ON THE SYMMETRY OF MANY-VALUED LOGICAL SYSTEMS

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Every many-valued logical algebra consists of the mutual inclusion of two-valued systems any one of which could be taken as a whole expression of the universe, ignoring the others. But the functioning of the calculus as many-valued entails the simultaneous consideration of all given two-valued components, arranged according to arbitrarily assigned sharing of the domain between the two complementary classes of each pair of values.

The number of one-variable functions allowed by the many-valued system depends then upon the number of two-valued sub-systems; any one of these functions may be taken as affecting in a mode different from that in which it likewise affects the others, certain pair of complementary values which are then designated as extreme cases of universe dichotomy. The interpretation of the values and of the modal functions is, of course, arbitrary, and there is no particular reason to call 'truth' the function specially affecting the first designated value and its negative or complement; but it could be a convenient expedient to make a corresponding characterization between pairs of values and modal functions, obtaining as many one-variable functions as allowed by the algebra.

The foregoing necessitates that the many-valued system be symmetric; that is, its values should be even-numbered. Every odd-number system is to be taken as having in fact as many values as its number's (even) successor.

Let us examine as an illustration the classical three-valued matrix calculus of Łukasiewicz enriched by Tarski's contribution, as expounded in Lewis' and Langford's *Symbolic Logic*. A mathematical method of determining the value $\emptyset$, 1/2 or 1 of a function is based on the primitive operations of implication

(C) and negation (N), the other functions being defined in terms of those. The method is: $pCq=1$ for $p\leqslant q$, $pCq=1-p\nrightarrow q$ for $p>q$; $Np = 1-p$. (Here '$\emptyset$' is numerically zero, not to confuse it with the disjunction sign '0').

The rule whereby the negation function Np has the intermediate value 1/2 (or ?) when the value of its variable, p, is likewise 1/2 (or ?), as pertaining to an asymmetric (odd-numbered) system, destroys the powerful equivalence between the implication function pCq and the disjunction function Np0q.

According to the definition p0q.=: pCq.Cq, the value of p0q is the larger of the two values of the variables when these are different, and the same as theirs when they have equal value. This corresponds to the two-value algebra rules.

Now, the absence of equivalence between p0q and Np0q is shown in the following table:

p	q	Np	pCq	Np0q
1	1	∅	1	1
1	?	∅	?	?
1	∅	∅	∅	∅
?	1	?	1	1
?	?	?	1	?
?	∅	?	?	?
∅	1	1	1	1
∅	?	1	1	1
∅	∅	1	1	1

The difference, the only one, is for value ? for both p and q. In this case, pCq has value 1 whereas Np0q has value ?.

The conjunction function pAq being defined as N(Np0Nq), its values are, similarly as in the two-valued algebra, the lesser of the two values of the variables when these are different, and the same when they have equal value. Then, if the equivalence function is pEq, and pEq .=: pCq.A.qCp, the lack of interchangeability between pCq and Np0q is explicitly expressed in the following table — offered for the sake of completeness — where pCq.E.Np0q :=:. pCq.C.Np0q:A:Np0q.C.pCq.

p	q	Np	pCq	Np0q	pCq.C.Np0q	Np0q.C.pCq	pCq.E.Np0q
1	1	∅	1	1	1	1	1
1	?	∅	?	?	1	1	1
1	∅	∅	∅	∅	1	1	1
?	1	?	1	1	1	1	1
?	?	?	1	?	?	1	?
?	∅	?	?	?	1	1	1
∅	1	1	1	1	1	1	1
∅	?	1	1	1	1	1	1
∅	∅	1	1	1	1	1	1

While Np0q.C.pCq is a law of the system, this is not the case with pCq.C.Np0q and, consequently, neither with pCq.E.Np0q.

The non-matrix method proof may be effected by *reductio ab absurdum.* We have that pCq.A.qCp:=:pEq and also pCq and N(qCp), and we take the hypothesis pEq; then we have to have pCq, which we do, and qCp; but the latter is not the case since we have N(qCp); hence, N(pEq).

In an *ad hoc* matrix pCq would be a law by definition, and also N(qCp). Therefore qCp would be null, and consequently, pCq.A.qCp too. Hence, pEq would be null and, therefore, N(pEq) would be a law.

Or: there is no equivalence between the two considered functions.

