscr{R}_i$ M-controls $\mathscr{R}_j$ and $\mathscr{R}_j$ falsifies α.
For B, S4 and S5. The same definition as for M, replacing the prefix "M" by the prefix "B", "S4" or "S5".

2.3. *Definition.* A row $\mathscr{R}_i$ of a table $\mathscr{T}_a$ is said to be M-*acceptable* [B-*acceptable*, S4-*acceptable*, S5-*acceptable*] iff for every natural number n $\mathscr{R}_i$ is n-M-acceptable [n-B-acceptable, n-S4-acceptable, n-S5-acceptable].

Remark. In what follows we shall often formulate definition-schemata, theorem-schemata and proof-schemata, where the prefix "X" will occur, and which will yield definitions, theorems and proofs if "X" is replaced everywhere by "M", "B", "S4" or "S5".

2.4. *Theorem-schema.* For every table $\mathscr{T}_a$ there is a natural number t such that if a row $\mathscr{R}_i$ of $\mathscr{T}_a$ is t-X-acceptable then $\mathscr{R}_i$ is X-acceptable. (The theorem does not state that the number t is the same for M, B, S4 and S5).
Proof-schema. The set of n + 1-X-acceptable rows is by definition a subset of the set of n-X-acceptable rows. But the number of rows of $\mathscr{T}_a$ being finite the successive sets of n-X-acceptable rows cannot decrease indefinitely with increasing n. So for some natural number t the set of t + 1-X-acceptable rows will be identifical with the set of t-X-acceptable rows. It is then easily seen that, for all r, if $\mathscr{R}_i$ is t-X-acceptable then $\mathscr{R}_i$ is t + r-X-acceptable. On the other hand for all s such that $s < t$ we have by condition (1) of n + 1-X-acceptability that if $\mathscr{R}_i$ is t-X-acceptable then $\mathscr{R}_i$ is s-X-acceptable. So for all n if $\mathscr{R}_i$ is t-X-acceptable then $\mathscr{R}_i$ is n-X-acceptable.

2.5. *Definition-schema.* The X-*characteristic number* of a table $\mathscr{T}_a$ is the smallest natural number t such that for every row $\mathscr{R}_i$ of $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ is t-X-acceptable then $\mathscr{R}_i$ is X-acceptable.

2.6. *Theorem-schema.* Let $\mathscr{R}_i$ be an X-acceptable row in $\mathscr{T}_a$. Then for every constituent $\Box\alpha$ in $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ falsifies $\Box\alpha$ then there is in $\mathscr{T}_a$ an X-acceptable row $\mathscr{R}_j$ such that $\mathscr{R}_i$ X-controls $\mathscr{R}_j$ and $\mathscr{R}_j$ falsifies α.

Proof-schema. Let t be the X-characteristic number of $\mathscr{T}_a$. $\mathscr{R}_i$ is X-acceptable and so t + 1-X-acceptable. So, by condition (2) of t + 1-X-acceptability, for every constituent $\Box\alpha$ in $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ falsifies $\Box\alpha$ then there is in $\mathscr{T}_a$ a t-X-acceptable row $\mathscr{R}_j$ such that $\mathscr{R}_i$ X-controls $\mathscr{R}_j$ and $\mathscr{R}_j$ falsifies α. But $\mathscr{R}_i$ being t-X-acceptable is X-acceptable.

2.7. *Theorem-schema.* Let $\mathscr{R}_i$ be an X-acceptable row in $\mathscr{T}_a$. Then for every constituent $\Box\alpha$ in T_a we have that R_i verifies $\Box\alpha$ iff all X-acceptable rows in $\mathscr{T}_a$ which are X-controled by $\mathscr{R}_i$ verifie α, and that $\mathscr{R}_i$ falsifies $\Box\alpha$ iff there is in $\mathscr{T}_a$ an X-acceptable row which is X-controled by $\mathscr{R}_i$ and which falsifies α.
Proof. Obvious by the fact that X-control implies the relation C0, and by **2.6**.

2.8. *Theorem-schema.* In every table $\mathscr{T}_a$ the set of X-acceptable rows is not empty.
Proof-schema. Let $\mathscr{R}_i$ be a row of $\mathscr{T}_a$ such that for every constituent $\Box\alpha$ in $\mathscr{T}_a$ (if any) $\mathscr{R}_i$ verifies $\Box\alpha$ iff $\mathscr{R}_i$ verifies α (and so $\mathscr{R}_i$ falsifies $\Box\alpha$ iff $\mathscr{R}_i$ falsifies α). Obviously we can construct such a row $\mathscr{R}_i$ by first associating arbitrarily the letters "T" and "F" with the constituents of degree 0 and by associating with every constituent $\Box\alpha$ of degree n + 1 the letters "T" and "F" according as α is verified or falsified by the subrow of $\mathscr{R}_i$ which contains all the constituents of $\mathscr{T}_a$ of degree not higher than n. From this construction it will follow that $\mathscr{R}_i$ is 0-X-acceptable. For n + 1-X-acceptability, by the hypothesis of an induction on definition 2.2, we may suppose that $\mathscr{R}_i$ is n-X-acceptable and so condition (1) for n + 1-X-acceptability holds. By the construction of $\mathscr{R}_i$ we have that $\mathscr{R}_i$ X-controls itself, and that for every constituent $\Box\alpha$ in $\mathscr{T}_a$, if $\mathscr{R}_i$ falsifies $\Box\alpha$ then $\mathscr{R}_i$ falsifies α. So condition (2) for n + 1-X-acceptability holds.

2.9. *Remark.* If we limit our considerations about control to X-acceptable rows, as every X-acceptable row is 0-X-acceptable, it is easily seen that C1 implies C0. This would allow us to simplify the definitions of S4-control and S5-control like this:

$\mathscr{R}_i$ S4-*controls* $\mathscr{R}_j$ iff $\mathscr{R}_i$ C1 $\mathscr{R}_j$,
$\mathscr{R}_i$ S5-*controls* $\mathscr{R}_j$ iff $\mathscr{R}_i$ C1 $\mathscr{R}_j$ and $\mathscr{R}_j$ C1 $\mathscr{R}_i$.
We shall however avoid making use of this facility for reasons which will become clear in the last section of this paper.

3. *Comparison between M, B, S4 and S5.*

3.0. *Theorem-schema.* For every table $\mathscr{T}_a$ the relation of X-control is reflexive within the set of X-acceptable rows in $\mathscr{T}_a$.
Proof-schema. The acceptable rows are 0-X-acceptable. For every 0-X-acceptable row $\mathscr{R}_i$ we have that, for every constituent $\Box\alpha$ in $\mathscr{T}_a$, if $\mathscr{R}_i$ verifies $\Box\alpha$ then $\mathscr{R}_i$ verifies α. So we have $\mathscr{R}_i$ C0 $\mathscr{R}_i$. For every row $\mathscr{R}_i$ (0-X-acceptable or not) we have obviously $\mathscr{R}_i$ C1 $\mathscr{R}_i$. The theorem follows then immediately by definitions **2.1**.

3.1. *Theorem.* For every table $\mathscr{T}_a$ the relation of B-control [S5-control] is symmetrical within the set of B-acceptable [S5-acceptable] rows in $\mathscr{T}_a$.
Proof. By definitions **2.1** it is obvious that B-control and S5-control are symmetrical within the set of all rows in $\mathscr{T}_a$. So these relations are symmetrical within the set of B-acceptable and S5-acceptable rows respectively.

3.2. *Theorem.* In every table $\mathscr{T}_a$ the relation of S4-control [S5-control] is transitive within the set of S4-acceptable [S5-acceptable] rows in $\mathscr{T}_a$ (in fact within the set of all rows).
Proof for S4. If in $\mathscr{T}_a$ $\mathscr{R}_i$ S4-controls $\mathscr{R}_{i'}$ and $\mathscr{R}_{i'}$ S4-controls $\mathscr{R}_{i''}$ then we have that for every constituent $\Box\alpha$ in $\mathscr{T}_a$ if $\mathscr{R}_i$ verifies $\Box\alpha$ then $\mathscr{R}_{i'}$ verifies $\Box\alpha$ and if $\mathscr{R}_{i'}$ verifies $\Box\alpha$, $\mathscr{R}_{i''}$ verifies α. So we have that for every constituent $\Box\alpha$ in $\mathscr{T}_a$, if $\mathscr{R}_i$ verifies $\Box\alpha$ then $\mathscr{R}_{i''}$ verifies α and so we have $\mathscr{R}_i$ C0 $\mathscr{R}_{i''}$. If in $\mathscr{T}_a$ $\mathscr{R}_i$ S4-controls $\mathscr{R}_{i'}$ and $\mathscr{R}_{i'}$ S4-controls $\mathscr{R}_{i''}$, then we have that for every constituent $\Box\alpha$ in $\mathscr{T}_a$ if $\mathscr{R}_i$ verifies $\Box\alpha$ then $\mathscr{R}_{i'}$ verifies $\Box\alpha$, and if $\mathscr{R}_{i'}$ verifies $\Box\alpha$ then $\mathscr{R}_{i''}$ verifies $\Box\alpha$. So we have that for every constituent $\Box\alpha$ in $\mathscr{T}_a$, if $\mathscr{R}_i$

verifies $\Box\alpha$ then $\mathscr{R}_{i''}$ verifies $\Box\alpha$ and so we have $\mathscr{R}_i$ C1 $\mathscr{R}_{i''}$. The proof for S5 is the same, a first time to show that $\mathscr{R}_i$ C0 $\mathscr{R}_{i''}$ and $\mathscr{R}_i$ C1 $\mathscr{R}_{i''}$ and a second time to show that $\mathscr{R}_{i''}$ C0 $\mathscr{R}_i$ and $\mathscr{R}_{i''}$ C1 $\mathscr{R}_i$.

3.3. *Theorem.* Let $\mathscr{R}_i$ and $\mathscr{R}_j$ be rows of a table $\mathscr{T}_a$. Then we have that a) if $\mathscr{R}_i$ S5-controls $\mathscr{R}_j$ then $\mathscr{R}_i$ S4-controls $\mathscr{R}_j$, b) if $\mathscr{R}_i$ S5-controls $\mathscr{R}_j$ then $\mathscr{R}_i$ B-controls $\mathscr{R}_j$, c) if $\mathscr{R}_i$ S4-controls $\mathscr{R}_j$ then $\mathscr{R}_i$ M-controls $\mathscr{R}_j$, d) if $\mathscr{R}_i$ B-controls $\mathscr{R}_j$ then $\mathscr{R}_i$ M-controls $\mathscr{R}_j$, and e) if $\mathscr{R}_i$ S5-controls $\mathscr{R}_j$ then $\mathscr{R}_i$ M-controls $\mathscr{R}_j$.
The theorem is trivial given definitions **2.1**.

3.4. *Theorem.* Let $\mathscr{R}_i$ be a row of a table $\mathscr{T}_a$. Then we have that a) if $\mathscr{R}_i$ is S5-acceptable then $\mathscr{R}_i$ is S4-acceptable, b) if $\mathscr{R}_i$ is S5-acceptable then $\mathscr{R}_i$ is B-acceptable, c) if $\mathscr{R}_i$ is S4-acceptable then $\mathscr{R}_i$ is M-acceptable, d) if $\mathscr{R}_i$ is B-acceptable then $\mathscr{R}_i$ is M-acceptable and e) if $\mathscr{R}_i$ is S5-acceptable then $\mathscr{R}_i$ is M-acceptable.
Proof-schema where X and Y can be replaced respectively by S5 and S4, or by S5 and B, or by S4 and M, or by B and M, or by S5 and M. The proofs are by induction on definition **2.2**.

If $\mathscr{R}_i$ is 0-X-acceptable $\mathscr{R}_i$ is 0-Y-acceptable by definition **2.2**. If $\mathscr{R}_i$ is $n + 1$-X-acceptable, then $\mathscr{R}_i$ is n-X-acceptable. By the hypothesis of the induction $\mathscr{R}_i$ is n-Y-acceptable and so condition (1) for $n + 1$-Y-acceptability holds for $\mathscr{R}_i$. If $\mathscr{R}_i$ is $n + 1$-X-acceptable then for every constituent $\Box\alpha$ in $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ falsifies $\Box\alpha$ there is in $\mathscr{T}_a$ an n-X-acceptable row $\mathscr{R}_j$ which is X-controled by $\mathscr{R}_i$ and which falsifies α. By the hypothesis of the induction $\mathscr{R}_j$ is n-Y-acceptable. By **3.3** $\mathscr{R}_i$ Y-controls $\mathscr{R}_j$. So condition (2) for $n + 1$-Y-acceptability holds for $\mathscr{R}_i$.

3.5. *Theorem.* There is a table $\mathscr{T}_a$ such that in it

a) there are two rows $\mathscr{R}_i$ and $\mathscr{R}_j$ such that $\mathscr{R}_i$ M-controls $\mathscr{R}_j$ and $\mathscr{R}_i$ does not B-control $\mathscr{R}_j$,
b) there are two rows $\mathscr{R}_i$ and $\mathscr{R}_j$ such that $\mathscr{R}_i$ M-controls $\mathscr{R}_j$ and $\mathscr{R}_i$ does not S4-control $\mathscr{R}_j$,
c) there are two rows $\mathscr{R}_i$ and $\mathscr{R}_j$ such that $\mathscr{R}_i$ B-controls $\mathscr{R}_j$ and $\mathscr{R}_i$ does not S4-control $\mathscr{R}_j$,

d) there are two rows $\mathscr{R}_i$ and $\mathscr{R}_j$ such that $\mathscr{R}_i$ S4-controls $\mathscr{R}_j$ and $\mathscr{R}_i$ does not B-control $\mathscr{R}_j$.

Proof by table 3.9 hereafter.

3.6. *Corollary*. There is a table $\mathscr{T}_a$ such that in it

a) there are two rows $\mathscr{R}_i$ and $\mathscr{R}_j$ such that $\mathscr{R}_i$ M-controls $\mathscr{R}_j$ and $\mathscr{R}_i$ does not S5-control $\mathscr{R}_j$,

b) there are two rows $\mathscr{R}_i$ and $\mathscr{R}_j$ such that $\mathscr{R}_i$ B-controls $\mathscr{R}_j$ and $\mathscr{R}_i$ does not S5-control $\mathscr{R}_j$,

c) there are two rows $\mathscr{R}_i$ and $\mathscr{R}_j$ such that $\mathscr{R}_i$ S4-controls $\mathscr{R}_j$ and $\mathscr{R}_i$ does not S5-control $\mathscr{R}_j$.

Proof by **3.3** and **3.5**.

