ON THE AXIOMS OF CHOICE AND REGULARITY

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0. *Introduction*

In § 1 a problem is posed which suggests a need for a very strong axiom of choice or some extra axiom of set theory. The Axiom of Regularity (REG) is the extra axiom that is usually used. In § 2 some implications of REG are given. In terms of a general definition of "set" given in § 3 it is shown in § 4 that REG seems to be true for classical mathematical structures. But the set theoretic mode of thought and modern mathematics are not limited to classical structures. In § 5 evidence against REG is given by introducing a new construction process which can be used to construct (reasonable ?) sets contradicting REG. The construction process of § 5 is clearly open to question and is still only at the intuitive level. But the main point of the article is not to prove REG false, but to show (as is done in § 5) that REG is a questionable rather than an obviously true axiom of set theory. So in § 6 a neutral viewpoint is adopted neither accepting REG nor its negation. However some limitation type axiom is necessary in order to eliminate such pathologies as $\in$-cycles like $x_1 \in x_2 \in \ldots \in x_n \in x_1$. An Axiom of Limitation (LIM) is introduced and shown to be equivalent to a "no$\in$-cycles axiom". Also LIM and REG are shown to be related to each other in a very natural way with LIM being a very weak and REG being a very strong limitation type axiom. Also since REG is no longer available to resolve the difficulty raised in § 1, a sufficiently strong axiom of choice (AC*) is introduced to do the job.

1. *A motivating problem*

Let NBG denote Von-Neumann-Bernays-Gödel set theory (specifically axioms A, B, C of [4]). To NBG we need to add some form of the axiom of choice (abbrev.: AC) in order to do mathematics as currently practiced. Let us use the following class form.

AC: For every class X of pairwise disjoint sets, there exists a choice class C which has exactly one point in common with each non-empty $x \in X$: in symbols,

$$(\exists C)(\forall x)_X[x \neq \emptyset \Rightarrow \exists u[C \cap x = \{u\}]].$$

If some of the classes $X_1, \ldots, X_n$ are proper, then we cannot form (in NBG) a collection having $X_1, \ldots, X_n$ as elements; however, we can intuitively think of such a collection. What if we have an intuitive collection $\{X_\lambda\}_{\lambda \in D}$ of pairwise disjoint classes ? Intuitively there exists a choice class C which has exactly one point in common with each X_λ Since some of the X_λ could be proper, clearly AC is not strong enough to justify the existence of such a choice class. Let us first discuss how such a situation arises very naturally.

We will define an *ordered* set to be any ordered pair $<x, r>$ of sets such that $r \subseteq x \times x$. We think of x as (partly) ordered by r and write "urv" for "$<u, v> \in r$".

For example, x could be quasi, partially, totally, or well-ordered by r. The case when $r = \emptyset$ corresponds to considering x as totally unordered, i.e., considering x abstracted from any ordering on it such as is done when we consider the cardinality of the set x.

Let us define two ordered sets $<x, r>$ and $<y, s>$as being *similar* or *order isomorphic* (abbrev.: $<x, r> \simeq <y, s>$) iff there is a bijective $f: x \to y$ which preserves ordering, i.e.,

$$urv \Leftrightarrow f(u)\, s\, f(v) \quad \text{for all } u, v \in x.$$

The equivalence classes with respect to the equivalence relation "$\simeq$" will be called *similarity classes*. Note that in the case when $r = s = \emptyset$ the notion of order isomorphism reduces to that of 1:1 correspondence as used in cardinality arguments.

THEOREM 1. If $<x, r>$ is an ordered set with $x \neq 0$, then the similarity class containing $<x, r>$ is a proper class: in symbols,

$$\{<y, s> \mid <x, r> \simeq <y, s>\} \notin V.$$

PROOF. Choose some element $u_0 \in x$. Then for any $u \notin x$ let x_u be the set obtained from x by replacing u_0 by u. Also let r_u denote the relation obtained from r by replacing all pairs $<u_0, v>$, $<v, u_0>$, or $<u_0, u_0> \in r$ by the corresponding new pairs $<u, v>$, $<v, u>$, or $<u, u>$ respectively. Then $<x, r> \simeq <x_u, r_u>$. Clearly F, defined on V-x by $F(u) = <x_u, r_u>$, is a 1:1 function on the proper class V-x, hence

$$\mathscr{R}(F) = \{F(u) \mid u \in V\text{-}x\} = \{<x_u, r_u> \mid u \in V\text{-}x\}$$

is a proper class. But

$$\{<y, s> \mid <x, r> \simeq <y, s>\} \supseteq \mathscr{R}(F),$$

hence is also a proper class, as claimed. QED.