The obstacle is found in the negation table:

p	Np
1	∅
?	?
∅	1

where the negation of ? is ? itself, when it should be a complementary expression.

Now, if we consider this system a four-valued calculus, its values are ∅, ?, —?, 1, where ? and —? (or 1/2 and —1/2) share the universe equitatively. In a 'regular' four-valued algebra, for example with values ∅, 1/4, 3/4, 1 there is no equal sharing (and no need of a negative sign) and its negation table is:

p	Np
1	Ø
3/4	1/4
1/4	3/4
Ø	1

In the four-valued system constructed from the three-valued one, the negation table is:

p	Np		p	Np
1	Ø		1	Ø
?	—?		1/2	—1/2
		or		
—?	?		—1/2	1/2
Ø	1		Ø	1

Here, as in any 'regular' even-valued algebra, including the two-valued calculus, ?0? (or 1/2 0 1/2) renders value ? (or 1/2), and —?0—? (or —1/2 0—1/2) renders value —? (or —1/2), while ?0—? (or 1/2 0 —1/2) renders value 1.

The description of p0q and pAq is amplified so that it covers all cases of many-valued systems as well as the two-valued algebra. The value of p0q or of pAq is the same as that of the variables when these have equal value. When the values of the variables are different, the following rules hold: The value of p0q is that of the logical sum of its variables, according to postulates, or —1 when one of the values of the variables is 1; or 1 when, neither being 1, numerically assigned fractional values add up to 1 or more; or whatever numerically assigned value corresponds to the addition of the number values when they add up to less than 1. The value of pAq is that of the logical product of its variables, according to postulates, or —Ø when one of the values of the variables is Ø; or, one being 1, then the value of the other; or Ø when the values of the variables are complementary; or Ø when the values of the variables, without being complementary, are disjoined (according to postulates); or the value of one variable when (according to postulates) it is totally included in the other; or whatever be the value of the logical product of the variables when (according to postulates) they overlap. (In the system here considered, ? and —? are, of course, complementary and both totally included in 1; as for ? in the three-valued calculus, it is, naturally, totally included in 1).

These descriptions take contradictory values as complementary classes within the universe; that is, every one of the pairs exhausts the domain by being all of them in mutual inclusion.

Two sample tables, of the three-valued calculus and the modified (four-valued) algebra, respectively, are:

p	Np	p0p	pAp	p0.Np	pA.Np
1	∅	1	1	1	∅
?	?	?	?	?	?
∅	1	∅	∅	1	∅

p	Np	p0p	pAp	p0.Np	pA.Np
1	∅	1	1	1	∅
?	—?	?	?	1	∅
—?	?	—?	—?	1	∅
∅	1	∅	∅	1	∅

If for a moment, to facilitate comparison, we attend only to the numerical or distributive character of the intermediate values ? and —?, and consider either of them as ?, the modified table would look like this:

p	Np	p0p	pAp	p0.Np	pA.Np
1	∅	1	1	1	∅
?	?	?	?	1	∅
∅	1	∅	∅	1	∅

But it is not only the distributive character of p as 1/2 and Np as 1/2 which intervenes, but also the complementary feature indicated by the negative sign, '—', which is now a logical, not a mathematical sign.

Now if we define pCq in terms of p0q and Np, so that pCq .=. Np0q, one table is:

p	q	Np	pCq	Np0q	p0q	pAq
1	1	∅	1	1	1	1
1	?	∅	?	?	1	?
1	—?	∅	—?	—?	1	—?
1	∅	∅	∅	∅	1	∅
?	1	—?	1	1	1	?
?	?	—?	1	1	?	?
?	—?	—?	—?	—?	1	∅
?	∅	—?	—?	—?	?	∅
—?	1	?	1	1	1	—?
—?	?	?	?	?	1	∅
—?	—?	?	1	1	—?	—?
—?	∅	?	?	?	—?	∅
∅	1	1	1	1	1	∅
∅	?	1	1	1	?	∅
∅	—?	1	1	1	—?	∅
∅	∅	1	1	1	∅	∅

If now we take the first table in this paper and modify it so that it represents a selection of the previous one, we have:

p	q	Np	pCq	Np0q
1	1	∅	1	1
1	?	∅	?	?
1	∅	∅	∅	∅
?	1	—?	1	1
?	?	—?	1	1
?	∅	—?	—?	—?
∅	1	1	1	1
∅	?	1	1	1
∅	∅	1	1	1

