GRADES OF MODALITY

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1. Despite its title, this paper does not deal with what Quine discussed in his "Three Grades of Modal Involvement" [3]. It may, however, apply somewhat to the sort of 'gradualism' with regard to modal matters which Goodman suggested ([2], p. 7) and which Quine further recommended ([5], p. 67). I shall develop here a notion of modality according to which propositions can be distinguished by degrees or grades of necessity or possibility. Thus it will be possible to say of two necessarily true propositions that one is more necessary than the other, or to say, simply, that one proposition is more or less necessary.

At this time I shall only be concerned with the formal behavior of these modalities; I shall not now discuss possible applications of these notions of necessity. Accordingly, after specifying the grammar I expect these modalities to follow, I will examine them from both syntactical and semantical points of view. The calculuses proposed will be shown to be consistent and complete with respect to their suggested semantics. I will only be concerned here with the degrees of modality in propositional logic; I expect no new problems to arise when the systems are extended to include quantifiers that do not arise with any quantified modal logic.

2. Since we are investigating propositional logics, it will be presumed that they are cast in a language that meets the usual conditions for such systems; that it contains atomic formulas, p, q, r... etc., and the truth-functional connectives, conjunction (&) and negation ($\neg$), obeying the usual grammatical rules. (Other truth-functional connectives, e.g. disjunction (v) and material implication ($\supset$) are to be defined in terms of conjunction and negation in the familiar ways.) In addition, it is supposed that the language contains monadic modal operators, N_1, N_2, ..., N_i, ..., such that for each of these operators, N_i, N_iA

is a well formed formula if and only if A is. These operators represent the different degrees of necessity; a formula, N_iA, should be read as saying that A is necessary to, at least, the degree i. This should be interpreted in such a way that the higher the degree, the stricter the necessity. Thus, for example, the necessity of A is stronger when N_2A is true than when just N_1A is true. Given these modal operators N_i for necessity, one can define possibility operators M_i as usual, so that M_iA is equivalent to $\neg N_i \neg A$.

Within this basic framework one can construct various languages, each containing a diferent number of modal operators N_i. The standard modal logics, of course, contain just one such operator. It might be, however, that one would prefer to distinguish two grades of necessity, say, logical necessity and physical necessity (assuming that if a proposition is logically necessary then it must also be physically necessary), with the former being the stricter of the two. Possibly one would want to introduce a third and weaker sort of 'practical' necessity, for when one says, ordinarily, of an event that it *must* occur. For other purposes, it might be appropriate to have still a greater number of distinct degrees of necessity. It is, of course, possible to have a language with denumerably many modal operators; indeed, the systems I describe below will be formulated in a way to allow for this case.

A language of the kind just described will be called $\mathscr{L}_k$, where k is the number of operators for (different degrees of) necessity in the language.

3. Given a language $\mathscr{L}_k$ with its operators N_i $(1 \leqslant i \leqslant k)$ to be interpreted as degrees of modality in as ascending order of strictness, it is fairly clear what axioms and principles of inference should govern these concepts. Accordingly, we define a calculus in $\mathscr{L}_k$ by the following axiom schemata and rules:

A.0 a set of axioms which with modus ponens generates the classical propositional calculus.

A.1 $N_iA \supset N_jA$, for every i and j such that $1 \leqslant j \leqslant i \leqslant k$.

A.2 $N_i(A \supset B) \supset . N_iA \supset N_iB$, for every i, $1 \leqslant i \leqslant k$.

A.3 $N_iA \supset A$, for every i, $1 \leqslant i \leqslant k$.

R.1 modus ponens: from A and $A \supset B$, to infer B.

R.2 necessitation: if $\vdash A$, then $\vdash N_i A$, for every i, $1 \leqslant i \leqslant k$.

A.1 is introduced to express the ranking of the modalities according to degree, the higher the degree, the stricter the necessity. The other postulates and rules are all familiar from the standard modal logics. [1]

Obviously, each individual modality, N_i, in this system will by itself behave like the necessity defined by the system T or M of Gödel-Feys-von Wright. Hence, I will call this system T_k, k being the number of modal operators in the language $\mathscr{L}_k$.