3.7. *Theorem*. There is a table $\mathscr{T}_a$ such that in it

a) there is a row which is M-acceptable and not B-acceptable,

b) there is a row which is M-acceptable and not S4-acceptable,

c) there is a row which is B-acceptable and not S4-acceptable,

d) there is a row which is S4-acceptable and not B-acceptable.

Proof by table **3.9** hereafter.

3.8. *Corollary*. There is a table $\mathscr{T}_a$ such that in it

a) there is a row which is M-acceptable and not S5-acceptable,

b) there is a row which is B-acceptable and not S5-acceptable,

c) there is a row which is S4-acceptable and not S5-acceptable.

Proof by **3.4** and **3.7**.

3.9. Let α be a variable. Let us consider the asc $\mathscr{S}_a$:
$(\alpha, \Box\alpha, \Box\Box\alpha, \Box\sim\Box\alpha)$. The table $\mathscr{T}_a$ from the asc $\mathscr{S}_a$ is the table required for the proof of **3.5.** and **3.7.**

The 0-X-acceptable rows of $\mathscr{T}_a$ are the following:

$\mathscr{R}0$: (α, T) $(\Box\alpha, T)$ $(\Box\Box\alpha, T)$ $(\Box\sim\Box\alpha, F)$.
$\mathscr{R}1$: (α, T) $(\Box\alpha, T)$ $(\Box\Box\alpha, F)$ $(\Box\sim\Box\alpha, F)$.
$\mathscr{R}2$: (α, T) $(\Box\alpha, F)$ $(\Box\Box\alpha, F)$ $(\Box\sim\Box\alpha, T)$.
$\mathscr{R}3$: (α, T) $(\Box\alpha, F)$ $(\Box\Box\alpha, F)$ $(\Box\sim\Box\alpha, F)$.
$\mathscr{R}4$: (α, F) $(\Box\alpha, F)$ $(\Box\Box\alpha, F)$ $(\Box\sim\Box\alpha, T)$.
$\mathscr{R}5$: (α, F) $(\Box\alpha, F)$ $(\Box\Box\alpha, F)$ $(\Box\sim\Box\alpha, F)$.

For M we have:

$\mathscr{R}0$ M-controls $\mathscr{R}0$ and $\mathscr{R}1$;
$\mathscr{R}1$ M-controls $\mathscr{R}0$, $\mathscr{R}1$, $\mathscr{R}2$ and $\mathscr{R}3$;
$\mathscr{R}2$ M-controls $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$;
$\mathscr{R}3$ M-controls $\mathscr{R}0$, $\mathscr{R}1$, $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$;
$\mathscr{R}4$ M-controls $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$;
$\mathscr{R}5$ M-controls $\mathscr{R}0$, $\mathscr{R}1$, $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$;
Are M-acceptable: $\mathscr{R}0$, $\mathscr{R}1$, $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$.

For B we have:

$\mathscr{R}0$ B-controls $\mathscr{R}0$ and $\mathscr{R}1$;
$\mathscr{R}1$ B-controls $\mathscr{R}0$, $\mathscr{R}1$ and $\mathscr{R}3$;
$\mathscr{R}2$ B-controls $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$;
$\mathscr{R}3$ B-controls $\mathscr{R}1$, $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$;
$\mathscr{R}4$ B-controls $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$;
$\mathscr{R}5$ B-controls $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$.
Are B-acceptable: $\mathscr{R}0$, $\mathscr{R}1$, $\mathscr{R}2$, $\mathscr{R}3$ and $\mathscr{R}4$.

For S4 we have:

$\mathscr{R}0$ S4-controls $\mathscr{R}0$;
$\mathscr{R}1$ S4-controls $\mathscr{R}0$ and $\mathscr{R}1$;
$\mathscr{R}2$ S4-controls $\mathscr{R}2$ and $\mathscr{R}4$;
$\mathscr{R}3$ S4-controls $\mathscr{R}0$, $\mathscr{R}1$, $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$;
$\mathscr{R}4$ S4-controls $\mathscr{R}2$ and $\mathscr{R}4$;
$\mathscr{R}5$ S4-controls $\mathscr{R}0$, $\mathscr{R}1$, $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$.
Are S4-acceptable: $\mathscr{R}0$, $\mathscr{R}2$, $\mathscr{R}3$, $\mathscr{R}4$ and $\mathscr{R}5$.

For S5 we have:

$\mathscr{R}0$ S5-controls $\mathscr{R}0$;
$\mathscr{R}1$ S5-controls $\mathscr{R}1$;
$\mathscr{R}2$ S5-controls $\mathscr{R}2$ and $\mathscr{R}4$;
$\mathscr{R}3$ S5-controls $\mathscr{R}3$ and $\mathscr{R}5$;
$\mathscr{R}4$ S5-controls $\mathscr{R}2$ and $\mathscr{R}4$;
$\mathscr{R}5$ S5-controls $\mathscr{R}3$ and $\mathscr{R}5$.
Are S5-acceptable $\mathscr{R}0$, $\mathscr{R}2$ and $\mathscr{R}4$.
(It is worth while noticing that although $\mathscr{R}3$ is B-acceptable and S4-acceptable, it is not S5-acceptable).

4. *Subtables and substitution-tables.*

4.0. *Theorem-schema.* Let $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ be rows of a table $\mathscr{T}_a$. Let $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ be rows of a table $\mathscr{T}_b$. Let $\mathscr{T}_a$ be a subtable of $\mathscr{T}_b$ and let $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ be subrows of $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ respectively. Then we have that if $\mathscr{R}_i$ does not X-control $\mathscr{R}_{i'}$, then $\mathscr{R}_j$ does not X-control $\mathscr{R}_{j'}$.
Proof. If $\mathscr{R}_i$ C0$\mathscr{R}_{i'}$ does not hold then there is a constituent $\Box\alpha$ in $\mathscr{T}_a$ such that $\mathscr{R}_i$ verifies $\Box\alpha$ and $\mathscr{R}_{i'}$ falsifies α. But then by **1.7**, in $\mathscr{T}_b$ $\mathscr{R}_j$ verifies $\Box\alpha$ and $\mathscr{R}_{j'}$ falsifies α, and so $\mathscr{R}_j$ C0 $\mathscr{R}_{j'}$ does not hold. If $\mathscr{R}_i$ C1 $\mathscr{R}_{i'}$ does not hold then there is a constituent $\Box\alpha$ in $\mathscr{T}_a$ such that $\mathscr{R}_i$ verifies $\Box\alpha$ and $\mathscr{R}_{i'}$ falsifies $\Box\alpha$. But then, by the definition of "subrow", in $\mathscr{T}_b$ $\mathscr{R}_j$ verifies $\Box\alpha$ and $\mathscr{R}_{j'}$ falsifies $\Box\alpha$, and so $\mathscr{R}_j$ C1 $\mathscr{R}_{j'}$ does not hold. A similar reasoning holds for the case where $\mathscr{R}_{i'}$ C0 $\mathscr{R}_i$ does not hold and for the case where $\mathscr{R}_{i'}$ C1 $\mathscr{R}_i$ does not hold. The theorem then follows immediately by definitions 2.1.

4.1. *Theorem-schema.* Let $\mathscr{R}_i$ be a row of a table $\mathscr{T}_a$ and let $\mathscr{R}_j$ be a row of a table $\mathscr{T}_b$. Let $\mathscr{T}_a$ be a subtable of $\mathscr{T}_b$ and let $\mathscr{R}_i$ be a subrow of $\mathscr{R}_j$. Then we have that if R_i is not X-acceptable then $\mathscr{R}_j$ is not X-acceptable.
Proof-schema. The proofs are by induction on definitions **2.2.** If $\mathscr{R}_i$ is not 0-X-acceptable then there is in $\mathscr{T}_a$ a constituent $\Box\alpha$ such that $\mathscr{R}_i$ verifies $\Box\alpha$ and falsifies α. But then, by **1.7**, in $\mathscr{T}_b$ $\mathscr{R}_j$ verifies $\Box\alpha$ and falsifies α and so $\mathscr{R}_j$ is not 0-X-acceptable. If $\mathscr{R}_i$ is not n + 1-X-acceptable because it is not n-X-acceptable, we have by the hypothesis of the induction that $\mathscr{R}_j$ is not n-X-acceptable and so not n + 1-X-acceptable. If $\mathscr{R}_i$ is not n + 1-X-acceptable because condition (2) of n + 1-X-acceptability does not hold, then there is a constituent $\Box\alpha$ in $\mathscr{T}_a$ such that $\mathscr{R}_i$ falsifies $\Box\alpha$ and there is no n-X-acceptable row in $\mathscr{T}_a$ which is X-controlled by $\mathscr{R}_i$ and which falsifies α. For every n-X-acceptable row $\mathscr{R}_{j'}$ of $\mathscr{T}_b$ which falsifies α, we have by **1.7** that the row $\mathscr{R}_{i'}$ of $\mathscr{T}_a$ which is a subrow of $\mathscr{R}_{j'}$ falsifies α, and by the hypothesis of the induction we have that $\mathscr{R}_{i'}$ is n-X-acceptable. By the hypothesis about$\mathscr{R}_i$, $\mathscr{R}_i$ does not X-control $\mathscr{R}_i$ and so by **4.0.** $\mathscr{R}_j$ does not X-control $R_{j'}$. But as $\mathscr{R}_i$ (which is a sub-

row of $\mathscr{R}_j$) falsifies $\Box\alpha$, so does R_j. It follows that condition (2) for n + 1-X-acceptability for $\mathscr{R}_j$ does not hold.

4.2. *Definition-schema.* Let $\mathscr{T}_a$ and $\mathscr{T}_b$ be tables such that $\mathscr{T}_a$ is a subtable of $\mathscr{T}_b$ and $\mathscr{T}_b$ contains just one constituent β more than $\mathscr{T}_a$. Let us notice that obviously if β is of the form $\Box\gamma$ then $\mathscr{T}_a$ is a table for γ. Let $\mathscr{R}_i$ be an X-acceptable row of $\mathscr{T}_a$. Let $\mathscr{R}_{it}$ and $\mathscr{R}_{if}$ be the two rows of $\mathscr{T}_b$ of which $\mathscr{R}_i$ is subrow. Let $\mathscr{R}_{it}$ verifie β and let $\mathscr{R}_{if}$ falsifie β. We define what it means to say that $\mathscr{R}_{it}$ or $\mathscr{R}_{if}$ is a $\mathscr{T}_a$-X-*selected row of* $\mathscr{T}_b$.
If β is a variable then both $\mathscr{R}_{it}$ and $\mathscr{R}_{if}$ are $\mathscr{T}_a$-X-selected.
If β is $\Box\gamma$, then $\mathscr{R}_{it}$ is $\mathscr{T}_a$-X-selected iff all X-acceptable rows of $\mathscr{T}_a$ which are X-controled by $\mathscr{R}_i$ verifie γ.
If β is $\Box\gamma$, then $\mathscr{R}_{if}$ is $\mathscr{T}_a$-X-selected iff $\mathscr{R}_{it}$ is not $\mathscr{T}_a$-X-selected.
It follows immediately from this that, if β is $\Box\gamma$, $\mathscr{R}_{if}$ is $\mathscr{T}_a$-X-selected iff there is an X-acceptable row in $\mathscr{T}_a$ which is X-controled by $\mathscr{R}_i$ and which falsifies γ.

4.3. *Theorem-schema.* Let $\mathscr{T}_a$ and $\mathscr{T}_b$ be tables such that $\mathscr{T}_a$ is a subtable of $\mathscr{T}_b$ and $\mathscr{T}_b$ contains just one constituent β more than $\mathscr{T}$. Let $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ be X-acceptable rows of $\mathscr{T}_a$ such that $\mathscr{R}_i$ X-controls $\mathscr{R}_{i'}$. Let $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ be subrows of the $\mathscr{T}$-X-selected rows $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ of $\mathscr{T}_b$ respectively. Then $\mathscr{R}_j$ X-controls $\mathscr{R}_{j'}$.
Proof for M. If $\mathscr{R}_i$ M-controls $\mathscr{R}_{i'}$ then $\mathscr{R}_i$ C0 $\mathscr{R}_{i'}$. So there is no constituent $\Box\alpha$ in $\mathscr{T}_a$ such that $\mathscr{R}_i$ verifies $\Box\alpha$ and $\mathscr{R}_{i'}$ falsifies α. By **1.7** we have that there is no constituent $\Box\alpha$ in $\mathscr{T}_b$ such that $\Box\alpha$ is in $\mathscr{T}_a$ and $\mathscr{R}_j$ verifies $\Box\alpha$ and $\mathscr{R}_{j'}$ falsifies α. It follows that, if β is a variable or if β is $\Box\gamma$ and $\mathscr{R}_{j'}$ falsifies $\Box\gamma$ then $\mathscr{R}_j$ C0 $\mathscr{R}_{j'}$ and so $\mathscr{R}_j$ M-controls $\mathscr{R}_{j'}$. If β is $\Box\gamma$ and $\mathscr{R}_j$ verifies $\Box\gamma$ then, as $\mathscr{R}_j$ is $\mathscr{T}_a$-M-selected, all the M-acceptable rows of $\mathscr{T}_a$ which are M-controled by $\mathscr{R}_i$ verifie γ. So $\mathscr{R}_{i'}$, and by **1.7**, $\mathscr{R}_{j'}$ verifie γ, and so again $\mathscr{R}_j$ C0 $\mathscr{R}_{j'}$.
Proof for B. If $\mathscr{R}_i$ B-controls $\mathscr{R}_{i'}$ we have $\mathscr{R}_i$ C0 $\mathscr{R}_{i'}$ and $\mathscr{R}_{i'}$ C0 $\mathscr{R}_i$. By reasoning like in the proof for M we can prove that if $\mathscr{R}_i$ C0 $\mathscr{R}_{i'}$ then $\mathscr{R}_j$ C0 $\mathscr{R}_{j'}$ and that if $\mathscr{R}_{i'}$ C0 $\mathscr{R}_i$ then $\mathscr{R}_{j'}$ C0 $\mathscr{R}_j$.
Proof for S4. If $\mathscr{R}_i$ S4-controls $\mathscr{R}_{i'}$ we have $\mathscr{R}_i$ C0 $\mathscr{R}_{i'}$ and $\mathscr{R}_i$ C1 $\mathscr{R}_{i'}$. By reasoning like in the proof for M we can prove that if $\mathscr{R}_i$ C0 $\mathscr{R}_{i'}$ then $\mathscr{R}_j$ C0 $\mathscr{R}_{j'}$. Furthermore if $\mathscr{R}_i$ C1 $\mathscr{R}_{i'}$ there is