Comparing the fourth and fith columns, we get:

p	q	Np	Three-valued pCq	Four-valued pCq	Three-valued Np0q	Four-valued Np0q
1	1	Ø	1	1	1	1
1	?	Ø	?	?	?	?
1	Ø	Ø	Ø	Ø	Ø	Ø
?	1	—?	1	1	1	1
?	?	—?	1	1	?	1
?	Ø	—?	?	—?	?	—?
Ø	1	1	1	1	1	1
Ø	?	1	1	1	1	1
Ø	Ø	1	1	1	1	1

The function pCq in the four-valued system takes value —? instead of ? for ? and Ø (for p and q respectively) while Np0q takes 1 instead of ? for ? and ?, and takes —? instead of ? for ? and Ø. The equivalence between pCq and Np0q is established.

The difference between the two-valued and a many-valued system as modified in this paper is thus in the obvious fact that the latter may assign a value other than 1 or Ø to a formula, and it allows more one-variable functions. But their laws are mutually corresponding. Thus, it is said in Lewis' and Langford's *Symbolic Logic* that whereas the triadic relations pq .⊃. r and p .⊃. q⊃r are equivalent in the two-valued algebra, the corresponding formulas of the three-valued calculus, pAqCr and pC.qCr, are not; but in our four-valued system they are. Let us examine first the tables for the two-valued calculus:

pq .⊃. r

p	q	r	pq	—(pq)	pq .⊃. r (or —(pq) ∨ r)
1	1	1	1	Ø	1
1	1	Ø	1	Ø	Ø
1	Ø	1	Ø	1	1
1	Ø	Ø	Ø	1	1
Ø	1	1	Ø	1	1
Ø	1	Ø	Ø	1	1
Ø	Ø	1	Ø	1	1
Ø	Ø	Ø	Ø	1	1

p .⊃. q⊃r

p	q	r	—p	—q	q⊃r (or —qVr)	p. .⊃. q⊃r (or —pV(⊃r)
1	1	1	Ø	Ø	1	1
1	1	Ø	Ø	Ø	Ø	Ø
1	Ø	1	Ø	1	1	1
1	Ø	Ø	Ø	1	1	1
Ø	1	1	1	Ø	1	1
Ø	1	Ø	1	Ø	Ø	1
Ø	Ø	1	1	1	1	1
Ø	Ø	Ø	1	1	1	1

Now the corresponding matrices for the four-valued system:

pAq.Cr

p	q	r	pAq	N(pAq)	pAq.Cr (or N(pAq)0r)
1	1	1	1	Ø	1
1	1	?	1	Ø	?
1	1	—?	1	Ø	—?
1	1	Ø	1	Ø	Ø
1	?	1	?	—?	1
1	?	?	?	—?	1
1	?	—?	?	—?	—?
1	?	Ø	?	—?	—?
1	—?	1	—?	?	1
1	—?	?	—?	?	?
1	—?	—?	—?	?	1
1	—?	Ø	—?	?	?
1	Ø	1	Ø	1	1
1	Ø	?	Ø	1	1
1	Ø	—?	Ø	1	1
1	Ø	Ø	Ø	1	1
?	1	1	?	—?	1
?	1	?	?	—?	1
?	1	—?	?	—?	—?
?	1	Ø	?	—?	—?
?	?	1	?	—?	1
?	?	?	?	—?	1
?	?	—?	?	—?	—?
?	?	Ø	?	—?	—?
?	—?	1	Ø	1	1
?	—?	?	Ø	1	1
?	—?	—?	Ø	1	1
?	—?	Ø	Ø	1	1
?	Ø	1	Ø	1	1
?	Ø	?	Ø	1	1
?	Ø	—?	Ø	1	1
?	Ø	Ø	Ø	1	1