Consider now the intuitive collection $\mathscr{X}$ of all similarity classes of well-ordered sets. We just showed that each such similarity class (with the one exception $\{<\emptyset, \emptyset>\}$) is a proper class. Thus we can't use AC to prove the existence of an NBG class C which has exactly one point in common with each $X \in \mathscr{X}$. But intuitively the class of ordinals, denoted by "On", is just such a class C. Likewise we can't use AC to prove the existence of a class K, called the class of cardinals, which consist of exactly one point from each similarity class or unordered sets. What to do ?

In the case of the ordinals we are lucky. We can actually define in NBG a particular (proper) class, denoted by On, of well-ordered sets and then prove (even without the use of AC) that every well-ordered set is similar to exactly one element of On; put another way, class On intersects each similarity class of well-ordered sets in exactly one point (cf. [9; p. 179]). In the case of cardinals we are almost as lucky. Using the theory of ordinals we can actually define in NBG a particular (proper) class K and prove (with the aid of AC) that every set can be put in 1:1 correspondence with exactly one element of K; put another way, class K intersects each similarity class of unordered sets in exactly one point. But what about the case of totally ordered sets, or partially ordered sets ?

2. *The Axiom of Regularity*

There is a candidate for an axiom of set theory, axiom D of [4], called the Axiom of Regularity (abbrev.: REG), which solves our difficulty. One standard formulation of REG says that any non-empty class X contains at least one set which has no element in common with X: in symbols

$$\forall X[X \neq \emptyset \Rightarrow \exists v[v \in X \text{ and } v \cap X = \emptyset]].$$

The implications of this not very obvious axiom are far reaching, but some easily understandable conclusions are now given. (Cf. [9, pp. 201-3] for a more detailed treatment.)

First, REG implies that there are no infinitely descending $\in$-chains; i.e., there is no sequence of sets $\{x_n\}_{n=0}^{\infty}$ such that

$$\ldots \in x_2 \in x_1 \in x_0 .$$

To prove this, assume the contrary. Let $A = \{x_0, x_1, x_2, \ldots\}$ where $\ldots \in x_1 \in x_0$. Then by REG there exists an $x_i \in A$ such that $x_i \cap A = \emptyset$, contradicting $x_{i+1} \in x_i \cap A$. (Using AC even the converse is provable.)

Next let us define a function $F{:}On \rightarrow V$ by

$$F(\beta) = P(\cup_{\alpha<\beta} F(\alpha))$$

where "P" denotes the power set operation, i.e., $P(Y) = \{y \mid y \subseteq Y\}$. Define $H = \cup_{\alpha \in On} F(\alpha)$. We will say that x is a *regular set* or x is *regular* iff $x \in H$. (In the literature these are usually called well-founded sets.) Then REG implies $V=H$, i.e., every set is a regular set. Before proving this let us give a suggestive interpretation. We can think of $V=H$ as giving us (to borrow a term from cosmology) a "big-bang" theory of sets. In the beginning there was only one set, the empty set.

Stage 0: The collection $F(0) = P(\emptyset) = \{\emptyset\}$ is made into a set.

Stage 1: The collection $F(1) = P(F(0)) = \{\emptyset, \{\emptyset\}\}$ is made into a set.

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Stage β: The collection $A = \bigcup_{\alpha<\beta} F(\alpha)$, if not already a set is made into a set. Then any subcollection of A not already a set is made into a set. Finally all subsets of A are collected and made into a set called $F(\beta)$.

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Thus $V=H=\bigcup_{\alpha\in On} F(\alpha)$ says that a class or collection is a set iff it is made into a set during one of these stages. Put another way, each set is constructed or built up from the empty set via iteration of the union and power set operations sufficiently many times. (Actually the construction procedure is much easier at successor ordinals since one can show that $F(\beta+1) = P(F(\beta))$.)

In order to prove $V=H$ and several other consequences of REG, let us define an ordinal valued function $\varrho : H \rightarrow On$ by $\varrho(x)$ = first α such that $x \in F(\alpha)$. We call $\varrho(x)$ the *rank of the set* x. In terms of our interpretation above the rank of a set x is closely related to the stage at which collection x was made into a set.