no constituent $\Box\alpha$ in $\mathscr{T}_a$ such that $\mathscr{R}_i$ verifies $\Box\alpha$ and $\mathscr{R}_{i'}$ falsi fies $\Box\alpha$. By the definition of subrow we have that there is no constituent $\Box\alpha$ in $\mathscr{T}_b$ such that $\Box\alpha$ is in $\mathscr{T}_a$ and $\mathscr{R}_j$ verifies $\Box\alpha$ and $\mathscr{R}_{j'}$ falsifies $\Box\alpha$. It follows that if β is a variable or if β is $\Box\gamma$ and $\mathscr{R}_j$ falsifies $\Box\gamma$ then $\mathscr{R}_j$ C1 $\mathscr{R}_{j'}$ and so, given the proof for C0, $\mathscr{R}_j$ S4-controls $\mathscr{R}_{j'}$. If β is $\Box\gamma$ and if $\mathscr{R}_j$ verifies $\Box\gamma$ then, as $\mathscr{R}_j$ is $\mathscr{T}_a$-S4-selected, all the S4-acceptable rows of $\mathscr{T}_a$ which are S4-controled by $\mathscr{R}_i$ verifie γ. By the transitivity of S4-control we have that all the S4-acceptable rows of $\mathscr{T}_a$ which are S4-controled by $\mathscr{R}_{i'}$ are S4-controled by $\mathscr{R}_i$. So all rows which are S4-controled by $\mathscr{R}_{i'}$ verifie γ. As $\mathscr{R}_{j'}$ is $\mathscr{T}_a$-S4-selected $\mathscr{R}_{j'}$ verifies $\Box\gamma$ and we have again that $\mathscr{R}_j$ C1 $\mathscr{R}_{j'}$.
Proof for S5. If $\mathscr{R}_i$ S5-controls $\mathscr{R}_{i'}$ we have $\mathscr{R}_i$ C0 $\mathscr{R}_{i'}$, $\mathscr{R}_{i'}$ C0 $\mathscr{R}_i$, $\mathscr{R}_i$ C1 $\mathscr{R}_{i'}$ and $\mathscr{R}_{i'}$ C1 $\mathscr{R}_i$. By reasoning like in the proof for S4 we have that if $\mathscr{R}_i$ C0 $\mathscr{R}_{i'}$ and $\mathscr{R}_i$ C1 $\mathscr{R}_{i'}$ then $\mathscr{R}_j$ C0 $\mathscr{R}_{j'}$ and $\mathscr{R}_j$ C1 $\mathscr{R}_{j'}$. By the same kind of proof we have that if $\mathscr{R}_{i'}$ C0 $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ C1 $\mathscr{R}_i$ then $\mathscr{R}_{j'}$ C0 $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ C1 $\mathscr{R}_j$.

4.4. *Theorem-schema.* Let $\mathscr{T}_a$ and $\mathscr{T}_b$ be tables such that $\mathscr{T}_a$ is a subtable of $\mathscr{T}_b$, and $\mathscr{T}_b$ contains just one constituent β more than $\mathscr{T}_a$. Let $\mathscr{R}_i$ be an X-acceptable row of $\mathscr{T}_a$. Let $\mathscr{R}_i$ be a subrow of the $\mathscr{T}_a$-X-selected row $\mathscr{R}_j$ of $\mathscr{T}_b$. Then $\mathscr{R}_j$ is X-acceptable. (The theorem does not state that the $\mathscr{T}_a$-X-selected rows of $\mathscr{T}_b$ are the sole X-acceptable rows of $\mathscr{T}_b$).
Proof-schema. By induction on definition 2.2. As $\mathscr{R}_i$ is 0-X-acceptable, $\mathscr{R}_i$ C0 $\mathscr{R}_i$. By **4.3** we have $\mathscr{R}_j$ C0 $\mathscr{R}_j$. So $\mathscr{R}_j$ is 0-X-acceptable. By the hypothesis of the induction $\mathscr{R}_j$ is n-X-acceptable and so condition (1) for $n + 1$-X-acceptability holds for $\mathscr{R}_j$. As $\mathscr{R}_i$ is X-acceptable, we have by **2.7** that for every constituent $\Box\alpha$ in $\mathscr{T}_a$ which is falsified by $\mathscr{R}_i$ there is in $\mathscr{T}_a$ an X-acceptable row $\mathscr{R}_{i'}$ such that $\mathscr{R}_i$ X-controls $\mathscr{R}_{i'}$ and $\mathscr{R}_{i'}$ falsifies α. Let $\mathscr{R}_{j'}$ be the $\mathscr{T}_a$-X-selected row of $\mathscr{T}_b$ such that $\mathscr{R}_{i'}$ is a subrow of $\mathscr{R}_{j'}$. By the hypothesis of the induction $\mathscr{R}_{j'}$ is n-X-acceptable. By **4.3** $\mathscr{R}_j$ X-controls $\mathscr{R}_{j'}$. By **1.7** $\mathscr{R}_{j'}$ falsifies α. So for every constituent $\Box\alpha$ in $\mathscr{T}_b$ which is in $\mathscr{T}_a$ and which is falsified by $\mathscr{R}_j$ there is in $\mathscr{T}_b$ a row $\mathscr{R}_{j'}$ such that $\mathscr{R}_{j'}$ is n-X-acceptable, $\mathscr{R}_j$ X-controls $\mathscr{R}_{j'}$ and $\mathscr{R}_{j'}$ falsifies α. It follows that if β is a variable or if β is $\Box\gamma$ and $\mathscr{R}_j$ verifies $\Box\gamma$ then condition (2) for

n + 1-X-acceptability holds for $\mathscr{R}_j$. If β is □γ and $\mathscr{R}_j$ falsifies □γ, then as $\mathscr{R}_j$ is $\mathscr{T}_a$-X-acceptable there is in $\mathscr{T}_a$ a row $\mathscr{R}_{i'}$ such that $\mathscr{R}_{i'}$ is X-acceptable, $\mathscr{R}_i$ X-controls $\mathscr{R}_{i'}$ and $\mathscr{R}_{i'}$ falsifies γ. Let $\mathscr{R}_{j'}$ be the $\mathscr{T}_a$-X-selected row of $\mathscr{T}_b$ such that $\mathscr{R}_{i'}$ is a subrow of $\mathscr{R}_{j'}$. By the hypothesis of the induction we have that $\mathscr{R}_{j'}$ is n-X-acceptable, by **4.3** we have that $\mathscr{R}_j$ X-controls $\mathscr{R}_{j'}$ and by **1.7** $\mathscr{R}_{j'}$ falsifies γ. So again condition (2) for n + 1-acceptability holds for $\mathscr{R}_j$.

4.5. *Theorem-schema.* Let $\mathscr{T}_a$ and $\mathscr{T}_b$ be tables such that $\mathscr{T}_a$ is a subtable of $\mathscr{T}_b$. Let $\mathscr{R}_i$ be an X-acceptable row of $\mathscr{T}_a$. Then in $\mathscr{T}_b$ there is an X-acceptable row $\mathscr{R}_j$ such that $\mathscr{R}_i$ is a subrow of $\mathscr{R}_j$.
Proof-schema. Let n be the number of constituents in $\mathscr{T}_b$ which are not in $\mathscr{T}_a$. The proofs are by induction on n. If n = 0 the theorem is trivial. If n > 0, we know by **1.3** that it is possible to construct a series of tables $\mathscr{T}_0$, ..., $\mathscr{T}_n$ such that $\mathscr{T}_0$ is $\mathscr{T}_a$, $\mathscr{T}_n$ is $\mathscr{T}_b$, and for each m such that $0 \leqslant m < n$ we have that $\mathscr{T}_{m+1}$ contains just one constituent more than $\mathscr{T}_m$. By the hypothesis of the induction there is in $\mathscr{T}_{n-1}$ an X-acceptable row $\mathscr{R}_k$ such that $\mathscr{R}_i$ is a subrow of $\mathscr{R}_k$. Let $\mathscr{R}_j$ be the $\mathscr{T}_{n-1}$-X-selected row of $\mathscr{T}_n$ (that is of $\mathscr{T}_b$) such that $\mathscr{R}_k$ is a subrow of$\mathscr{R}_j$. By **4.4** $\mathscr{R}_j$ is X-acceptable and obviously $\mathscr{R}_i$ is a subrow of $\mathscr{R}_j$.

4.6. *Theorem-schema.* Let $\mathscr{T}_a$ and $\mathscr{T}_b$ be tables such that $\mathscr{T}_a$ is a subtable of $\mathscr{T}_b$. Let $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ be X-acceptable rows of $\mathscr{T}_a$ such that $\mathscr{R}_i$ X-controls $\mathscr{R}_{i'}$. Then there are in $\mathscr{T}_b$ X-acceptable rows $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ such that $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ are subrows of $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ respectively and $\mathscr{R}_j$ X-controls $\mathscr{R}_{j'}$.
Proof-schema. Le n be the number of constituents in $\mathscr{T}_b$ which are not in $\mathscr{T}_a$. The proofs are by induction on n. If n = 0 the theorem is trivial. If n > 0 let us consider the series of tables $\mathscr{T}_0$, ..., $\mathscr{T}_n$ of **4.5**. By the hypothesis of the induction there are in $\mathscr{T}_{n-1}$ X-acceptable rows $\mathscr{R}_k$ and $\mathscr{R}_{k'}$ such that $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ are subrows of $\mathscr{R}_k$ and $\mathscr{R}_{k'}$ respectively and $\mathscr{R}_k$ X-controls $\mathscr{R}_{k'}$. Let $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ be the $\mathscr{T}_{n-1}$-X-selected rows in $\mathscr{T}_n$ (that is in $\mathscr{T}_b$) such that $\mathscr{R}_k$ and $\mathscr{R}_{k'}$ are subrows of $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ respectively. By **4.4** $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ are X-acceptable, by **4.3** $\mathscr{R}_j$ X-con-

trols $\mathscr{R}_{j'}$ and obviously $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ are subrows of $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ respectively.

4.7. *Theorem-schema.* Let α be a variable, let β be any formula, let $\mathscr{T}_b$ be the β/α-table from $\mathscr{T}_a$, let $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ be rows of $\mathscr{T}_b$, let $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ be rows of $\mathscr{T}_a$ such that $\mathscr{R}_j$ and $\mathscr{R}_{j'}$ are β/α-rows from $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ respectively. Then we have that if $\mathscr{R}_i$ does not X-control $\mathscr{R}_{i'}$ then $\mathscr{R}_j$ does not X-control $\mathscr{R}_{j'}$.
Proof. If $\mathscr{R}_i$ C0 $\mathscr{R}_{i'}$ does not hold then there is a constituent $\Box\gamma$ in $\mathscr{T}_a$ such that $\mathscr{R}_i$ verifies $\Box\gamma$ and $\mathscr{R}_{i'}$ falsifies γ. Let δ be the result of substituting β for α in γ. Then by **1.9** we have that $\mathscr{R}_j$ verifies $\Box\delta$ and $\mathscr{R}_{j'}$ falsifies δ. So $\mathscr{R}_j$ C0 $\mathscr{R}_{j'}$ does not hold. If $\mathscr{R}_i$ C1 $\mathscr{R}_{j'}$ does not fold· then there is a constituent $\Box\gamma$ in $\mathscr{T}_a$ such that $\mathscr{R}_i$ verifies $\Box\gamma$ and $\mathscr{R}_{i'}$ falsifies $\Box\gamma$. Let $\Box\delta$ be the result of substituting β for α in $\Box\gamma$. Then by the definition of "β/α-row" we have that $\mathscr{R}_j$ verifies $\Box\delta$ and $\mathscr{R}_{j'}$ falsifies $\Box\delta$. So $\mathscr{R}_j$ C1 $\mathscr{R}_{j'}$ does not hold. A similar reasoning holds for the case where $\mathscr{R}_{i'}$ C0 R_i does not hold and for the case where $\mathscr{R}_{i'}$ C1 $\mathscr{R}_i$ does not hold. The theorem then follows immediately by definitions **2.1**.

4.8. *Theorem-schema.* Let α be a variable, let β be any formula, let $\mathscr{T}_b$ be the β/α-table from $\mathscr{T}_a$, let $\mathscr{R}_i$ be a not X-acceptable row of $\mathscr{T}_b$ such that $\mathscr{R}_j$ is a β/α-row from $\mathscr{R}_i$. Then $\mathscr{R}_j$ is not X-acceptable.
Proof-schema. The proofs are by induction on definitions **2.2**. If $\mathscr{R}_i$ is not 0-X-acceptable then there is in $\mathscr{T}_a$ a constituent $\Box\gamma$ such that $\mathscr{R}_i$ verifies $\Box\gamma$ and falsifies γ. Let δ be the result of substituting β for α in γ. Then by 1.9 we have that $\mathscr{R}_j$ verifies $\Box\delta$ and falsifies δ, and so $\mathscr{R}_j$ is not 0-X-acceptable. If $\mathscr{R}_i$ is not n + 1-X-acceptable because it is not n-X-acceptable, we have by the hypothesis of the induction that $\mathscr{R}_j$ is not n-X-acceptable and so not n + 1-X-acceptable. If $\mathscr{R}_i$ is not n + 1-X-acceptable because condition (2) for n + 1-X-acceptability does not hold, then there is a constituent $\Box\gamma$ in $\mathscr{T}_a$ such that $\mathscr{R}_i$ falsifies $\Box\gamma$ and there is no n-X-acceptable row in $\mathscr{T}_a$ which is X-controled by $\mathscr{R}_i$ and which falsifies γ. Let δ be the result of substituting β for α in γ. Then for every n-X-acceptable

row $\mathscr{R}_{j'}$ of $\mathscr{T}_b$ which falsifies δ, we have by 1.9 that the row $\mathscr{R}_{i'}$ of $\mathscr{T}_a$, which is such that $\mathscr{R}_{j'}$ is a β/α-row from $\mathscr{R}_{i'}$, falsifies γ, and by the hypothesis of the induction we have that $\mathscr{R}_{i'}$ is n-X-acceptable. So by the hypothesis about $\mathscr{R}_i$, $\mathscr{R}_i$ does not X-control $\mathscr{R}_{i'}$, and so by **4.7** $\mathscr{R}_j$ does not X-control $\mathscr{R}_{j'}$. But as $\mathscr{R}_j$ is β/α-row from $\mathscr{R}_i$ and as $\mathscr{R}_i$ falsifies $\Box\gamma$, we have that $\mathscr{R}_i$ falsifies $\Box\delta$. It follows that condition (2) for n + 1-X-acceptability for $\mathscr{R}_j$ does not hold.

5. *Tautologies.*

5.0. *Definition-schema.* Let $\mathscr{T}_a$ be a table for α. Then α is said to be a $\mathscr{T}_a$-P-*tautology* [a $\mathscr{T}_a$-X-*tautology*] if α is verified by all [by all X-acceptable] rows of $\mathscr{T}_a$.