p	q	r	pAq	N(pAq)	pAq.Cr (or N(pAq)0r)
—?	1	1	—?	?	1
—?	1	?	—?	?	?
—?	1	—?	—?	?	1
—?	1	∅	—?	?	?
—?	?	1	∅	1	1
—?	?	?	∅	1	1
—?	?	—?	∅	1	1
—?	?	∅	∅	1	1
—?	—?	1	—?	?	1
—?	—?	?	—?	?	?
—?	—?	—?	—?	?	1
—?	—?	∅	—?	?	?
—?	∅	1	∅	1	1
—?	∅	?	∅	1	1
—?	∅	—?	∅	1	1
—?	∅	∅	∅	1	1
∅	1	1	∅	1	1
∅	1	?	∅	1	1
∅	1	—?	∅	1	1
∅	1	∅	∅	1	1
∅	?	1	∅	1	1
∅	?	?	∅	1	1
∅	?	—?	∅	1	1
∅	?	∅	∅	1	1
∅	—?	1	∅	1	1
∅	—?	?	∅	1	1
∅	—?	—?	∅	1	1
∅	—?	∅	∅	1	1
∅	∅	1	∅	1	1
∅	∅	?	∅	1	1
∅	∅	—?	∅	1	1
∅	∅	∅	∅	1	1

pC.qCr

p	q	r	Np	Nq	qCr (or Np0r)	pC.qCr (or Np0(qCr))
1	1	1	Ø	Ø	1	1
1	1	?	Ø	Ø	?	?
1	1	—?	Ø	Ø	—?	—?
1	1	Ø	Ø	Ø	Ø	Ø
1	?	1	Ø	—?	1	1
1	?	?	Ø	—?	1	1
1	?	—?	Ø	—?	—?	—?
1	?	Ø	Ø	—?	—?	—?
1	—?	1	Ø	?	1	1
1	—?	?	Ø	?	?	?
1	—?	—?	Ø	?	1	1
1	—?	Ø	Ø	?	?	?
1	Ø	1	Ø	1	1	1
1	Ø	?	Ø	1	1	1
1	Ø	—?	Ø	1	1	1
1	Ø	Ø	Ø	1	1	1
?	1	1	—?	Ø	1	1
?	1	?	—?	Ø	?	1
?	1	—?	—?	Ø	—?	—?
?	1	Ø	—?	Ø	Ø	—?
?	?	1	—?	—?	1	1
?	?	?	—?	—?	1	1
?	?	—?	—?	—?	—?	—?
?	?	Ø	—?	—?	—?	—?
?	—?	1	—?	?	1	1
?	—?	?	—?	?	?	1
?	—?	—?	—?	?	1	1
?	—?	Ø	—?	?	?	1
?	Ø	1	—?	1	1	1
?	Ø	?	—?	1	1	1
?	Ø	—?	—?	1	1	1
?	Ø	Ø	—?	1	1	1

p	q	r	Np	Nq	qCr (or Np0r)	pC.qCr (or Np0(qCr))
—?	1	1	?	Ø	1	1
—?	1	?	?	Ø	?	?
—?	1	—?	?	Ø	—?	1
—?	1	Ø	?	Ø	Ø	?
—?	?	1	?	—?	1	1
—?	?	?	?	—?	1	1
—?	?	—?	?	—?	—?	1
—?	?	Ø	?	—?	—?	1
—?	—?	1	?	?	1	1
—?	—?	?	?	?	?	?
—?	—?	—?	?	?	1	1
—?	—?	Ø	?	?	?	?
—?	Ø	1	?	1	1	1
—?	Ø	?	?	1	1	1
—?	Ø	—?	?	1	1	1
—?	Ø	Ø	?	1	1	1
Ø	1	1	1	Ø	1	1
Ø	1	?	1	Ø	?	1
Ø	1	—?	1	Ø	—?	1
Ø	1	Ø	1	Ø	Ø	1
Ø	?	1	1	—?	1	1
Ø	?	?	1	—?	1	1
Ø	?	—?	1	—?	—?	1
Ø	?	Ø	1	—?	—?	1
Ø	—?	1	1	?	1	1
Ø	—?	?	1	?	?	1
Ø	—?	—?	1	?	1	1
Ø	—?	Ø	1	?	?	1
Ø	Ø	1	1	1	1	1
Ø	Ø	?	1	1	1	1
Ø	Ø	—?	1	1	1	1
Ø	Ø	Ø	1	1	1	1

pAq.Cr is equivalent to pC.qCr, as can be noticed by inspection of their corresponding columns.