Additional postulates may be added to define systems corresponding to S.4 and S.5. Thus, $S.4_k$ is the result of adding the axiom schemata:

A.4 $N_i A \supset N_i N_i A$, for every i, $1 \leqslant i \leqslant k$,

to T_k and $S.5_k$ is the result of adding:

A.5 $\neg N_i A \supset N_i \neg N_i A$, for every i, $1 \leqslant i \leqslant k$,

to T_k. Other systems, corresponding to other standard modal logics, may be defined in corresponding ways, but I shall not be concerned with them here.

It might be suggested that a stronger axiom than **A.2** should be postulated for these systems. One might propose in its place:

A.2' $N_i(A \supset B) \supset . N_j A \supset N_m$ where $m = \min(i, j)$.

This corresponds to the widely accepted principle that the conclusion of an inference (B) is at least as sure (necessary) as the weakest of its premisses (A and $A \supset B$). **A.2** would then follow as a special case of **A.2'** when $i = j$.

A.2' is, however, already derivable in T_k, given **A.2** and **A.1**. Thus, suppose that $j < i$, so that **A.2'** appears as $N_i(A \supset B) \supset . N_j A \supset N_j B$. By **A.1** $\vdash N_i(A \supset B) \supset . N_j(A \supset B)$ and by **A.2** $\vdash N_j (A \supset B) \supset . N_j A \supset N_j B$; hence $N_i(A \supset B) \supset . N_j A \supset N_j B$ is derivable

([1]) It is worth noting that the rule R.2 could be eliminated in favor of a principle to the effect that if a formula, A, is an axiom, then so is $N_i A$ an axiom for every i, $1 \leqslant i \leqslant k$. In proofs of later theorems it will often be convenient to thing of the systems formulated in this way, with modus ponens as the sole rule of inference.

by transitivity. Similarly, if $i<j$, so that **A.2'** is $N_i(A\supset B)\supset . N_jA \supset N_iB$; $N_jA\supset N_iA$ holds by **A.1**; $N_i(A\supset B)\supset . N_iA\supset N_iB$ (**A.2**) is equivalent to $N_iA\supset . N_i(A\supset B)\supset . N_iB$; so $\vdash N_jA\supset . N_i(A\supset B) \supset . N_iB$, which is equivalent to the desired **A.2'**.

But, by the same token, if **A.2'** be postulated as an axiom, not only is **A.2** derivable as a special case, but **A.1** is redundant, given **R.2**. Thus suppose that $i\geqslant j$; $N_j(A\supset A)\supset . N_iA\supset N_jA$ is then an instance of **A.2'**. Since $\vdash A\supset A$, $\vdash N_j(A\supset A)$ by **R.2**. Hence, $N_iA\supset N_jA$ is provable whenever $i\geqslant j$. Thus, the systems obtained from T_k ($S.4_k$, $S.5_k$) by replacing **A.1** and **A.2** by **A.2'** is equivalent to T_k ($S.4_k$, $S.5_k$).

4. In another paper [1], I showed how one could construct a general semantical theory for the standard modal logics in the manner of Kripke's semantics, but in such a way that it was not necessary to assume among the primitives of the semantics a relation, **R**, of relative possibility between possible worlds. Those methods will now be extended to provide an interpretation for the different modalities N_i in a language $\mathscr{L}_k$.

In [1] I suggested that we conceive a possible world, w_m, as defined by the ordered pair $<\mathbf{P}_m, \mathbf{B}_m>$, where $\mathbf{B}_m$ is a complete and consistent set of formulas (intuitively, the atomic formulas in $\mathbf{B}_m$ may be thought to be all the sentences asserting atomic truths of w_m) and $\mathbf{P}_m$ is a consistent, but not necessarily complete, set of formulas included in $\mathbf{B}_m$. $\mathbf{P}_m$ was called the set of 'fundamental postulates' governing w_m; any atomic formula in $\mathbf{P}_m$ was necessary in w_m, i.e., if $p\in\mathbf{P}_m$, then Np is true in w_m.