5.1. *Definition-schema.* The formula α is said to be *a* P-*tautology* [*an* X-*tautology*] iff, for every table $\mathscr{T}_a$ such that $\mathscr{T}_a$ is a table for α, α is a $\mathscr{T}_a$-P-tautology [a $\mathscr{T}_a$-X-tautology].

5.2. *Theorem-schema.* Let $\mathscr{T}_a$ and $\mathscr{T}_b$ be tables for α such that $\mathscr{T}_a$ is a subtable of $\mathscr{T}_b$. Then we have that if α is a $\mathscr{T}_a$-P-tautology [a $\mathscr{T}_a$-X-tautology] then α is a $\mathscr{T}_b$-P-tautology [a $\mathscr{T}_b$-X-tautology].
Proof for P. Let $\mathscr{R}_j$ be any row of $\mathscr{T}_b$. Let $\mathscr{R}_i$ be the row of $\mathscr{T}_a$ which is a subrow of $\mathscr{R}_j$. By the hypothesis of the theorem $\mathscr{R}_i$ verifies α. By **1.7** $\mathscr{R}_j$ verifies α.
Proof-schema for M, B, S4 and S5. Let $\mathscr{R}_j$ be any X-acceptable row of $\mathscr{T}_b$. Let $\mathscr{R}_i$ be the row of $\mathscr{T}_a$ which is a subrow of $\mathscr{R}_j$. By **4.1** $\mathscr{R}_i$ is X-acceptable. By the hypothesis of the theorem $\mathscr{R}_i$ verifies α. By **1.7** $\mathscr{R}_j$ verifies α.

5.3. *Theorem-schema.* Let $\mathscr{T}_a$ and $\mathscr{T}_b$ be tables for α such that $\mathscr{T}_a$ is a subtable of $\mathscr{T}_b$. Then we have that if α is a $\mathscr{T}_b$-P-tautology [a $\mathscr{T}_b$-X-tautology] then α is a $\mathscr{T}_a$-P-tautology [a $\mathscr{T}_a$-X-tautology].
Proof for P. Let $\mathscr{R}_i$ be any row of $\mathscr{T}_a$. Let $\mathscr{R}_j$ be a row of $\mathscr{T}_b$ such that $\mathscr{R}_i$ is a subrow of $\mathscr{R}_j$. By the hypothesis of the theorem $\mathscr{R}_j$ verifies α. By **1.7** $\mathscr{R}_i$ verifies α.

Proof-schema for M, B, S4 and S5.

Let $\mathscr{R}_i$ be any X-acceptable row of $\mathscr{T}_d$. By **4.5** there is in $\mathscr{T}_b$ an X-acceptable row $\mathscr{R}_j$ such that $\mathscr{R}_i$ is a subrow of $\mathscr{R}_j$. By the hypothesis of the theorem $\mathscr{R}_j$ verifies α. By **1.7** $\mathscr{R}_i$ verifies α.

5.4. *Theorem-schema.* If there is a table $\mathscr{T}_b$ for α such that α is a $\mathscr{T}_b$-P-tautology [a $\mathscr{T}_b$-X-tautology] then α is a P-tautology [an X-tautology].

Proof-schema for P, M, B, S4 and S5. Let $\mathscr{T}_a$ be the minimum table for α. Let $\mathscr{T}_c$ be any table for α. $\mathscr{T}_a$ is a subtable of $\mathscr{T}_b$. By the hypothesis of the theorem α is a $\mathscr{T}_b$-P-tautology [a $\mathscr{T}_b$-X-tautology]. So by **5.3** α is a $\mathscr{T}_a$-P-tautology [a $\mathscr{T}_a$-X-tautology]. But $\mathscr{T}_a$ is a subtable of $\mathscr{T}_c$. So by **5.2** α is a $\mathscr{T}_c$-P-tautology [a $\mathscr{T}_c$-X-tautology].

5.5. *Theorem.* All P-tautologies are M-tautologies. All M-tautologies are B-tautologies. All M-tautologies are S4-tautologies. All B-tautologies are S5-tautologies. All S4-tautologies are S5-tautologies.

Proof. Let $\mathscr{T}_a$ be a table for α. If α is verified by all rows of $\mathscr{T}_a$, then α is verified by all M-acceptable rows of $\mathscr{T}_a$. If α is verified by all M-acceptable rows of $\mathscr{T}_a$ then by **3.4** α is verified by all B-acceptable rows of $\mathscr{T}_a$. If α is verified by all M-acceptable rows of $\mathscr{T}_a$, then by **3.4** α is verified by all S4-acceptable rows of $\mathscr{T}_a$. If α is verified by all B-acceptable rows of $\mathscr{T}_a$, then by **3.4** α is verified by all S5-acceptable rows of $\mathscr{T}_a$. If α is verified by all S4-acceptable rows of $\mathscr{T}_a$ then by **3.4** α is verified by all S5-acceptable rows of $\mathscr{T}_a$.

5.6. *Theorem.* (1) There is a formula which is an S5-tautology and not an S4-tautology. (2) There is a formula which is an S5-tautology and not a B-tautology. (3) There is a formula which is an S4-tautology and not a B-tautology. (4) There is a formula which is a B-tautology and not an S4-tautology. (5) There is a formula which is an S4-tautology and not an M-tautology. (6) There is a formula which is a B-tautology and not an M-tautology. (7) There is a formula which is an M-tautology and not a P-tautology.

Proof. Let α be a variable. The table for α, $\Box\alpha$, $\Box\Box\alpha$ and $\Box\sim\Box\alpha$ and the following formulas yield a proof of the theorem. (See 3.9).
For (1) and (2), $\sim\Box\alpha\rightarrow\Box\sim\Box\alpha$ is falsified by $\mathscr{R}3$ of table **3.9**, but is verified by all S-5-acceptable rows.
For (3) and (5), $\Box\alpha\rightarrow\Box\Box\alpha$ is falsified by $\mathscr{R}1$ of table **3.9**, but is verified by all S4-acceptable rows.
For (4) and (6), $\sim\alpha\rightarrow\Box\sim\Box\alpha$ is falsified by $\mathscr{R}5$ of table **3.9**, but is verified by all B-acceptable rows.
For (7), $\Box\alpha\rightarrow\alpha$ is falsified by the following not 0-M-acceptable row: (α, F) ($\Box\alpha$, T) ($\Box\Box\alpha$, T) ($\Box\sim\Box\alpha$, T), but is verified by all M-acceptable rows.

5.7. *Theorem-schema.* Let α be a variable, let β be any formula, let γ be any formula, let δ be the result of substituting β for α in γ. Then we have that if γ is a P-tautology [an X-tautology] then δ is a P-tautology [an X-tautology].
Proof for P. Let $\mathscr{T}_a$ be a table for γ, let $\mathscr{T}_b$ be the β/α-table from $\mathscr{T}_a$, let $\mathscr{R}_j$ be any row of $\mathscr{T}_b$ and let $\mathscr{R}_i$ be the row of $\mathscr{T}_a$ such that $\mathscr{R}_j$ is a β/α-row from $\mathscr{R}_i$. Then we have by the hypothesis about γ, that $\mathscr{R}_i$ verifies γ. By **1.9** we have that $\mathscr{R}_j$ verifies δ.

Proof-schema for M, B, S4 and S5. Let $\mathscr{T}_a$ be a table for γ, let $\mathscr{T}_b$ be the β/α-table from $\mathscr{T}_a$, let $\mathscr{R}_j$ be any X-acceptable row of $\mathscr{T}_b$ and let $\mathscr{R}_i$ be the row of $\mathscr{T}_a$ such that $\mathscr{R}_j$ is a β/α-row from $\mathscr{R}_i$.Then we have by **4.8** that $\mathscr{R}_i$ is X-acceptable and so, by the hypothesis about γ, $\mathscr{R}_i$ verifies γ. By **1.9** we have that $\mathscr{R}_j$ verifies δ.

PART II

6. *Plain proposition logic.*

6.0. *Definitions.* An *uncovered occurence of a constituent* β *in a formula* α is an occurence of β in α not as a part of a constituent.

An *uncovered constituent* of a formula α is a constituent of α which has an uncovered occurence in α.

A *value-assignment for a formula* α is a monadic function which takes the uncovered constituents of α as arguments and the letters "T" or "F" as values.

6.1 *Definition.* Let $\mathscr{V}_i$ be a value-assignment for α. We define under what conditions $\mathscr{V}_i$ *verifies* or *falsifies* α.

If α is a constituent (and so an uncovered constituent of itself) then $\mathscr{V}_i$ verifies α iff $\mathscr{V}_i$ applied to α takes the value "T", and $\mathscr{V}_i$ falsifies α iff $\mathscr{V}_i$ applied to α takes the value "F".

If α is of the form $\sim\beta$, then $\mathscr{V}_i$ *verifies* α iff $\mathscr{V}_i$ falsifies β, and $\mathscr{V}_i$ *falsifies* α iff $\mathscr{V}_i$ verifies β.

If α is of the form $\beta\rightarrow\gamma$, then $\mathscr{V}_i$ *verifies* α iff $\mathscr{V}_i$ falsifies β or verifies γ, and $\mathscr{V}_i$ *falsifies* α iff $\mathscr{V}_i$ verifies β and falsifies γ.

6.2. *Definition.* A formula α is said to be *a plain tautology* iff all value-assignments for α verifie α.

6.3. Let $\mathscr{T}_a$ be a table for α, let $\mathscr{R}_i$ be a row of $\mathscr{T}_a$ and let $\mathscr{V}_j$ be a value-assignment for α. We say that $\mathscr{R}_i$ and $\mathscr{V}_j$ are α-*equivalent* iff, for every uncovered constituent β of α, $\mathscr{R}_i$ verifies β iff $\mathscr{V}_j$ verifies β (and so $\mathscr{R}_i$ falsifies β iff $\mathscr{V}_j$ falsifies β).

6.4. *Theorems.* Let $\mathscr{T}_a$ be a table for α, let $\mathscr{R}_i$ be a row of $\mathscr{T}_a$ and let $\mathscr{V}_j$ be a value-assignment for α, such that $\mathscr{R}_i$ and $\mathscr{V}_j$ are α-equivalent. Then (1) $\mathscr{R}_i$ verifies [falsifies] α iff $\mathscr{V}_j$ verifies [falsifies] α. (2) The formula α is a $\mathscr{T}_a$-P-tautology (and so a P-tautology) iff it is a plain tautology.

Proof for (1). By definition **6.3** and the parallelism between **1.6** and **6.1**.

Proof for (2). If α is a $\mathscr{T}_a$-P-tautology then α is verified by all rows of $\mathscr{T}_a$. Obviously every value-assignment for α is α-equivalent with some row of $\mathscr{T}_a$. So by (1) all value-assignments for α verifie α. If α is a plain tautology then α is verified by all value-assignments for α. Obviously every row of $\mathscr{T}_a$ is α-equivalent with some value-assignment for α. So by (1) all rows of $\mathscr{T}_a$ verifie α.

6.5. *Definition.* Let $\alpha_1, \ldots, \alpha_n$ be the n distinct constituents of an asc $\mathscr{S}_a$. Let $\mathscr{R}_i$ be a row of the table $\mathscr{T}_a$ from $\mathscr{S}_a$. Then the *representing formula* (the rf) of $\mathscr{R}_i$ is the formula $\beta_1 \wedge \ldots \wedge \beta_n$, where β_m $(1 \leqslant m \leqslant n)$ is α_m or $\sim\alpha_m$ according as $\mathscr{R}_i$ verifies or falsifies α_m.

6.6. *Theorems.* Let $\mathscr{R}_i$ be a row of a table $\mathscr{T}_a$ and let α be the rf of $\mathscr{R}_i$. Then (1) $\mathscr{R}_i$ verifies α and (2) every other row $\mathscr{R}_j$ of $\mathscr{T}_a$ falsifies α.

Proof. By plain proposition logic it is easily seen that for every row $\mathscr{R}_k$ of $\mathscr{T}_a$ we have that $\mathscr{R}_k$ verifies α iff $\mathscr{R}_k$ verifies all the members of the continuous conjunction α, and that $\mathscr{R}_k$ falsifies α iff $\mathscr{R}_k$ falsifies one of the members of the continuous conjunction α. By the hypothesis of the theorem we have that $\mathscr{R}_i$ verifies all the members of the continuous conjunction α and that every other row $\mathscr{R}_j$ falsifies one of the members of the continuous conjunction α.

6.7. *Theorem.* Let $\mathscr{T}_a$ be a table with r distinct rows. Then the following formulas are $\mathscr{T}_a$-P-tautologies (and so P-tautologies):

(1) $\alpha_1 \vee \ldots \vee \alpha_r$, where $\alpha_1, \ldots, \alpha_r$ are the rfs of the r distinct rows of $\mathscr{T}_a$;

(2) $(\sim\alpha_1 \wedge \ldots \wedge \sim\alpha_i) \rightarrow (\alpha_j \vee \ldots \vee \alpha_r)$ where $\alpha_1, \ldots, \alpha_i$ are the rfs of i distinct rows of $\mathscr{T}_a$ and where $\alpha_j, \ldots, \alpha_r$ are the rfs of the r-i other rows of $\mathscr{T}_a$;

(3) $\alpha \rightarrow \beta$ where α is the rf of some row $\mathscr{R}_i$ in $\mathscr{T}_a$, and where β is a formula such that $\mathscr{T}_a$ is a table for β and $\mathscr{R}_i$ verifies β;

(4) $\alpha \rightarrow \sim\beta$ where α is the rf of some row $\mathscr{R}_i$ in $\mathscr{T}_a$, and where β is a formula such that $\mathscr{T}_a$ is a table for β and $\mathscr{R}_i$ falsifies β:

(5) $\beta \rightarrow (\alpha_1 \vee \ldots \vee \alpha_i)$ where β is a formula such that $\mathscr{T}_a$ is a table for β, and $\alpha_1, \ldots, \alpha_i$ are the rfs of all the rows in $\mathscr{T}_a$ which verifie β;
(6) $\sim\beta \rightarrow (\alpha_1 \vee \ldots \vee \alpha_i)$ where β is a formula such that $\mathscr{T}_a$ is a table for β, and $\alpha_1, \ldots, \alpha_i$ are the rfs of all the rows in $\mathscr{T}_a$ which falsifie β.
Proofs. For (1). Every row of $\mathscr{T}_a$ verifies one of the members of the disjunction and so verifies the disjunction.
For (2). Every row of $\mathscr{T}_a$ either falsifies one of the members of the antecedent of the implication and so falsifies that antecedent and verifies the implication, or verifies one of the members of the consequent of the implication, and so verifies that consequent and the implication.
For (3) and (4). Every row $\mathscr{R}_j$ in $\mathscr{T}_a$ other than $\mathscr{R}_i$ falsifies α, and so verifies $\alpha \rightarrow \beta$ and $\alpha \rightarrow \sim\beta$. $\mathscr{R}_i$ verifies $\alpha \rightarrow \beta$ or $\alpha \rightarrow \sim\beta$ according as $\mathscr{R}_i$ verifies or falsifies β.
For (5). Every row in $\mathscr{T}_a$ which falsifies β falsifies the antecedent of the implication and so verifies the implication. Every row in $\mathscr{T}_a$ which verifies β verifies one of the members of the consequent of the implication and so verifies that consequent and the implication. For (6). Every row in $\mathscr{T}_a$ which verifies β falsifies the antecedent of the implication and so verifies the implication. Every row in $\mathscr{T}_a$ which falsifies β verifies one of the members of the consequent of the implication and so verifies that consequent and the implication.