Likewise the principle of inference is the same for both calculuses, being in each a law of the system. In the two-valued algebra the two forms $(p(p \supset q)) \supset q$ and $p \supset ((p \supset q) \supset q)$ are equivalent, and in our four-valued system the corresponding functions pA.pCq:Cq and pC:pCq.Cq are also equivalent. Let us examine first the two-valued form:

(p(p⊃q))⊃q

p	q	—p	p⊃q (or —pVq)	p(p⊃q)	—(p(p⊃q))	(p(p⊃q))⊃q (or —(p(p⊃q))Vq)
1	1	Ø	1	1	Ø	1
1	Ø	Ø	Ø	Ø	1	1
Ø	1	1	1	Ø	1	1
Ø	Ø	1	1	Ø	1	1

p⊃((p⊃q)⊃q)

p	q	—p	p⊃q (or —pVq)	—(p⊃q)	(p⊃q)⊃q (or —(p⊃q)Vq)	p⊃((p⊃q)⊃q) (or —pV((p⊃q)⊃q)
1	1	Ø	1	Ø	1	1
1	Ø	Ø	Ø	1	1	1
Ø	1	1	1	Ø	1	1
Ø	Ø	1	1	Ø	Ø	1

Now for our four-valued system:

pA.pCq:Cq

p	q	Np	pCq (or Np0q)	pA.pCq	N(pA.pCq)	pA.pCq:Cq (or N(pA.pCq)0q)
1	1	Ø	1	1	Ø	1
1	?	Ø	?	?	—?	1
1	—?	Ø	—?	—?	?	1
1	Ø	Ø	Ø	Ø	1	1
?	1	—?	1	?	—?	1
?	?	—?	1	?	—?	1
?	—?	—?	—?	Ø	1	1
?	Ø	—?	—?	Ø	1	1
—?	1	?	1	—?	?	1
—?	?	?	?	Ø	1	1
—?	—?	?	1	—?	?	1
—?	Ø	?	?	Ø	1	1
Ø	1	1	1	Ø	1	1
Ø	?	1	1	Ø	1	1
Ø	—?	1	1	Ø	1	1
Ø	Ø	1	1	Ø	1	1

pC:pCq.Cq

p	q	Np	pCq (or Np0q)	N(pCq)	pCq.Cq (or N(pCq)0q)	pC:pCq.Cq (or Np0:pCq.Cq)
1	1	∅	1	∅	1	1
1	?	∅	?	—?	1	1
1	—?	∅	—?	?	1	1
1	∅	∅	∅	1	1	1
?	1	—?	1	∅	1	1
?	?	—?	1	∅	?	1
?	—?	—?	—?	?	1	1
?	∅	—?	—?	?	?	1
—?	1	?	1	∅	1	1
—?	?	?	?	—?	1	1
—?	—?	?	1	∅	—?	1
—?	∅	?	?	—?	—?	1
∅	1	1	1	∅	1	1
∅	?	1	1	∅	?	1
∅	—?	1	1	∅	—?	1
∅	∅	1	1	∅	∅	1

As an illustation let us prove that the other principles shown in the cited book as not holding in the three-valued system do hold in our four-valued calculus, making a parallel with their corresponding laws of the two-valued algebra.

One is Np.Cp:Cp. In the two-valued algebra we have $(—p \supset p) \supset p$:

p	—p	$—p \supset p$ (or pVp)	$—(—p \supset p)$	$(—p \supset p) \supset p$ (or $—(—p \supset p)$Vp)
1	∅	1	∅	1
∅	1	∅	1	1

In the four-valued system:

Np.Cp:Cp

p	Np	Np.Cp. (or p0p)	N(Np.Cp)	Np.Cp:Cp (or N(Np.Cp)0p)
1	∅	1	∅	1
?	—?	?	—?	1
—?	?	—?	?	1
∅	1	0	1	1

Another is p0.Np. We have pV—p, the Law of the Excluded Middle:

p	—p	pV—p
1	0	1
∅	1	1

Now for p0.Np

p	Np	p0.Np
1	∅	1
?	—?	1
—?	?	1
∅	1	1

A third one is pC.N(pC.Np), with its analogue p⊃—(p⊃—p):

p	—p	p⊃—p (or —pV—p)	—(p⊃—p)	p⊃—(p⊃—p) (or —pV—(p⊃—p))
1	∅	∅	1	1
∅	1	1	∅	1

For pC.N(pC.Np):

p	Np	pC.Np (or Np0Np)	N(pC.Np)	pC.N(pC.Np) (or Np0.N (pC.Np))
1	∅	∅	1	1
?	—?	—?	?	1
—?	?	?	—?	1
∅	1	1	∅	1