Since we now wish to distinguish grades of necessity, we must distinguish grades amongst the fundamental postulates. Thus, for a world, w_m, we should have different sets of formulas, $\mathbf{P}^1_m$, $\mathbf{P}^2_m$, ..., $\mathbf{P}^i_m$, ... etc., so that a formula in $\mathbf{P}^2_m$, for example, is 'fundamental to, at least, the degree 2'. We should then define evaluation rules for formulas in such a way that if an atomic formula, p, belongs to $\mathbf{P}^i_m$, then it is necessary to, at least, the degree i (i.e., N_ip is true in w_m).

Let us put this more precisely. A *normal world*, w_m, is an ordered set, π_m, of sets of formulas in $\mathscr{L}_k$, $\mathbf{P}^0_m$, $\mathbf{P}^1_m$, ..., $\mathbf{P}^i_m$, ...

$(1 \leqslant i \leqslant k)$, satisfying these conditions: (i) that as there are k many distinct modal operators N_i in $\mathscr{L}_k$, there are at least k many sets $\mathbf{P}^j_m$ $(j > 0)$ in π_m besides $\mathbf{P}^0_m$; (ii) that each set $\mathbf{P}^j_m$ in π_m is a consistent set of formulas (i.e. there is no formula in $\mathscr{L}_k$ such that both it and its denial belong to $\mathbf{P}^j_m$); (iii) moreover, that $\mathbf{P}^0_m$ is a complete set (i.e. for every formula in $\mathscr{L}_k$ either it or its denial belongs to $\mathbf{P}^0_m$); $\mathbf{P}^0_m$ thus plays the role played by $\mathbf{B}_m$ in the account of [1]; and (iv) that for each $\mathbf{P}^i_m$ and $\mathbf{P}^j_m$ in π_m, if $i \geqslant j$, then $\mathbf{P}^i_m \subseteq \mathbf{P}^j_m$ (thus all the $\mathbf{P}^i_m$ must be included in $\mathbf{P}^0_m$).

Given this notion of a 'possible world' it is now possible to define models by which formulas can be evaluated. As in [1] let a *model*, μ, be a pair $< w_0, \mathbf{W} >$, where $\mathbf{W}$ is a set of worlds as defined above and $w_0 \in \mathbf{W}$. For each model, $\mu = < w_0, \mathbf{W} >$, let the *evaluation function*, φ_μ, which maps pairs of formulas, A, and possible worlds, w_m, to truth values, be defined as follows:

(a) If A is an atomic formula, $\varphi_\mu (A, w_m) = T$ if and only if $A \in P^0_m$.

Suppose than φ_μ is defined for formulas B and C and all worlds w_m in $\mathbf{W}$.

(b) $\varphi_\mu(B \,\&\, C, w_m) = T$ if and only if $\varphi_\mu(B, w_m) = T$ and $\varphi_\mu (C, w_m) = T$.

(c)$\varphi_\mu(\neg B, w_m) = T$ if and only if $\varphi_\mu(B, w_m) \neq T$.

To determine truth values for necessitative formulas, N_iB, we define relations between the possible worlds in W, different relations being used in the evaluation of different modalities. Following the pattern of [1], let us define relations R^i_1 as follows:

$w_m \, R^i_1 \, w_n$ if and only if $\mathbf{P}^i_m \subseteq \mathbf{P}^0_n$ $(1 \leqslant i \leqslant k)$.

We can then give a comprehensive definition of φ_μ for all formulas of the sort N_iB:

(d) $\varphi_\mu(N_iB, wn) = T$ if and only if $\varphi_\mu(B, w_n) =$ for every w_n in $\mathbf{W}$ such that $w_m \, R^i_1 \, w_n$.

The clause (d) above defines a T-ish necessity, as will be shown; notice that all the relations R^i_1 are reflexive, but not necessarily transitive or symmetric. If an S.4 necessity is to be obtained, relations R^i which are transitive as well as reflexive must be defined. For an S.5 necessity relations which are symmetric, transitive and reflexive are required. These conditions are met by R^i_2 and R^i_3 respectively:

$w_m R^i_2 w_n$ if and only if $\mathbf{P}^j_m \subseteq \mathbf{P}^j_n$, for every j such that $i \leqslant j \leqslant k$.