6.8. *Definition.* The *deductive system* P is the system which contains the following three axiom-schemata and the following deduction rule.

A0. $\alpha \rightarrow (\beta \rightarrow \alpha)$.
A1. $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$.
A2. $(\sim\alpha \rightarrow \sim\beta) \rightarrow (\beta \rightarrow \alpha)$.
MP. Modus ponens for material implication.

6.9. *Theorem.* A formula α is a theorem of the system P (a P-theorem) iff α is a P-tautology.
Proof. We can consider the language, constructed from propositional variables and the three constants for negation, material

implication and necessity, as a language for plain propositional logic by considering all constituents, modal or not, as being variables for propositions and by considering that only the uncovered occurences of the constituents are occurences of these variables. Such a convention is quite similar to the convention according to which, in a language where we have the variables " p ", " p' ", " p'' " etc., the occurences of the expressions " p " and " p' " within the variable " p'' " are not considered as occurences of the variables " p " and " p' ". Once we have made such a convention we can see in Church's [8] that a formula α is a P-theorem iff α is a plain tautology. The theorem then follows by **6.4**.

7. *The deductive systems M, B, S4 and S5.*

7.0. We give ourselves the following five axiom-schemata and the following deduction rule.

AD. $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$.

AR. $\Box\alpha \to \alpha$.

AS. $\sim\alpha \to \Box \sim \Box\alpha$.

AT. $\Box\alpha \to \Box\Box\alpha$.

AST. $\sim\Box\alpha \to \Box \sim \Box\alpha$.

NE. Necessitation. From α to deduce $\Box\alpha$.

(The letters "D", "R", "S" and "T" are suggested by the words "distribution", "reflexive", "symmetrical" and "transitive".)

The deductive systems M, B, S4 and S5 contain the following axiom-schemata and deduction rules respectively.

M. The same as P plus **AD**, **AR** and **NE**.

B. The same as M plus **AS**.

S4. The same as M plus **AT**.

S5. The same as M plus **AS** and **AT**.

To indicate that α is a theorem of one of these systems or of some other system we write "$\vdash \alpha$", relying on the context to avoid ambiguities.

7.1. *Theorem.* In M, B, S4 and S5 if $\vdash \alpha \equiv \beta$, and if δ is the result of replacing an occurrence of α in γ by an occurrence of β, then $\vdash \gamma \equiv \delta$.

Proof by induction on the construction of γ from α. If γ is α the theorem is trivial. If γ is (1) $\sim\varepsilon$, or (2) $\varepsilon \rightarrow \zeta$, or (3) $\zeta \rightarrow \varepsilon$ or (4) $\Box\varepsilon$, where ε contains α, and if η is the result of replacing an occurrence of α in ε by an occurrence of β, then by the hypothesis of the induction $\vdash \varepsilon \equiv \eta$. By P we have then $\vdash \sim\varepsilon \equiv \sim\eta$, or $(\varepsilon \rightarrow \zeta) \equiv (\eta \rightarrow \zeta)$, or $\vdash (\zeta \rightarrow \varepsilon) \equiv (\zeta \rightarrow \eta)$ and so the theorem is proved for cases (1), (2) and (3). For case (4) we have by P, **NE** and **AD** $\vdash \varepsilon \rightarrow \eta$, $\vdash \eta \rightarrow \varepsilon$, $\vdash \Box(\varepsilon \rightarrow \eta)$, $\vdash \Box(\eta \rightarrow \varepsilon)$, $\vdash \Box\varepsilon \rightarrow \Box\eta$, $\vdash \Box\eta \rightarrow \Box\varepsilon$ and $\vdash \Box\varepsilon \equiv \Box\eta$.

7.2. *Theorem.* In M, and so in B, S4 and S5: $\vdash \Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$.
Proof. (In this and in the following proofs obvious steps will be omitted).
(1) $\vdash (\alpha \wedge \beta) \rightarrow \alpha$. (P).
(2) $\vdash \Box((\alpha \wedge \beta) \rightarrow \alpha)$. (1)(NE).
(3) $\vdash \Box(\alpha \wedge \beta) \rightarrow \Box\alpha$. (2)(AD)(MP).
(4) $\vdash (\alpha \wedge \beta) \rightarrow \beta$. (P).
(5) $\vdash \Box((\alpha \wedge \beta) \rightarrow \beta)$. (4)(NE).
(6) $\vdash \Box(\alpha \wedge \beta) \rightarrow \Box\beta$. (5)(AD)(MP).
(7) $\vdash \Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$. (3)(6)(P).

7.3. *Theorem.* In M, and so in B, S4 and S5 we have:
$\vdash \Box(\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \wedge \alpha_n) \rightarrow (\Box\alpha_0 \wedge \ldots \wedge \Box\alpha_{n-1} \wedge \Box\alpha_n)$.
Proof by induction on n.
7.2 treats the case where $n=1$.
For $n > 1$ we have by **7.2**:
(1) $\vdash \Box(\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \wedge \alpha_n) \rightarrow (\Box(\alpha_0 \wedge \ldots \wedge \alpha_{n-1}) \wedge \Box\alpha_n)$.
By the hypothesis of the induction we have:
(2) $\vdash \Box(\alpha_0 \wedge \ldots \wedge \alpha_{n-1}) \rightarrow (\Box\alpha_0 \wedge \ldots \wedge \Box\alpha_{n-1})$.
By (1), (2) and P we have:
(3) $\vdash \Box(\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \wedge \alpha_n) \rightarrow (\Box\alpha_0 \wedge \ldots \wedge \Box\alpha_{n-1} \wedge \Box\alpha_n)$.

7.4. *Theorem.* In M, and so in B, S4 and S5: $\vdash (\Box\alpha \wedge \Box\beta) \rightarrow \Box(\alpha \wedge \beta)$.
Proof.
(1) $\vdash \alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$. (P).
(2) $\vdash \Box(\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta)))$. (1)(NE).

(3) $\vdash \Box\alpha \to \Box(\beta \to (\alpha \wedge \beta))$. (2)(AD)(MP).
(4) $\vdash \Box(\beta \to (\alpha \wedge \beta)) \to (\Box\beta \to \Box(\alpha \wedge \beta))$. (AD).
(5) $\vdash \Box\alpha \to (\Box\beta \to \Box(\alpha \wedge \beta))$. (3)(4)(P).
(6) $\vdash (\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)$. (5)(P).

7.5. *Theorem.* In M, and so in B, S4 and S5 we have:
$\vdash (\Box\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \wedge \Box\alpha_n) \to \Box(\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \wedge \alpha_n)$.
Proof by induction on n.
7.4 treats the case where $n = 1$.
For $n > 1$ we have by the hypothesis of the induction:
(1) $\vdash (\Box\alpha_0 \wedge \ldots \wedge \Box\alpha_{n-1}) \to \Box(\alpha_0 \wedge \ldots \wedge \alpha_{n-1})$.
(2) $\vdash (\Box\alpha_0 \wedge \ldots \wedge \Box\alpha_{n-1} \wedge \Box\alpha_n) \to (\Box(\alpha_0 \wedge \ldots \wedge \alpha_{n-1}) \wedge \Box\alpha_n)$. (1)(P).
(3) $\vdash (\Box(\alpha_0 \wedge \ldots \wedge \Box\alpha_{n-1}) \wedge \Box\alpha_n) \to \Box(\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \wedge \alpha_n)$. (7.4).
(4) $\vdash (\Box\alpha_0 \wedge \ldots \wedge \Box\alpha_{n-1} \wedge \Box\alpha_n) \to \Box(\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \wedge \alpha_n)$. (2)(3)(P).

7.6 *Theorem.* In S5: $\vdash \sim\Box\alpha \to \Box\sim\Box\alpha$.
Proof.
(1) $\vdash \Box\alpha \to \Box\Box\alpha$. (AT).
(2) $\vdash \sim\Box\Box\alpha \to \sim\Box\alpha$. (1)(P).
(3) $\vdash \Box(\sim\Box\Box\alpha \to \sim\Box\alpha)$. (2)(NE).
(4) $\vdash \Box\sim\Box\Box\alpha \to \Box\sim\Box\alpha$. (3)(AD)(MP).
(5) $\vdash \sim\Box\alpha \to \Box\sim\Box\Box\alpha$. (AS).
(6) $\vdash \sim\Box\alpha \to \Box\sim\Box\alpha$. (5)(4)(P).

7.7. *Theorem.* In the system which contains the axiom-schemata and deduction rules of M plus **AST**: $\vdash \sim\alpha \to \Box\sim\Box\alpha$.
Proof.
(1) $\vdash \sim\Box\alpha \to \Box\sim\Box\alpha$. (AST).
(2) $\vdash \Box\alpha \to \alpha$. (AR).
(3) $\vdash \sim\alpha \to \sim\Box\alpha$. (2)(P).
(4) $\vdash \sim\alpha \to \Box\sim\Box\alpha$. (3)(1)(P).

7.8. *Theorem.* In the system of **7.7**: $\vdash \Box\alpha \to \Box\Box\alpha$.
Proof.
(1) $\vdash \sim\Box\alpha \to \Box\sim\Box\alpha$. (AST).
(2) $\vdash \sim\Box\sim\Box\alpha \to \Box\alpha$. (1)(P).
(3) $\vdash \Box(\sim\Box\sim\Box\alpha \to \Box\alpha)$. (2)(NE).
(4) $\vdash \Box\sim\Box\sim\Box\alpha \to \Box\Box\alpha$. (3)(AD)(MP).

(5) $\vdash \sim \Box \sim \Box\alpha \rightarrow \Box \sim \Box \sim \Box\alpha$. (AST).
(6) $\vdash \sim \Box \sim \Box\alpha \rightarrow \Box\Box\alpha$. (5)(4)(P).
(7) $\vdash \Box \sim \Box\alpha \rightarrow \sim \Box\alpha$. (AR).
(8) $\vdash \Box\alpha \rightarrow \sim \Box \sim \Box\alpha$. (7)(P).
(9) $\vdash \Box\alpha \rightarrow \Box\Box\alpha$. (8)(6)(P).

7.9. *Theorem*. S5 and the system of **7.7** are equivalent, that is they yield the same set of theorems.
Proof. By **7.6**, **7.7** and **7.8**.

Remark. The system of **7.7** is the Gödel formulation of S5. It should be noticed that in **7.6** no use is made of **AR**, but that use is made of **AR** in **7.7** and **7.8**.

8. *Consistency*.

8.0. *Theorem-schema*. The formulas which are axioms by **A0**, **A1** or **A2** are X-tautologies.
Proof. The formulas in question are plain tautologies. So by **6.4** they are P-tautologies, and by **5.5** they are X-tautologies.

8.1. *Theorem-schema*. **MP** preserves X-tautologyhood.
Proof-schema. Let $\mathscr{T}_a$ be a table for $\alpha \rightarrow \beta$. If α and $\alpha \rightarrow \beta$ are X-tautologies then they are $\mathscr{T}_a$-X-tautologies and so α and $\alpha \rightarrow \beta$ are verified by all X-ecceptable rows of $\mathscr{T}_a$. By **1.6** this supposes that all X-acceptable rows of $\mathscr{T}_a$ verifie β. So β is a $\mathscr{T}_a$-X-tautology and by **5.4** an X-tautology.

8.2. *Theorem-schema*. The formulas which are axioms by AD are X-tautologies.
Proof-schema. Let $\mathscr{T}_a$ be a table for $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$. A row $\mathscr{R}_i$ of $\mathscr{T}_a$ will not falsifie this formula unless $\mathscr{R}_i$ verifies $\Box(\alpha \rightarrow \beta)$ and $\Box\alpha$ and falsifies $\Box\beta$. This is impossible if $\mathscr{R}_i$ is X-acceptable. Indeed if $\mathscr{R}_i$ is X-acceptable and verifies $\Box(\alpha \rightarrow \beta)$ and $\Box\alpha$, then by **2.7** in $\mathscr{T}_a$ every X-acceptable row $\mathscr{R}_{i'}$ which is X-controled by $\mathscr{R}_i$ verifies $\alpha \rightarrow \beta$ and α. But then by **1.6** every such row $\mathscr{R}_{i'}$ must verifie β and so by **2.7** $\mathscr{R}_i$ verifies $\Box\beta$.

8.3. *Theorem-schema.* **NE** preserves X-tautologyhood.
Proof-schema. Let $\mathscr{T}_a$ be a table for $\Box\alpha$. If α is an X-tautology then all X-acceptable rows of $\mathscr{T}_a$ verifie α. So for every X-acceptable row $\mathscr{R}_i$ of $\mathscr{T}_a$ all X-acceptable rows of $\mathscr{T}_a$ which are X-controled by $\mathscr{R}_i$ verifie α. So by **2.7** every X-acceptable row $\mathscr{R}_i$ of $\mathscr{T}_a$ verifies $\Box\alpha$.

8.4. *Theorem-schema.* The formulas which are axioms by AR are X-tautologies.
Proof-schema. Let $\mathscr{T}_a$ be a table for $\Box\alpha$. A row $\mathscr{R}_i$ of $\mathscr{T}_a$ will not falsifie $\Box\alpha \rightarrow \alpha$ unless $\mathscr{R}_i$ verifies $\Box\alpha$ and falsifies α. This is impossible if $\mathscr{R}_i$ is X-acceptable. Indeed if $\mathscr{R}_i$ is X-acceptable it is 0-X-acceptable and then by definition **2.2** if $\mathscr{R}_i$ verifies $\Box\alpha$, $\mathscr{R}_i$ verifies α.