And a fourth is pC.Np:C.Np, or (p⊃—p)⊃—p:

	p⊃p	p⊃—p (or —pV —p)	—(p⊃—p)	(p⊃—p)⊃—p (or —(p⊃—p)V —p)
1	∅	∅	1	1
∅	1	1	∅	1

For pC.Np:C.Np:

p	Np	pC.Np	N(pC.Np)	pC.Np:C.Np
1	∅	∅	1	1
?	—?	—?	?	1
—?	?	?	—?	1
∅	1	1	∅	1

Let us explore now how the many-valued systems take up one-variable functions

It is obvious that a many-valued system, being richer in possibilities of value combinations than the minimum (two-valued) one and other many-valued systems of fewer values that may be, has possibly more one-variable and many-variable functions than its predecessors in an arrangement of growing sequence as to the number of values.

Now the designated values of a many-valued system that includes 1 and ∅ are these two values which explicitly embrace the universe in its entirety. (All other pairs of complementary values do not need to make ∅ a part of their formulation, as ∅ is always included. Thus, the complement of ? is —?; but if we want to introduce ∅ in the formulation, we will have to make —? $\not\leftarrow$ ∅ the complement of ?, and ? $\not\leftarrow$ ∅ the complement of —?. But —? $\not\leftarrow$ ∅ = —?, and ? $\not\leftarrow$ ∅ ≠ ?; therefore, the complement of ? is —?, and *vice versa*). (∅ is antidesignated.)

In a many-valued system, the relationships among values are given in the postulates of the system. In our four-valued form, the assumptions are necessary that ? is totally included in 1, and that —? is totally included in 1. In higher order algebras, the mutual relations are more complex. Let us study an (arbitrary) eight-valued system with 1 and ∅ as its designated values.

To facilitate comprehension (as well as for philosophical reasons whose explanation would be out of place in this paper), let us consider values and variables as classes, their functional (and propositional) interpretations fitting that of classes. The table functions are then taken as compound classes.

There is in this connection a formation rule: Whenever a class-variable is totally included in a class-value, even if the inclusion is not mutual, it is equated with the class-value. The logical reason for this is that, when the field is restricted to the values, there is no room for other classes, and whatever foreign element we introduce is spread over the whole class-value to which it belongs. In case the class-variable covers parts of two complementary class-values, we must recognize that it obviously is coincident with the more restricted class-value in which it is totally included. This reference to 'more restricted' means that if a class-variable is (totally) included in a class-value that is in turn (totally) included in another class-value, the class-variable is equated with the included class-value, not with the including one, unless it be formulated that the class-variable is included in both. Conversely, a class-variable totally included (according to the formulas) in a class-value which totally includes another class-value, is not necessarily (and not at all formally) included in the included class-vaule, and is therefore equated with the including class-value, not with the included one, unless it be formulated, again, that it is included in both.

When this is the case, namely that a class-variable is included in two class-values of which one is included in the other, the class-variable is coincident with each as we consider the values severally, but is equated with the largest, including class-value when the whole picture is considered. Thus, the complement of a class-variable is the class-value that is the complement of the most comprehensive class-value in which the class-variable is explicitly included according to the formulas, although when the pertinent class-values are severally considered, the complement is, in each case, the complementary class-value.

The natural repugnance apt to be found before the fact that a class seems a proper subclass of itself in our strongly intensional account, may be eased by the interpretation that the subclass represents the class in the included class-value, being a *mapping* of the whole class-variable, a mapping entailing an infinite regress.

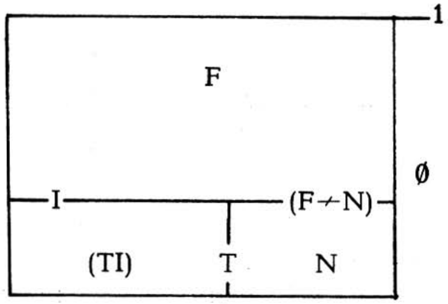
The inclusion of a class-variable in each class-value, or the equation of a class-variable with the class-values, determines then in each case the class-function (a qualified class-variable), and the assignment of a value to the function. This assignment is then interpreted in the tables as an equation between the assigned value and the function, and therefore, as the total inclusion (and spread) of the class-function in the (assigned) class-value.