$w_m R^i_3 w_n$ if and only if $\mathbf{P}^j_m = \mathbf{P}^j_n$, for every j such that $i \leqslant j \leqslant k$.

(Notice that the R^i_3 are included in the R^i_2 which are included in the R^i_1.) For the S.4 and S.5 kinds of necessity, (d) above should be modified by replacing 'R^i_1' in its statement by 'R^i_2' and 'R^i_3' respectively. In what follows I shall treat these three cases as one, letting 'φ' and correlatively 'N_i' be ambiguous, except as otherwise mentioned. The definitions of 'R^i_2' and 'R^i_3' may seem unduly complex. However, it is necessary to introduce talk of the sets $\mathbf{P}^j_m$ and $\mathbf{P}^j_n$ for *every* $j \geqslant i$, in order to insure the validity of A.1 as well as A.4 and A.5. Notice that the analogous condition for R^i_1, that $w_n R^i_1 w_n$ iff $\mathbf{P}^j_m \subseteq \mathbf{P}^0_n$, for every $j \geqslant i$, is already met since $\mathbf{P}^j_m \subseteq \mathbf{P}_{im}$ when $j \geqslant i$.

$\varphi_\mu(A, w_m) = T$' may be read as 'A is true in w_m on μ'. We say go on to say that A is true on $\mu(= \langle w_0, \mathbf{W} \rangle)$ iff $\varphi_\mu(A, w_0) = T$, since w_0 is supposed to be the actual world (for μ). Finally let us say that A is valid if and only if A is true on every model μ. These formulations are ambiguous, depending on which relations R^i are used in clause (d) of the definition of φ. To be accurate, we should speak of a formula's being T_k-valid, $S.4_k$-valid, or $S.5_k$-valid according as these relations are R^i_1, R^i_2 or R^i_3 respectively (and the models are defined for $\mathscr{L}_k$.)

5. We are now in a position to establish consistency and completeness theorems for the calculuses defined in section 3 relative to the semantics just given.

Theorem 1. If A is provable in T_k ($S.4_k$, $S.5_k$), then A is T_k ($S.4_k$, $S.5_k$)-valid.

I leave the proof of this to the reader; it is easily shown that all the axioms are valid and that the rules preserve this property.

Theorem 2. If A is T_k ($S.4_k$, $S.5_k$)-valid, then A is provable in T_k ($S.4_k$, $S.5_k$).

This may be established along the lines given in [1] utilizing a Henkin-type argument. We observe first that some quite general results hold for these systems:

Lemma 1. If X is any system containing the classical propositional calculus, and if $\neg A$ is not provable in X, then the system X' obtained from X by the addition of A as an axiom is consistent (i.e there is no formula B such that both B and $\neg B$ are provable in X').

Lemma 2. If X is any consistent system containing T_k ($S.4_k$, $S.5_k$), and if A is not provable in X, then there is a complete and consistent extension of X in which A is not provable.

We can now show that any non-theorem of these systems is falsifiable, which gives us Theorem 2.

Let K be any consistent extension of T_k ($S.4_k$, $S.5_k$) in which both R.1 and R.2 are admissible, and let L, M, N, ... etc. be complete and consistent extensions of K (lemma 2). For any such system M, let the *world determined by M*, $w_M = \pi_M = \langle \mathbf{P}^0_M, P^1_M, \ldots, \mathbf{P}^i_M \ldots \rangle$, where $\mathbf{P}^0_M$ is the set of all formulas, B, provable in M, and each $\mathbf{P}^i_M$ ($i > 0$) is the set of all formulas, B, such that $\vdash_M N_i B$. Clearly each set $\mathbf{P}^i_M$ ($i \geqslant 0$) is consistent, since M is consistent, and $\mathbf{P}^0_M$ is complete, since M is complete. There will be one set $\mathbf{P}^i_M$ for each operator, N_i, in the system, and moreover, if $i \geqslant j$, then $\mathbf{P}_M \subseteq \mathbf{P}^j_M$ by virtue of **A.1.** Hence, each each w_M so defined qualifies as a possible world in the sense of section **4**. Let us speak of a *model determined by K*, μ_K, as a pair $\langle w_L, \mathbf{W} \rangle$, where **W** is the set of all worlds determined by complete and consistent extensions of K, and w_L is a world determined by L, any complete and consistent extension of K. Given μ_K defined this way, we have:

Lemma 3. For all $w_M \in \mathbf{W}$, $\varphi_{\mu K}(B, w_M) = T$, if and only if $\vdash_M B$.

Proof is by induction on B. (a) If B is an atomic formula, then $\vdash_M B$ iff $B \in \mathbf{P}^0_M$ iff $\varphi_{\mu K}(B, w_M) = T$. Suppose then for the induction that the lemma holds for formulas C and D. (b) If B = C&D, $\vdash_M C \,\&\, D$ iff $\vdash_M C$ and $\vdash_M D$ iff $\varphi_{\mu K}(C, w_M) = T$ and $\varphi_{\mu K}(D, w_M) = T$ (inductive hypothesis) iff $\varphi_{\mu K}(C \,\&\, D, w_M) = T$. (c) If $B = \neg C$, then if $\neg C$ is provable in M, then not $\vdash_M C$ (by the consistency of M), so $\varphi_{\mu K}(C, w_M) \neq T$ by the inductive hypothesis; hence, $\varphi_{\mu K}(\neg C, w_M) = T$. But, if it is assumed that $\varphi_{\mu K}(\neg C, w_M) = T$, then $\varphi_{\mu K}(C, w_M) \neq T$, so not $\vdash_M C$. But if not $\vdash_M C$ then $\vdash_M \neg C$ (by the completeness of M). (d) If $B = N_i C$, $C \in \mathbf{P}^i_M$. Let N be any complete and consistent extension of K such that $w_M R^i w_N$. $w_N \in \mathbf{W}$. Since $P^i_M \subseteq P^0_N$, $C \in P^0_N$ which is to say $\vdash_N C$; so $\varphi_{\mu K}(C, w_N) = T$ inductive hypothesis). Hence, $\varphi_{\mu K}(N_i C, w_M) = T$.

Suppose that $\varphi_{\mu K}(N_i C, w_M) = T$, but that not $\vdash_M N_i C$. Let N be the system all of whose axioms are the formulas, A, such that $\vdash_M N_i A$. **R.1** is admissible in N, so C is not provable in N. N is a consistent extension of K, for if $\vdash_K A$, $\vdash_K N_i A$ by the admissibility of **R.2** in K, so $\vdash_M N_i A$ and $\vdash_N A$. N thus has a complete and consistent extension, N^*, in which C is not provable (lemma 2). $w_{N^*} \in \mathbf{W}$. $w_M R^i w_{N^*}$, for if (i) R^i is R^i_1 and K is from T_k, then $\mathbf{P}^i_M \subseteq \mathbf{P}^0_{N^*}$, since if $\vdash_M N_i A$ then $\vdash_{N^*} A$, or if (ii) R^i is R^i_2 and K is from $S.4_k$, then for any $j \geqslant i$, $\mathbf{P}^j_M \subseteq \mathbf{P}^j_{N^*}$, since if $A \in \mathbf{P}^j_M$ then $A \in \mathbf{P}^j_{N^*}$ as for (ii), and also if $A \in \mathbf{P}^j_{N^*}$, i.e. $\vdash_{N^*} N_j A$, then not $\vdash_{N^*} \neg N_j A$, by the consistency of N^*, so not $\vdash_M N_i \neg N_j A$, in which case not $\vdash_M N_j \neg N_j A$ by A.1. If that be so, then $\vdash_M \neg N_j \neg N_j A$ by the completeness of M, and so $\vdash_M N_j \neg N_j A$ by A.5. Thus $A \in \mathbf{P}^j_M$ and therefore $w_M R^i_3 w_{N^*}$. Now, given that $\varphi_{\mu K}(N_i C, w_M) = T$, $\varphi_{\mu K}(C, w_N) = T$ for every world w_N in $\mathbf{W}$ related by R^i to w_M. In particular, $\varphi_{\mu K}(C, w_{N^*}) = T$. But by the inductive hypothesis, this implies that $\vdash_{N^*} C$, contrary to the specification of N^*. Consequently, we must conclude that $\vdash_M N_i C$. This completes the proof of lemma 3.