8.5. *Theorem.* The formulas which are axioms by **AS** are B-tautologies and S5-tautologies.
Proof for B. Let $\mathscr{T}_a$ be a table for $\Box\sim\Box\alpha$. A row $\mathscr{R}_i$ of $\mathscr{T}_a$ will not falsifie $\sim\alpha \rightarrow \Box\sim\Box\alpha$ unless $\mathscr{R}_i$ falsifies α and $\Box\sim\Box\alpha$. This is impossible if $\mathscr{R}_i$ is B-acceptable. Indeed if $\mathscr{R}_i$ is B-acceptable and falsifies $\Box\sim\Box\alpha$, then by **2.7** there is a row $\mathscr{R}_{i'}$ in $\mathscr{T}_a$ such that $\mathscr{R}_{i'}$ is B-acceptable, $\mathscr{R}_i$ B-controls $\mathscr{R}_{i'}$ and $\mathscr{R}_{i'}$ falsifies $\sim\Box\alpha$ and so verifies $\Box\alpha$. By definition **2.1** $\mathscr{R}_{i'}$ C0 $\mathscr{R}_i$ and so $\mathscr{R}_i$ must verifie α.
For S5 the theorem follows by **5.5**.

8.6. *Theorem.* The formulas which are axioms by AT are S4-tautologies and S5-tautologies.
Proof for S4. Let $\mathscr{T}_a$ be a table for $\Box\Box\alpha$. A row $\mathscr{R}_i$ of $\mathscr{T}_a$ will not falsifie $\Box\alpha \rightarrow \Box\Box\alpha$ unless $\mathscr{R}_i$ verifies $\Box\alpha$ and falsifies $\Box\Box\alpha$. This is impossible if $\mathscr{R}_i$ is S4-acceptable. Indeed if $\mathscr{R}_i$ is S4-acceptable and falsifies $\Box\Box\alpha$, then by **2.7** there is a row $\mathscr{R}_{i'}$ in $\mathscr{T}_a$ such that $\mathscr{R}_{i'}$ is S4-acceptable, $\mathscr{R}_i$ S4-controls $\mathscr{R}_{i'}$ and $\mathscr{R}_{i'}$ falsifies $\Box\alpha$. By definition **2.1** $\mathscr{R}_i$ C1 $\mathscr{R}_{i'}$ and so $\mathscr{R}_i$ cannot verifie $\Box\alpha$.
For S5 the theorem follows by **5.5**.

8.7. *Theorem-schema.* All X-theorems are X-tautologies. Proof. For M: by **8.0** — **8.4**. For B: by **8.0** — **8.5**. For S4: by **8.0** — **8.4** and **8.6**. For S5: by **8.0** — **8.6**. By saying that the deductive systems M, B, S4 and S5 are consistent we mean that theorem **8.7** holds.

8.8. *Theorem-schema.* If $\mathscr{R}_i$ is an X-acceptable row of a table $\mathscr{T}_a$ and if α is the rf of $\mathscr{R}_i$, then $\sim\alpha$ is not an X-theorem. Proof-schema. By **6.6** we see that $\mathscr{R}_i$ falsifies $\sim\alpha$. So $\sim\alpha$ is falsified by an X-acceptable row in $\mathscr{T}_a$ and is not an X-tautology. By **8.7** $\sim\alpha$ is not an X-theorem.

9. *Completeness*

9.0. *Theorem.* Let $\mathscr{R}_i$ be a row of a table $\mathscr{T}_a$ and let ψ be the rf of $\mathscr{R}_i$. Then we have that, if $\mathscr{R}_i$ is not S5-acceptable, $\sim\psi$ is an S5-theorem.

Proof by induction on definition **2.2**.

a) If $\mathscr{R}_i$ is not 0-S5-acceptable, then there is a constituent $\Box\alpha$ in $\mathscr{T}_a$ such that $\mathscr{R}_i$ verifies $\Box\alpha$ and falsifies α. By **6.6**, **6.7** and **6.9** $\psi\rightarrow\Box\alpha$ and $\psi\rightarrow\sim\alpha$ are P-theorems and so S5-theorems. So we have in S5:

(1) $\vdash\psi\rightarrow(\Box\alpha\wedge\sim\alpha)$.

(2) $\vdash\sim(\Box\alpha\wedge\sim\alpha)$. (AR)(P).

(3) $\vdash\sim\psi$. (1)(2)(P).

b) If $\mathscr{R}_i$ is not n + 1-S5-acceptable and if condition (1) for n + 1-S5-acceptability does not hold, then $\mathscr{R}_i$ is not n-S5-acceptable. In this case we have by the hypothesis of the induction:

(1) $\vdash\sim\psi$.

c) If $\mathscr{R}_i$ is not n + 1-S5-acceptable although condition (1) for n + 1-S5-acceptability holds, then condition (2) for n + 1-S5-acceptability does not hold. So there is in $\mathscr{T}_a$ a constituent $\Box\alpha$ such that $\mathscr{R}_i$ falsifies $\Box\alpha$ and there is no row $\mathscr{R}_j$ in $\mathscr{T}_a$ such that $\mathscr{R}_j$ is n-S5-acceptable, $\mathscr{R}_i$ S5-controls $\mathscr{R}_j$ and $\mathscr{R}_j$ falsifies α.

So we have:

(1) $\vdash \psi \rightarrow \sim \Box \alpha$.

If there are in $\mathscr{T}_a$ no rows at all which falsifie α, then α is a P-tautology and so a P-theorem and an S5-theorem. So we have:

(2) $\vdash \alpha$.

(3) $\vdash \Box \alpha$. (2)(**NE**).

(4) $\vdash \sim \psi$. (1)(3)(P).

d) Let us suppose we have the same case as in c) except that some rows of $\mathscr{T}_a$ falsifie α. Let $\varkappa_1, \ldots, \varkappa_r$ be the rfs of all the r distinct rows of $\mathscr{T}_a$ which falsifie α. Then we have:

(1) $\vdash \psi \rightarrow \sim \Box \alpha$.

(2) $\vdash \sim \alpha \rightarrow (\varkappa_1 \vee \ldots \vee \varkappa_r)$.

If none of the rows of $\mathscr{T}_a$ which falsifie α are n-S5-acceptable, then by the hypothesis of the induction we have:

(3) $\vdash \sim \varkappa_1$; ... ; $\vdash \sim \varkappa_r$.

(4) $\vdash \sim (\varkappa_1 \vee \ldots \vee \varkappa_r)$. (3)(P).

(5) $\vdash \alpha$. (2)(4)(P).

(6) $\vdash \Box \alpha$. (5)(**NE**).

(7) $\vdash \sim \psi$. (1)(6)(P).

e) Let us suppose we have the same case as in d) except that some rows of $\mathscr{T}_a$ which falsifie α are n-S5-acceptable. Then these rows are not S5-controled by $\mathscr{R}_i$. S5-control of a row $\mathscr{R}_j$ by $\mathscr{R}_i$ implies the following four conditions: (1) $\mathscr{R}_i$ C0 $\mathscr{R}_j$, (2) $\mathscr{R}_j$ C0 $\mathscr{R}_i$, (3) $\mathscr{R}_i$ C1 $\mathscr{R}_j$ and (4) $\mathscr{R}_j$ C1 $\mathscr{R}_i$.

Let $\varkappa_1, \ldots, \varkappa_r$ be the rfs of all the r distinct rows of $\mathscr{T}_a$ which falsifie α and which are not n-S5-acceptable.

Let $\gamma_1, \ldots, \gamma_s$ be the rfs of all the s distinct rows of $\mathscr{T}_a$ which falsifie α, which are n-S5-acceptable and for which condition (1) for S5-control by $\mathscr{R}_i$ falls.

Let $\mu_1, \ldots, \mu_t$ be the rfs of all the t distinct rows of $\mathscr{T}_a$ which falsifie α, which are n-S5-acceptable and for which condition (2) for S5-control by $\mathscr{R}_i$ falls.

Let $\varrho_1, \ldots, \varrho_u$ be the rfs of all the u distinct rows of $\mathscr{T}_a$ which falsifie α, which are n-S5-acceptable and for which condition (3) for S5-control by $\mathscr{R}_i$ fails.

Let $\varphi_1, \ldots, \varphi_v$ be the rfs of all the v distinct rows of $\mathscr{T}_a$ which falsifie α, which are n-S5-acceptable and for which condition (4) for S5-control by $\mathscr{R}_i$ fails. Then we have:

(1) $\vdash \psi \rightarrow \sim \Box\alpha$.

(2) $\vdash \sim\alpha \rightarrow (\gamma_1 \vee \ldots \vee \gamma_s \vee \mu_1 \vee \ldots \vee \mu_t \vee \varrho_1 \vee \ldots \vee \varrho_u \vee \varphi_1 \vee \ldots \vee \varphi_v \vee \varkappa_1 \vee \ldots \vee \varkappa_r)$.

By the hypothesis of the induction and P we have:

(3) $\vdash \sim (\varkappa_1 \vee \ldots \vee \varkappa_r)$.

(4) $\vdash \sim\alpha \rightarrow (\gamma_1 \vee \ldots \vee \gamma_s \vee \mu_1 \vee \ldots \vee \mu_t \vee \varrho_1 \vee \ldots \vee \varrho_u \vee \varphi_1 \vee \ldots \vee \varphi_v)$. (2)(3)(P).

(We can have $r=0$ and then we have immediately (4). We can have $s=0$, or $t=0$, or $u=0$, or $v=0$, but by the hypothesis of the case $s+t+u+v>0$).

For all m such that $1 \leqslant m \leqslant s$, if $\mathscr{R}_m$ is the row of $\mathscr{T}_a$ which is represented by γ_m we have that there is a constituent $\Box\beta_m$ in $\mathscr{T}_a$ such that $\mathscr{R}_i$ verifies $\Box\beta_m$ and $\mathscr{R}_m$ falsifies β_m. So we have:

(5) $\vdash \psi \rightarrow \Box\beta_1$; ... ; $\vdash \psi \rightarrow \Box\beta_s$.

(6) $\vdash \gamma_1 \rightarrow \sim\beta_1$; ... ; $\vdash \gamma_s \rightarrow \sim\beta_s$.

For all m such that $1 \leqslant m \leqslant t$, if $\mathscr{R}_m$ is the row of $\mathscr{T}_a$ which is represented by μ_m, we have that there is a constituent $\Box\zeta_m$ in $\mathscr{T}_a$ such that $\mathscr{R}_m$ verifies $\Box\zeta_m$ and $\mathscr{R}_i$ falsifies ζ_m. So we have:

(7) $\vdash \psi \rightarrow \sim\zeta_1$; ... ; $\vdash \psi \rightarrow \sim\zeta_t$.

(8) $\vdash \mu_1 \rightarrow \Box\zeta_1$; ... ; $\vdash \mu_t \rightarrow \Box\zeta_t$.

(9) $\vdash \psi \rightarrow \Box \sim \Box\zeta_1$; ... ; $\vdash \psi \rightarrow \Box \sim \Box\zeta_t$. (7)(AS).

For all m such that $1 \leqslant m \leqslant u$, if $\mathscr{R}_m$ is the row of $\mathscr{T}_a$ which is representer by ϱ_m, we have that there is a constituent $\Box\vartheta_m$ in $\mathscr{T}_a$ such that $\mathscr{R}_i$ verifies $\Box\vartheta_m$ and $\mathscr{R}_m$ falsifies $\Box\vartheta_m$. So we have:

(10) $\vdash \psi \rightarrow \Box\vartheta_1$; ... ; $\vdash \psi \rightarrow \Box\vartheta_u$.

(11) $\vdash \varrho_1 \rightarrow \sim \Box\vartheta_1$; ... ; $\vdash \varrho_u \rightarrow \sim \Box\vartheta_u$.

(12) $\vdash \psi \rightarrow \Box\Box\vartheta_1$; ... ; $\vdash \psi \rightarrow \Box\Box\vartheta_u$. (10)(AT).

For all m such that $1 \leqslant m \leqslant v$, if $\mathscr{R}_m$ is the row of $\mathscr{T}_a$ which is represented by φ_m, we have that there is a constituent $\Box\lambda_m$ in $\mathscr{T}_a$ such $\mathscr{R}_m$ verifies $\Box\lambda_m$ and $\mathscr{R}_i$ falsifies $\Box\lambda_m$. So we have:

(13) $\vdash \psi \rightarrow \sim \Box\lambda_1$; ... ; $\vdash \psi \rightarrow \sim \Box\lambda_v$.

(14) $\vdash \varphi_1 \rightarrow \Box\lambda_1$; ... ; $\vdash \varphi_v \rightarrow \Box\lambda_v$.

(15) $\vdash \psi \rightarrow \Box \sim \Box\lambda_1$; ... ; $\vdash \psi \rightarrow \Box \sim_v \Box\lambda_v$. (13)(AST).

By (4), (6), (8), (11), (14) and P we have:

(16) $\vdash \sim\alpha \rightarrow (\sim\beta_1 \vee \ldots \vee \sim\beta_s \vee \Box\zeta_1 \vee \ldots \vee \Box\zeta_t \vee \sim \Box\vartheta_1 \vee \ldots \vee \sim \Box\vartheta_u \vee \Box\lambda_1 \vee \ldots \vee \Box\lambda_v)$.

(17) $\vdash (\beta_1 \wedge \ldots \wedge \beta_s \wedge \sim \Box\zeta_1 \wedge \ldots \wedge \sim \Box\zeta_t \wedge \Box\vartheta_1 \wedge \ldots \wedge \Box\vartheta_u \wedge \sim \Box\lambda_1 \wedge \ldots \wedge \sim \Box\lambda_v) \rightarrow \alpha$. (16)(P).
(18) $\vdash \Box(\beta_1 \wedge \ldots \wedge \beta_s \wedge \sim \Box\zeta_1 \wedge \ldots \wedge \sim \Box\zeta_t \wedge \Box\vartheta_1 \wedge \ldots \wedge \Box\vartheta_u \wedge \sim \Box\lambda_1 \wedge \ldots \wedge \sim \Box\lambda_v) \rightarrow \Box\alpha$. (17)(NE)(AD)(MP).
(19) $\vdash (\Box\beta_1 \wedge \ldots \wedge \Box\beta_s \wedge \Box \sim \Box\zeta_1 \wedge \ldots \wedge \Box \sim \Box\zeta_t \wedge \Box\Box\vartheta_1 \wedge \ldots \wedge \Box\Box\vartheta_u \wedge \Box \sim \Box\lambda_1 \wedge \ldots \wedge \Box \sim \Box\lambda_v) \rightarrow \Box\alpha$. (18)(7.5)(P).
By (5), (9), (12), (15) and P we have:
(20) $\vdash \psi \rightarrow \delta$ where δ is the antecedent of (19).
(21) $\vdash \psi \rightarrow \Box\alpha$. (20)(19)(P).
(22) $\vdash \sim\psi$. (1)(21)(P).