Belonging of a class-variable or class-function to a class-value means inclusion. And also membership, the class-value being impure. Thus a (say) *true* class-function has something true as its membership, which is then part of the membership of the class-value 'truth', the class-function being therefore included in the class-value. But the class-function, having a *true* membership, is in itself also true, figuring likewise as a member in the class-value, the class-values being members of themselves.

One of the first postulates of the system is that the explicit equation $A=A$ represents the universe class, 1. The one-variable functions are made up by equations of the variable with some value, and their respective complements.

In our sample eight-valued system, the values, by complementary pairs, are 1, $\emptyset$; T, F; N, I; $F \not- N$, TI, of which the last two are compound in order to exhibit their make-ups. All these need not be interpreted, but it might help to imagine them respectively as possibility and self-contradiction; truth and possible falsehood; necessity and possible non-necessity; either possible falsehood or necessity, and both truth and possible non-necessity.

The postulates include those of the class calculus, and the relevant ones for us now are: $T.1 = T$, $F.1 = F$, $N.1 = N$, $I.1 = I$, $(F \not- N)1 = F \not- N$, $(TI)1 = TI$; $1.\emptyset = \emptyset$, $TF = \emptyset$, $NI = \emptyset$, $(F \not- N)(TI) = \emptyset$, $TN = N$, $(TI)N = \emptyset$, $FI = F$. Three obvious theorems would be: $(TI)T = TI$, $(TI)I = TI$, $FN = \emptyset$. In other words, N is totally included in T, (TI) is totally included in T but excluding N, I is the logical sum of (TI) and F, and, obviously, $F \not- N$ is the sum of F and N, as shown in the following Venn diagram:



In a two-valued system, a one-variable function $-(A=\emptyset)$ would be equivalent to function $A=1$. Let us call the former A_p, and the latter A_x. The table for these functions and their respective complements is:

		A_p	$-A_p$	A_x	$-A_x$
A	$-A$	$-(A=\emptyset)$	$(A=\emptyset)$	$(A=1)$	$-(A=1)$
1	∅	1	∅	1	∅
∅	1	∅	1	∅	1

In a many-valued-system, say four-valued, the two pairs of functions would be different as to the assigned values corresponding to the values that do not enter into the make-up of the functions (we are going to mark with horizontal lines the repetitive blocks of assigned values as the range of successive functions is expanded):

		A_p	$-A_p$	A_x	$-A_x$
A	$-A$	$-(A=\emptyset)$	$(A=\emptyset)$	$(A=1)$	$-(A=1)$
1	∅	1	∅	1	∅
∅	1	∅	1	∅	1
?	—?	1	∅	∅	1
—?	?	1	∅	∅	1

And here we could introduce another one-variable function, say A_y or $(A=1) \not- (A=?)$, and its complement, $-A_y$ or $-(A=1)$ $-(A=?)$:

		A_p	$—A_p$	A_x	$—A_x$	A_y	$—A_y$
A	—A	—(A=∅)	(A=∅)	(A=1)	—(A=1)	(A=1) $\not-$ (A=?)	—A(=1)—(A=?)
1	∅	1	∅	1	∅	1	∅
∅	1	∅	1	∅	1	∅	1
?	—?	1	∅	∅	1	1	∅
—?	?	1	∅	∅	1	∅	1

Again, A_y and $—A_y$ differ from the others by the assigned values corresponding to ? and —?.

In a six-valued system the table for those functions would be:

		A_p	$—A_p$	A_x	$—A_x$	A_y	$—A_y$
A	—A	—(A=∅)	(A=∅)	(A=1)	—(A=1)	(A=1) $\not-$ (A=?)	—A(=1)—(A=?)
1	∅	1	∅	1	∅	1	∅
∅	1	∅	1	∅	1	∅	1
?	—?	1	∅	∅	1	1	∅
—?	?	1	∅	∅	1	∅	1
#	—#	1	∅	∅	1	∅	1
—#	#	1	∅	∅	1	∅	1

But here again, there is room for yet another one-variable function (and its complement), A_z (and $—A_z$), which differs from A_y (and $—A_y$) as to be assigned values corresponding to the new pair of values (A_z is (A=1 $\not-$ (A=?) $\not-$ (A=#), and $—A_z$ is —(A=1)—(A=?)—(A=#):