Theorem 2 now follows immediately. For, if a formula, A, is not provable in T_k ($S.4_k$, $S.5_k$), there is a system L such that L is a complete and consistent extension of T_k ($S.4_k$, $S.5_k$) and

A is not provable in L (lemma 2). Let the model μ be $<w_L, \mathbf{W}>$ where $\mathbf{W}$ is the set of all worlds determined by complete and consistent extensions of T_k ($S.4_k$, $S.5_k$) and w_L is the world determined by the aforementioned L. By lemma 3, φ_μ (A, w_L) = T iff $\vdash_L A$; so φ_μ (A, w_L) $\neq$ T. The non-theorem A is thus falsified by this μ. Therefore, if A is T_k ($S.4_k$, $S.5_k$)-valid, it must be provable in T_k ($S.4_k$, $S.5_k$).

6. Thus far we have constructed equivalent syntactical and semantical frameworks in which true propositions can be distinguished according to the degrees of their necessity. We could say that A is more necessary than B if A's greatest degree of necessity is greater than the greatest necessity of B. That is to say, A is more necessary than B when N_iA and N_jB are true and $i > j$ and there is no m such that $m \geqslant i$ and N_mB is true. Nevertheless, not all (true) propositions belong to the spectrum of necessity. There remains a gulf between those propositions which a necessary to some degree, and those true but contingent propositions which are in no wise necessary. It would be desirable to arrange all (true) propositions along a single continuum of degrees of necessity.

Along these lines it would seen natural to introduce into the language of these systems another operator, N_0, which should satisfy the principle

A.3′ $A \equiv N_0A$.

This postulate would replace **A.3** in T_k ($S.4_k$, $S.5_k$), its being understood that all the other postulates of the system would now apply to N_0 as well as the other operators for necessity. Since N_0A would be equivalent to A, we could then say that any true proposition was necessary to, at least, the degree 0. This is not to say very much, to be sure; it does not commit one to saying that contingent propositions are necessary in any orthodox sense, but it does allow one to locate them within the spectrum of modality.

From the semantical point of view presented in section **4**, one would like to introduce a relation, $R^0{}_1$, such that

$w_m\, R^0{}_1\, w_n$ if and only if $\mathbf{P}^0{}_m \subseteq \mathbf{P}^0{}_n$

(similarly for R^0_2 and R^0_3) and then evaluate formulas of the form N_0C in the same manner as other formulas N_iC.

This will not do, however. The axiom **A.3**′ is not valid by this account. Those instances of **A.3**′ in which A contains no well formed parts of the form N_iB for $i > 0$ are valid, but if an N_i $(i > 0)$ does occur in A, the axiom might be falsifiable. Thus the formula $N_1p \supset N_0N_1p$ is falsified by a model $\mu = \langle w_0, \{w_0, w_1, w_2\} \rangle$, where $\mathbf{P}^0_0 = \mathbf{P}^0_1 = \{p, q, r, s, \ldots\}$; $\mathbf{P}^1_0 = \{p\}$; $\mathbf{P}^1_1 = \{q\}$; and $\mathbf{P}^0_2 = \{\neg p, q, \ldots\}$. This describes enough to show that $\varphi_\mu(N_1p, w_0) = T$ since $\varphi_\mu(p, w_0) = T$ and $\varphi_\mu(p, w_1) = T$ and w_0 and w_1 are the only worlds related by R^1 to w_0 for this μ), but $\varphi_\mu(N_0N_1p, w_0) \neq T$ (since $w_0 R^0 w_1$ but $\varphi_\mu(N_1p, w_1) \neq T$ as $W_1 R^1 w_2$ and $\varphi_\mu(p, w_2) \neq T$).