9.1 *Theorem.* Let $\mathscr{R}_i$ be a row of a table $\mathscr{T}_a$ and let ψ be the rf of $\mathscr{R}_i$. Then we have that, if $\mathscr{R}_i$ is not S4-acceptable, $\sim\psi$ is an S4-theorem.

Proof by replacing in the proof of **9.0** the prefix "S5" by the prefix "S4" and by introducing the following simplifications in case e).
S4-control of a row $\mathscr{R}_j$ by $\mathscr{R}_i$ implies the following two conditions: (1) $\mathscr{R}_i$ C0 $\mathscr{R}_j$ and (2) $\mathscr{R}_i$ C1 $\mathscr{R}_j$. Considerations about $\mu_1, \ldots, \mu_t, \varphi_1, \ldots, \varphi_v, \Box\zeta_1, \ldots, \Box\zeta_t$ and $\Box\lambda_1, \ldots, \Box\lambda_v$ are dropped and the schemata (2), (4) and (16) — (19) are shortened in accordance.
(7) — (9) are dropped and so no use is made of **AS**.
(13) — (15) are dropped and so no use is made of **AST**.

9.2. *Theorem.* Let $\mathscr{R}_i$ be a row of a table $\mathscr{T}_a$ and let ψ be the rf of $\mathscr{R}_i$. Then we have that, if $\mathscr{R}_i$ is not B-acceptable, $\sim\psi$ is a B-theorem.

Proof by replacing in the proof of **9.0** the prefix "S5" by the prefix "B" and by introducing the following simplifications in case e).
B-control of a row $\mathscr{R}_j$ by $\mathscr{R}_i$ implies the following two conditions: (1) $\mathscr{R}_i$ C0 $\mathscr{R}_j$ and (2) $\mathscr{R}_j$ C0 $\mathscr{R}_i$. Considerations about $\varrho_1, \ldots, \varrho_u, \varphi_1, \ldots, \varphi_v, \Box\vartheta_1, \ldots, \Box\vartheta_u$ and $\Box\lambda_1, \ldots, \Box\lambda_v$ are dropped and the schemata (2), (4), and (16) — (19) are shortened in accordance.
(10) — (12) are dropped and so no use is made of **AT**.
(13) — (15) are dropped and so no use is made of **AST**.

9.3. *Theorem.* Let $\mathcal{R}_i$ be a row of a table $\mathcal{T}_a$ and let ψ be the rf of $\mathcal{R}_i$. Then we have that, if $\mathcal{R}_i$ is not M-acceptable, $\sim\psi$ is an M-theorem.

Proof by replacing in the proof of 9.0 the prefix "S5" by the prefix "M" and by introducing the following simplifications in case e).

M-control of a row $\mathcal{R}_j$ by $\mathcal{R}_i$ implies the following unique condition: $\mathcal{R}_i$ C0 $\mathcal{R}_j$. Considerations about $\mu_1, \ldots, \mu_t, \varrho_1, \ldots, \varrho_u, \varphi_1, \ldots, \varphi_v, \Box\zeta_1, \ldots \Box\zeta_t, \Box\vartheta_1, \ldots, \Box\vartheta_u$ and $\Box\lambda_1, \ldots, \lambda_v$ are dropped and the schemata (2), (4) and (16) — (19) are shortened in accordance.

(7) — (9) are dropped and so no use is made of **AS**.

(10) — (12) are dropped and so no use is made of **AT**.

(13) — (15) are dropped and so no use is made of **AST**.

9.4. *Theorem-schema.* If α is an X-tautology then α is an X-theorem.

Proof-schema. Let $\mathcal{T}_a$ be a table for α. If there is no row in $\mathcal{T}_a$ which falsifies α, then α is a P-tautology and so a P-theorem and an X-theorem. If some rows of $\mathcal{T}_a$ falsifie α then these rows are not X-acceptable. Let $\psi_1, \ldots, \psi_r$ be the rfs of the r distinct rows of $\mathcal{T}_a$ which falsifie α. Then we have:

(1) $\vdash \sim\alpha \rightarrow (\psi_1 \vee \ldots \vee \psi_r)$. **(6.7)**.

(2) $\vdash \sim(\psi_1 \vee \ldots \vee \psi_r)$. (by **9.0**, **9.1**, **9.2** or **9.3** and by P).

(3) $\vdash \alpha$. (1)(2)(P).

By saying that the deductive systems M, B, S4 and S5 are complete we mean that theorem **9.4** holds.

10. *Models*

10.0. *Definition.* Let us denote by "$\mathcal{L}$" the language of modal propositional logic with negation, material implication and necessity as primitives.

A *model* $\mathcal{M}$ *for* $\mathcal{L}$ is a set of three elements namely (1) a not empty set, the elements of which are called "*worlds*", (2) a dyadic relation R defined on the worlds of $\mathcal{M}$ and (3) a dyadic function $\mathcal{V}$, which takes the variables of $\mathcal{L}$ as first arguments and the worlds of $\mathscr{M}$ as second arguments and which takes as values the letters "T" and "F".

We refer to models and to worlds by means of the letters "$\mathscr{M}$" and "$\mathscr{W}$" respectively, eventually followed by suitable indexes.

10.1. *Definition.* Let α be a formula of $\mathscr{L}$, let $\mathscr{M}_a$ be a model containing the relation R and the function $\mathscr{V}$, and let $\mathscr{W}_i$ be a world of $\mathscr{M}_a$. We define by induction on the construction of α under what conditions "$\mathscr{M}_a\mathscr{W}_i$ *verifie* α"["$\mathscr{M}_a\mathscr{W}_i$ *falsifie*" α"].
If α is a variable, $\mathscr{M}_a\mathscr{W}_i$ verifie α iff $\mathscr{V}$ applied to α and $\mathscr{W}_i$ takes "T" as value, and $\mathscr{M}_a\mathscr{W}_i$ falsifie α iff $\mathscr{V}$ applied to α and $\mathscr{W}_i$ takes "F" as value.
If α is of the form $\sim\beta$, $\mathscr{M}_a\mathscr{W}_i$ verifie α iff $\mathscr{M}_a\mathscr{W}_i$ falsifie β, and $\mathscr{M}_a\mathscr{W}_i$ falsifie α iff $\mathscr{M}_a\mathscr{W}_i$ verifie β.
If α is of the form $\beta \rightarrow \gamma$, $\mathscr{M}_a\mathscr{W}_i$ verifie α iff $\mathscr{M}_a\mathscr{W}_i$ falsifie β or verifie γ, and $\mathscr{M}_a\mathscr{W}_i$ falsifie α iff $\mathscr{M}_a\mathscr{W}_i$ verifie β and falsifie γ.
If α is of the form $\Box\beta$, $\mathscr{M}_a\mathscr{W}_i$ verifie α iff for every world $\mathscr{W}_j$ of $\mathscr{M}_a$ such that $\mathscr{W}_i \mathrm{R} \mathscr{W}_j$ we have that $\mathscr{M}_a\mathscr{W}_j$ verifie β, and $\mathscr{M}_a\mathscr{W}_i$ falsifie α iff for some world $\mathscr{W}_j$ of $\mathscr{M}_a$ such that $\mathscr{W}_i \mathrm{R} \mathscr{W}_j$ we have that $\mathscr{M}_a\mathscr{W}_j$ falsifie β.

10.2. *Definition.* Let α be a formula of $\mathscr{L}$, and let $\mathscr{M}_a$ be a model. Then we say that α is $\mathscr{M}_a$-*valid* iff for every world $\mathscr{W}_i$ of $\mathscr{M}_a$ we have that $\mathscr{M}_a\mathscr{W}_i$ verifie α.

10.3. *Definitions.* A model is an M-*model* iff its relation R is reflexive. A model is a B-*model* iff its relation R is reflexive and symmetrical. A model is an S4-*model* iff its relation R is reflexive and transitive. A model is an S5-*model* iff its relation R is reflexive, symmetrical and transitive.

10.4 *Definition-schema.* The formula α is X-*valid* iff for every X-model $\mathscr{M}_a$ the formula α is $\mathscr{M}_a$-valid.

11. *Truth-tables and models.*

11.0. *Definition-schema.* An X-*image* of a truth-table $\mathscr{T}_a$ is a set of three elements namely (1) a set, the elements of which are called "worlds", (2) a two-place relation R defined on the worlds in question and (3) a two-argument function $\mathscr{V}$, which takes the variables of $\mathscr{L}$ as first argument and the worlds in question as second argument and which takes as values the letters "T" and "F", the set of worlds, the relation R and the function $\mathscr{V}$ in question being such that there is a one-onerelation I between the X-acceptable rows of $\mathscr{T}_a$ and the worlds in question such that (1) if among the worlds in question $\mathscr{W}_i$ and $\mathscr{W}_j$ correspond by I respectively to the rows $\mathscr{R}_i$ and $\mathscr{R}_j$ of $\mathscr{T}_a$ then $\mathscr{W}_i R \mathscr{W}_j$ iff $\mathscr{R}_i$ X-controls $\mathscr{R}_j$ and (2) if among the worlds in question $\mathscr{W}_i$ corresponds by I to the row $\mathscr{R}_i$ of $\mathscr{T}_a$, then for every variable α in $\mathscr{T}_a$ we have that $\mathscr{V}$ applied to α and $\mathscr{W}_i$ takes "T" ("F")as value iff $\mathscr{R}_i$ verifies [falsifies] α.

11.1. *Theorem-schema.* For every table $\mathscr{T}_a$ the X-images of $\mathscr{T}_a$ are X-models.
Proof-schema. By **2.8** we see that the X-images of $\mathscr{T}_a$ are models, because the set of X-acceptable rows of $\mathscr{T}_a$ is not empty, and so the set of worlds in the X-images of $\mathscr{T}_a$ will not be empty. By **3.0, 3.1** and **3.2** we see that in every X-image of $\mathscr{T}_a$ the relation R will be reflexive,and will be symmetrical and/or transitive as needed to be a B-model, an S4-model or an S5-model.

11.2. *Theorem-schema.* Let $\mathscr{M}_a$ be an X-image of $\mathscr{T}_a$, let $\mathscr{W}_i$ be the world of $\mathscr{M}_a$ which corresponds by the one-one-relation I to the row $\mathscr{R}_i$ of $\mathscr{T}_a$, and let α be a formula such that $\mathscr{T}_a$ is a table for α. Then we have that $\mathscr{M}_a\mathscr{W}_i$ verifie (falsifie) α iff $\mathscr{R}_i$ verifies (falsifies) α.

Proof-schema. The proofs are by induction on the construction of α. If α is a variable, then the theorem holds by definitions **10.1** and **11.0.** If α is of the form $\sim\beta$, we have that by the hypothesis of the induction that $\mathscr{M}_a\mathscr{W}_i$ verifie (falsifie) β iff

$\mathscr{R}_i$ verifies (falsifies) β and from this the theorem follows for this case by definitions **1.6** and **10.1**. If α is of the form $\beta \rightarrow \gamma$, we have by the hypothesis of the induction that $\mathscr{M}_a \mathscr{W}_i$ verifie (falsifie) β iff $\mathscr{R}_i$ verifies (falsifies) β and that $\mathscr{M}_a \mathscr{W}_i$ verifie (falsifie) γ iff $\mathscr{R}_i$ verifies (falsifies) γ. From this the theorem follows for this case by definitions **1.6** and **10.1**. If α is of the form $\Box\beta$ and if $\mathscr{R}_i$ verifies $\Box\beta$, then by **2.7** we have that every X-acceptable row in $\mathscr{T}_a$ which is X-controled by $\mathscr{R}_i$ verifies β. By the definition of "X-image" we have that for every world $\mathscr{W}_j$ in $\mathscr{M}_a$ such that $\mathscr{W}_i \mathrm{R} \mathscr{W}_j$ there is in $\mathscr{T}_a$ an X-acceptable row $\mathscr{R}_j$ which corresponds by the one-one-relation I to $\mathscr{W}_j$, which is X-controled by $\mathscr{R}_i$, and so which verifies β. By the hypothesis of the induction we have that for every such world $\mathscr{W}_j$, $\mathscr{M}_a \mathscr{W}_j$ verifie β. By the definition **10.1** we have that $\mathscr{M}_a \mathscr{W}_i$ verifie $\Box\beta$. If α is of the form $\Box\beta$ and if $\mathscr{R}_i$ falsifies $\Box\beta$, then by **2.7** we have that there is an X-acceptable row $\mathscr{R}_j$ in $\mathscr{T}_a$ which is X-controled by $\mathscr{R}_i$ and which falsifies β. By the definition of "X-image" we have that in $\mathscr{M}_a$ there is a world $\mathscr{W}_j$ such that $\mathscr{W}_i \mathrm{R} \mathscr{W}_j$ and $\mathscr{W}_j$ corresponds by the one-one-relation I to $\mathscr{R}_j$. By the hypothesis of the induction we have that $\mathscr{M}_a \mathscr{W}_j$ falsifie β. By definition **10.1** we have that $\mathscr{M}_a \mathscr{W}_i$ falsifie $\Box\beta$.

11.3. *Theorem-schema.* If a formula α is X-valid then α is an X-tautology.
Proof-schema. If α is not an X-tautology then, if $\mathscr{T}_a$ is a table for α, there is in $\mathscr{T}_a$ an X-acceptable row $\mathscr{R}_i$ which falsifies α. Let $\mathscr{M}_a$ be an X-image of $\mathscr{T}_a$ and let $\mathscr{W}_i$ be the world in $\mathscr{M}_a$ which by the one-one-relation I corresponds to $\mathscr{R}_i$. Then by **11.2** we have that $\mathscr{M}_a \mathscr{W}_i$ falsifie α. So there is an X-model $\mathscr{M}_a$ such that α is not $\mathscr{M}_a$ valid. So α is not X-valid. The theorem then follows by contraposition.

11.4 *Definition.* Given a table $\mathscr{T}_a$, a model $\mathscr{M}_b$, and a world $\mathscr{W}_j$ of $\mathscr{M}_b$, we say that *the representing row of* $\mathscr{W}_j$ *in* $\mathscr{T}_a$ is the row $\mathscr{R}_j$ of $\mathscr{T}_a$ such that for every constituent α in $\mathscr{T}_a$, $\mathscr{R}_i$ verifies (falsifies) α iff $\mathscr{M}_b \mathscr{W}_j$ verifie (falsifie) α.