A	—A	A_p	$—A_p$	A_x	$—A_x$	A_y	$—A_y$	A_z	$—A_z$
1	∅	1	∅	1	∅	1	∅	1	∅
∅	1	∅	1	∅	1	∅	1	∅	1
?	—?	1	∅	∅	∅	1	∅	1	∅
—?	?	1	∅	∅	∅	∅	1	∅	1
#	—#	1	∅	∅	∅	∅	1	1	∅
—#	#	1	∅	∅	∅	∅	1	∅	1

In our eight-valued sample system, making the notation of functions to conform with that of the values as to the restrictest one, the definitions would be:

$$A_p = \mathrm{df} -(A = \emptyset)$$
$$-A_p = \mathrm{df}\ (A = \emptyset)$$
$$A_x = \mathrm{df}\ (A = 1)$$
$$-A_x = \mathrm{df} -(A = 1)$$
$$A_t = \mathrm{df}\ (A = 1) \not{-} (A = T)$$
$$-A_t = \mathrm{df} -(A = 1) -(A = T)$$
$$A_n = \mathrm{df}\ (A = 1) \not{-} (A = T) \not{-} (A = N)$$
$$-A_n = \mathrm{df} -(A = 1) -(A = T) -(A = N)$$

Table:

A	$-A$	A_p	$-A_p$	A_x	$-A_x$	A_x	$-A_x$	A_n	$-A_n$
1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$
$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
T	F	1	$\emptyset$	$\emptyset$	1	1	$\emptyset$	1	$\emptyset$
F	T	1	$\emptyset$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
N	I	1	$\emptyset$	$\emptyset$	1	$\emptyset$	1	1	$\emptyset$
I	N	1	$\emptyset$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$F \not{-} N$	TI	1	$\emptyset$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
TI	$F \not{-} N$	1	$\emptyset$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1

The functions may be respectively interpreted as 'A is possible', 'A is self-contradictory'; 'A is the universe', 'A is not the universe'; 'A is true (and possible)', 'A is false (and not the universe)'; 'A is necessary (and true and possible)', 'A is non-necessary (and not true and not the universe)'.

We still may introduce the function pair A_{f+n} and $-A_{f+n}$, or respectively, $(A=1) \not{-} (A=T) \not{-} (A=N)\ (A=(F \not{-} N))$ and $-(A=1)-(A=T)-(A=N)-(A=(F \not{-} N))$, and obtain:

A	$-A$	A_p	$-A_p$	A_x	$-A_x$	A_t	$-A_t$	A_n	$-A_n$	A_{f+n}	$-A_{f+n}$
1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$
$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
T	F	1	$\emptyset$	$\emptyset$	1	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$
F	T	1	$\emptyset$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
N	I	1	$\emptyset$	$\emptyset$	1	$\emptyset$	1	1	$\emptyset$	1	$\emptyset$
I	N	1	$\emptyset$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$F \not{-} N$	TI	1	$\emptyset$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	1	$\emptyset$
TI	$F \not{-} N$	1	$\emptyset$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1

And, of course, we may introduce other one-variable functions until the combinatory possibilities of the system are exhausted, taking into account that the terms of each pair of complementary functions follows the commutative property. For example, another important function is $(A=1) \not- (A=\emptyset)$, and $—(A=1)—(A=\emptyset)$, of which the assigned values would be, in our table:

1	$\emptyset$
1	$\emptyset$
$\emptyset$	1
$\emptyset$	1
$\emptyset$	1
$\emptyset$	1
$\emptyset$	1
$\emptyset$	1

On account of the complementary property of the values, the two-variable function $A \not- —A$ is a tautology, whereas $A—A$ is an inconsistency. The one-variable function $A=A$ and its complement $—(A=A)$ are also, respectively, a tautology and an inconsistency.

The tables for many-variable functions are, of course, highly complex. While such a function as $A \not- B$ is neither a tautology nor an inconsistency (the two designated values appearing — as in the included two-valued calculus —, and some assigned values corresponding to non-designated ones), a function like $(AB=A)=(A \not- B=B)$ is a tautology, its complement being, naturally, an inconsistency. It is worth noting that the same is valid for the two-valued system.

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Author's note: The interpretation set forth in this paper attends to the relationships among values, wherein negation is taken as complementarity. This approach does not preclude propositional interpretations — which may be even quasi-truth-functional — whereby negation is taken as contradiction. It is intended te provide a basis framework of truth-functional value relations upon a dichotomous principle.