A.3′ fails in cases like this because distinct worlds w_m and w_n agree in all formulas in their initial sets, $\mathbf{P}^0_m$ and $\mathbf{P}^0_n$, but differ in some fundamental postulates, $\mathbf{P}^i_m$ and $\mathbf{P}^i_n$. N_0 as a 'truth operator' says, in effect, when evaluating formulas N_0A in w_m look at A only in the world w_m. R^0 as given above does not narrow one's vision enough; it allows one to shift one's view to some other w_n. Accordingly, let us instead stipulate that

$w_m R^0_1 w_n$ if and only if $w_m = w_n$.

A.3′ is valid on this account, as are all the other axioms for N_0. The proof of semantic completeness is unaffected by this addition to the semantics for T_k. Thus in the proof of lemma 3, case (d) (2nd part) in which B might have the form N_0C, we observe that for the systems M, N, and N* described, if $C \in \mathbf{P}^i_m$ $(i \geqslant 0)$, then $\vdash_M N_iC$, so by **A.3**′ $\vdash_M N_0N_iC$, in which case $\vdash_N N_iC$ and $\vdash_{N^*} N_iC$ and so $C \in \mathbf{P}^i_{N^*}$. Therefore, $\mathbf{P}^i_M \subseteq \mathbf{P}^i_{N^*}$ for every $i \geqslant 0$. Similarly, $\mathbf{P}^i_N \subseteq \mathbf{P}^i_M$; so $w_M = w_{N^*}$, and $w_M R^0_1 w_{N^*}$.

It is unfortunate that in the semantics for T_k as enriched by N_0 and **A.3**′ R^0_1 must be given special treatment, that it must be defined significantly differently than the other relations R^i_1. Perhaps one would expect this, however, as one notices that in T_k $N_0A \supset N_0N_0A$ and $\neg N_0A \supset N_0$—N_0A are both derivable. And there's **A.3**′ itself. R_2 must also be given special handling in the semantics for $S.4_k$ (extended). Thus let us also say:

$w_m R^0_2 w_n$ if and only if $w_m = w_n$.

For $S.5_k$, however, because of the restrictions already on all the relations R^i_3, one can define R^0_3 in the same way as these other relations:

7. At the opening of this paper I suggested that the account of modalities given here might have some application to the modal 'gradualism' of Goodman, Quine and others. I should now like to caution that suggestion somewhat. I believe some progress has been made here in developing the idea of a spectrum of modal degrees; we can explicate how one proposition, p, is *more necessary than* another, q, in terms of p's possessing a higher degree of necessity than q. Moreover, one could say simply that a proposition, p, is *more or less necessary* according as the degree of necessity on p is more or less high. Nevertheless, I doubt that the critics of modal logic would find these notions much more congenial than they found the more familiar ideas of (absolute) necessity and its correlative, analyticity. For arguments similar to those wrought against strict necessity and analyticity might be turned against the present graded modalities.

Thus, it might be argued against the absolute necessity as formalized by the familiar modal logics, that no clear sense can be made of this concept since there are no useful criteria which unequivocally locate propositions some as 'fundamental postulates' and so as necessary, others as true but contigent (as described in [1]). This problem need not arise within the present framework. All that is required is that one be able to say of a proposition, p, that it is more fundamental than another, q, or that p is more central to one's conceptual framework than q. p should then be located in a set $\mathbf{P}^i$ and q in $\mathbf{P}^j$ where $i > j$. It does not matter if this grants some degree of necessity to statements like 'there have been black dogs' as well as to those like '$2 + 2 = 4$', though one would expect it to be a lesser degree.

Some sort of relative centrality of propositions in conceptual frameworks does belong within the picture sketched by Quine in, e.g. [4]. Nevertheless, it is not clear that the behavioral criteria which Quine would allow to determine how central or how fundamental a statement is, would define the linear order among classes of statements which is required by the present semantical account. If not, then the theory of grades of modality proposed here would not provide a true explication of that sort of gradualism.

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