11.5. *Theorem.* Let $\mathscr{R}_i$ be the representing row in the table $\mathscr{T}_a$ of the world $\mathscr{W}_j$ in the model $\mathscr{M}_b$. Let α be a formula such

that $\mathscr{T}_a$ is a table for α. Then we have that $\mathscr{R}_i$ verifies (falsifies) α iff $\mathscr{M}_b\mathscr{W}_j$ verifie (falsifie) α.
Proof. By an induction on the construction of α like in **1.6.**

11.6. *Theorems.* Given the table $\mathscr{T}_a$ and the model $\mathscr{M}_b$, let $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ be respectively the representing rows in $\mathscr{T}_a$ of the worlds $\mathscr{W}_j$ and $\mathscr{W}_{j'}$ of $\mathscr{M}_b$. Then we have that (1) if $\mathscr{W}_j \, R \, \mathscr{W}_{j'}$ then $\mathscr{R}_i \, C0 \, \mathscr{R}_{i'}$ and (2) if $\mathscr{W}_j \, R \, \mathscr{W}_{j'}$ and if R is transitive then $\mathscr{R}_i \, C1 \, \mathscr{R}_{i'}$.
Proof for (1). For every constituent $\Box\alpha$ in $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ verifies $\Box\alpha$ then by **11.4** $\mathscr{M}_b\mathscr{W}_j$ verifie $\Box\alpha$. By **10.1** $\mathscr{M}_b\mathscr{W}_{j'}$ verifie α and by 11.5 it follows that $\mathscr{R}_{i'}$ verifies α. So $\mathscr{R}_i \, C0 \, \mathscr{R}_{i'}$.
Proof for (2). For every constituent $\Box\alpha$ in $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ verifies $\Box\alpha$ then by **11.4** $\mathscr{M}_b\mathscr{W}_j$ verifie $\Box\alpha$. By the transitivity of R, for every world $\mathscr{W}_{j''}$ in $\mathscr{M}_b$ such that $\mathscr{W}_{j'} R \mathscr{W}_{j''}$ we have that $\mathscr{W}_j \, R \, \mathscr{W}_{j''}$ and by **10.1**, for every such world $\mathscr{W}_{j''}$ we have that $\mathscr{M}_b\mathscr{W}_{j''}$ verifie α, and so that $\mathscr{M}_b\mathscr{W}_{j'}$ verifie $\Box\alpha$. By **11.4** it follows that $\mathscr{R}_{i'}$ verifies $\Box\alpha$ and so $\mathscr{R}_i \, C1 \, \mathscr{R}_{i'}$.

11.7. *Theorem-schema.* Given the table $\mathscr{T}_a$ and the model $\mathscr{M}_b$, let $\mathscr{R}_i$ and $\mathscr{R}_{i'}$ be respectively the representing rows in $\mathscr{T}_a$ of the worlds $\mathscr{W}_j$ and $\mathscr{W}_{j'}$ of $\mathscr{M}_b$. Then we have that if $\mathscr{M}_b$ is an X-model and if $\mathscr{W}_j \, R \, \mathscr{W}_{j'}$ then $\mathscr{R}_i$ X-controls $\mathscr{R}_{i'}$.
Proof for M. By **11.6** for any model and so for all M-models, if $\mathscr{W}_j \, R \, \mathscr{W}_{j'}$, then $\mathscr{R}_i \, C0 \, \mathscr{R}_{i'}$ and so $\mathscr{R}_i$ M-controls $\mathscr{R}_{i'}$.
Proof for B. By the symmetry of R if $\mathscr{W}_j \, R \, \mathscr{W}_{j'}$ then $\mathscr{W}_{j'} \, R \, \mathscr{W}_j$. It follows by 11.6 that if $\mathscr{W}_j \, R \, \mathscr{W}_{j'}$ then $\mathscr{R}_i \, C0 \, \mathscr{R}_{i'}$ and $\mathscr{R}_i \, C0 \, \mathscr{R}_{i}$, and so $\mathscr{R}_i$ B-controls $\mathscr{R}_{i'}$.
Proof for S4. If R is transitive then, by **11.6** we have that if $\mathscr{W}_j \, R \, \mathscr{W}_{j'}$ then $\mathscr{R}_i \, C0 \, \mathscr{R}_{i'}$ and $\mathscr{R}_i C1 \, \mathscr{R}_{i'}$ and so $\mathscr{R}_i$ S4-controls $\mathscr{R}_{i'}$.
Proof for S5. By the symmetry and the transitivity of R we have by **11.6** that if $\mathscr{W}_j \, R \, \mathscr{W}_{j'}$ then $\mathscr{R}_i \, C0 \, \mathscr{R}_{i'}$ and $\mathscr{R}_{i'} \, C0 \, \mathscr{R}_i$ and $\mathscr{R}_i \, C1 \, \mathscr{R}_{i}$, and $\mathscr{R}_{i}$, $C1 \, \mathscr{R}_{i}$, and so $\mathscr{R}_i$ S5-controls $\mathscr{R}_{i'}$.

11.8. *Theorem-schema.* Given the table $\mathscr{T}_a$ and the model $\mathscr{M}_b$, let $\mathscr{R}_i$ be the representing row in $\mathscr{T}_a$ of the world $\mathscr{W}_j$ in $\mathscr{M}_b$. Then we have that if $\mathscr{M}_b$ is an X-model then $\mathscr{R}_i$ is X-acceptable.

Proof-schema by an induction on definition 2.2. In X-models R is reflexive and so $\mathscr{W}_j \mathrm{R} \mathscr{W}_j$. By 11.6 we have $\mathscr{R}_i$ C0 $\mathscr{R}_i$ which amounts to the same as saying that $\mathscr{R}_i$ is 0-X-acceptable. By the hypothesis of the induction $\mathscr{R}_i$ is n-X-acceptable and so condition (1) for n + 1-X-acceptability holds for $\mathscr{R}_i$. For every constituent $\Box\alpha$ in $\mathscr{T}_a$ which is falsified by $\mathscr{R}_i$ we have by definition **11.4** that $\mathscr{M}_b\mathscr{W}_j$ falsifie $\Box\alpha$. By definition **10.1** there is in $\mathscr{M}_b$ a world $\mathscr{W}_{j'}$ such that $\mathscr{W}_j \mathrm{R} \mathscr{W}_{j'}$ and $\mathscr{W}_b\mathscr{W}_{j'}$ falsifie α. Let $\mathscr{R}_{i'}$ be the representing row in $\mathscr{T}_a$ of $\mathscr{W}_{j'}$. By **11.5** $\mathscr{R}_{i'}$ falsifies α, by the hypothesis of the induction $\mathscr{R}_{i'}$ is n-X-acceptable and by 11.7 $\mathscr{R}_i$ X-controls $\mathscr{R}_{i'}$. So condition (2) for n + 1-X-acceptability holds for $\mathscr{R}_i$.

11.9. *Theorem-schema.* If a formula α is an X-tautology then α is X-valid.
Proof-schema. If α is not X-valid then there is a model $\mathscr{M}_b$ and a world $\mathscr{W}_j$ in $\mathscr{M}_b$ such that $\mathscr{M}_b\mathscr{W}_j$ falsifie α. Let $\mathscr{T}_a$ be a table for α and let $\mathscr{R}_i$ be the representing row in $\mathscr{T}_a$ of the world $\mathscr{W}_j$ in $\mathscr{M}_b$. By **11.5** $\mathscr{R}_i$ falsifies α and by **11.8** $\mathscr{R}_i$ is X-acceptable. So α is not a $\mathscr{T}_a$-X-tautology and so not an X-tautology. The theorem then follows by contraposition.

12. *Not reflexive systems.*

12.0. In [7] considerations can be found about systems which can be called "*not reflexive*" because about the models for these systems it is not stipulated that the relation R should be reflexive. We shall call these systems "M°", "B°", "S4°", "S5°", "M'", "B'", "S4'" and "S5'". It will however appear that S5' is identical with S5. The whole preceding theory is easily adapted to these eight systems once the following basic adaptations are made.

12.1. *Definitions.* All models as defined in **10.0** are M°-*models*. Models where the relation R is symmetrical are B°-*models*. Models where the relation R is transitive are S4°-*models*. Models where the relation R is symmetrical and transitive are S5°-*models*.

12.2. *Definitions.* The *deductive systems* M°, B°, S4° and S5° contain the following axiom-schemata and deduction-rules respectively.

M°. The same as P plus **AD** and **NE**.

B°. The same as M° plus **AS**.

S4°. The same as M° plus **AT**.

S5°. The same as M° plus **AS** and **AT**.

Remark. In S5° we have $\vdash \sim \Box\alpha \rightarrow \Box \sim \Box\alpha$ because no use is made of **AR** in theorem **7.6**. But as in theorems **7.7** and **7.8** use is made of **AR**, S5° cannot be defined as containing the same postulates as M° plus **AST**.

12.3. *Definitions.* M°-*control,* B°-*control,* S4°-*control* and S5°-*control* are identical with M-control, B-control, S4-control and S5-control respectively.

12.4. *Definition-schema,* in which the prefix "X°" can be replaced by "M°", "B°", "S4°" or "S5°".

In every table $\mathscr{T}_a$ all rows are 0-X°-*acceptable*. A row $\mathscr{R}_i$ in $\mathscr{T}_a$ is n + 1-X°-acceptable iff (1) $\mathscr{R}_i$ is n-X°-acceptable and (2) for every constituent $\Box\alpha$ in $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ falsifies $\Box\alpha$ there is in $\mathscr{T}_a$ an n-X°-acceptable row $\mathscr{R}_j$ such that $\mathscr{R}_i$ X°-controls $\mathscr{R}_j$ and $\mathscr{R}_j$ falsifies α.

Remark. As we do not stipulate any more that for every 0-X°-acceptable row $\mathscr{R}_i$ we must have $\mathscr{R}_i$ C0 $\mathscr{R}_i$, we can not say anymore that, between X°-acceptable rows, C1 implies C0. So no use can be made here of the simplified definition of S4-control and S5-control considered in **2.9**.

12.5. *Definition.* A model $\mathscr{M}_a$ is an M'-model iff for every world $\mathscr{W}_i$ in $\mathscr{M}_a$ there is a world $\mathscr{W}_j$ in $\mathscr{M}_a$ such that $\mathscr{W}_i$ R $\mathscr{W}_j$. M'-models where the relation R is symmetrical are B'-models. M'-models where the relation R is transitive are S4'-models. M'-models where the relation R is symmetrical and transitive are S5'-models.

Remark. It is easily seen that in S5'-models R is reflexive and so S5'-models are S5-models. As in S5-models R is reflexive, S5-models are obviously S5'-models.

12.6. *Definitions.* We give ourselves the following axiom-schema:

AC. $\Box \sim \alpha \rightarrow \sim \Box\alpha$.

(the letter "C" is suggested by the word "control"). The deductive systems M', B', S4' and S5' contain the following axiom-schemata and deduction rules respectively.

M'. The same as P plus **AD, AC** and **NE.**

B'. The same as M' plus **AS.**

S4'. The same as M' plus **AT.**

S5'. The same as M' plus **AS** and **AT.**

Remarks. a) For the same reason as in **12.2,** S5' cannot be defined as containing the same postulates as M' plus AST.

b) In S5 we have $\vdash \Box \sim \alpha \rightarrow \sim \Box\alpha$.

In S5' we have $\vdash \Box\alpha \rightarrow \alpha$ as follows.

(1) $\vdash \sim \alpha \rightarrow \Box \sim \Box\alpha$. (**AS**).

(2) $\vdash \Box \sim \Box\alpha \rightarrow \sim \Box\Box\alpha$. (**AC**).

(3) $\vdash \sim \alpha \rightarrow \sim \Box\Box\alpha$. (1)(2)(P).

(4) $\vdash \Box\Box\alpha \rightarrow \alpha$. (3)(P).

(5) $\vdash \Box\alpha \rightarrow \Box\Box\alpha$. (**AT**).

(6) $\vdash \Box\alpha \rightarrow \alpha$. (5)(4)(P).

So S5 and S5' yield the same theorems.

12.7. *Definitions.* M'-*control,* B'-*control,* S4'-*control* and S5'-*control* are identical with M-control, B-control, S4-control and S5-control respectively.

12.8. *Definition-schema,* in which the prefix " X' " can be replaced by " M' ", " B' ", " S4' " or " S5' ".

In every table $\mathscr{T}_a$ all rows are 0-X'-*acceptable.*

A row $\mathscr{R}_i$ in $\mathscr{T}_a$ is n + 1-X'-*acceptable* iff (1) $\mathscr{R}_i$ is n-X'-acceptable, (2) for every constituent $\Box\alpha$ is $\mathscr{T}_a$ we have that if $\mathscr{R}_i$ falsifies $\Box\alpha$ there is in $\mathscr{T}_a$ an n-X'-acceptable row $\mathscr{R}_j$ such that $\mathscr{R}_i$ X'-controls $\mathscr{R}_j$ and $\mathscr{R}_j$ falsifies α and (3) in any case there is in $\mathscr{T}_a$ an n-X'-acceptable row $\mathscr{R}_j$ such that $\mathscr{R}_i$ X'-controls $\mathscr{R}_j$.

Remarks. a) for the same reason as in **12.4** S4'-control and S5'-control cannot receive the simplified definition considered in **2.9.**

b) By an induction on definition **2.2** one can prove that in every table $\mathscr{T}_a$ all S5'-acceptable rows are S5-acceptable. By an induction on definition **12.8** one can prove that in every table $\mathscr{T}_a$ all S5-acceptable rows are S5'-acceptable.

12.9. *Final remark.* The technique proposed in this paper can be called the technique of "*clean truth-tables*".

REFERENCES

[1] KURT GÖDEL, Eine Interpretation des intuitionistischen Aussagenkalküls, *Ergebnisse eines mathematischen Kolloquiums*, Heft 4, pp. 39-40 (1933).

[2] HENRY S. LEONARD, Two-valued truth-tables for modal functions, *Structure, method and meaning, Essays in honor of Henry M. Sheffer*, pp. 42-67, New-York, 1951.

[3] ALAN ROSS ANDERSON, Improved decision procedures for Lewis's calculus S4 and von Wright's calculus M, *J.S.L.*, Vol. 19 (1954), pp. 201-214.

[4] ALAN ROSS ANDERSON, Correction to a paper on modal logic, *J.S.L.*, Vol. 20 (1955), p. 150.

[5] WILLIAM H. HANSON, On some alledged decision procedures for S4, *J.S.L.*, Vol. 31 (1966) pp. 641-643.

[6] SAUL A. KRIPKE, Semantical analysis of modal logic I. Normal modal propositional calculi. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, Vol. 9, (1963), pp. 67-96.

[7] G. E. HUGHES and M. J. CRESSWELL, *An introduction to modal logic*, London, 1968.

[8] ALONZO CHURCH, *Introduction to mathematical logic, Vol. I.*, Princeton, 1956.