To prove REG$\Rightarrow V=H$, assume $V - H \neq \emptyset$. Then by REG there is some $x \in V - H$ such that $x \cap (V - H) = \emptyset$, i.e., $x \subseteq H$. Then $\{\varrho(u) \mid u \in x\}$ is a set of ordinals, hence is strictly bounded above by some ordinal β, i.e., $u \in x \Rightarrow \varrho(u) < \beta$. Then by definition of ϱ and β we have

$$u \in F(\varrho(u)) \subseteq \bigcup_{\alpha<\beta} F(\alpha) \text{ for all } u \in x.$$

Hence

$$x \subseteq \bigcup_{\alpha<\beta} F(\alpha),$$

hence

$$x \in P(\bigcup_{\alpha<\beta} F(\alpha)) = F(\beta) \subseteq H$$

contradicting

$$x \in V - H.$$

The rank function has nice properties, the important one for us being

(*) $x \in y \Rightarrow \varrho(x) < \varrho(y)$ for $x, y \in H$

where "<" denotes the standard ordering among ordinals. To see this, assume $\varrho(y) = \beta$. Then

$$\begin{aligned} & x \in y \in F(\beta) = P(\cup_{\alpha<\beta} F(\alpha)) \\ \Rightarrow\ & x \in y \subseteq \cup_{\alpha<\beta} F(\alpha) \\ \Rightarrow\ & x \in F(\alpha) \text{ for some } \alpha < \beta \\ \Rightarrow\ & \varrho(x) \leqslant \alpha < \beta = \varrho(y), \text{ as claimed.} \end{aligned}$$

The property (*) above of the rank function can be used to show that REG implies that there are no finite $\in$-cycles; i.e., we can't find any sets $x_1, \ldots, x_n$ such that

$$x_1 \in_2 \in \ldots \in x_n \in x_1.$$

In particular, if we had such a finite $\in$-cycle, then (*) would give us

$$\varrho(x_1) < \varrho(x_2) < \ldots < \varrho(x_n) < \varrho(x_1)$$

hence $\varrho(x_1) < \varrho(x_1)$, a contradiction.

Finally another application of REG is to solve our originally stated problem: to justify the choosing of one element from each similarity class of an intuitive collection of similarity classes. In fact REG+AC implies that given any equivalence relation R, there exists a class which consists of exactly one element from each non-empty R-equivalence class. (The equivalence relation "$\simeq$" of similarity is of course just a special example.) The idea of the proof is simple. Using REG we can cut each non-empty R-equivalence class X down to a unique set $X_R \subseteq$; in particular, if

$$\alpha_X = \inf\{\varrho(u) \mid u \in X\}$$

then

$$X_R =_{Df} \{u \in X \mid \varrho(u) = \alpha_X\} \subseteq F(\alpha_X).$$

Then we use AC to choose one element from each of these disjoint sets X_R, hence from each of the R-equivalence classes.

In summary, NBG+REG+AC implies the following:

(1) no infinitely descending $\in$-chains;
(2) a "big-bang" theory of sets (all sets are regular);
(3) no finite $\in$-cycles;
(4) for any equivalence relation R, there exists a choice class C_R which consists of exactly one element from each non-empty R-equivalence class.

Conclusions (3) and (4) are intuitively true statements of set theory. On the other hand (1) and (2) — which in NBG+AC are each equivalent to REG — are not so obviously true. Many logicians, however, do consider REG to be obvious (cf. [7; p. 56], [14]). But to discuss whether REG is true or not is to discuss the question, "What is a set?"

3. *A definition of "set"*

Let us consider the following intuitive definition which we will refer to as the *"generalized big-bang"* definition of a set (abbrev.: GBB). One starts with certain physical and/or conceptual objects, called atoms. We build new sets by stages. At any stage we form new sets by taking collections of objects (i.e., atoms and/or sets) formed at previous stages. Also at any stage we can add new atoms. We iterate this procedure over all possible stages (one stage for each ordinal α). The collection of sets so obtained is the collection of all sets.

Classically an *urelement* is defined as an object which isn't a collection (hence of course not a set) but which can be an element of a set. We adopt a broader outlook and define an atom as an object which is discovered or formed at some stage by some process other than collecting objects formed at previous stages. Often atoms will be objects defined by the properties they satisfy. It is even possible that an atom is itself a set. However, an atom x can't be a proper class since if **x** is introduced in stage α, then x is an element of sets formed during stage $\alpha+1$. Hence proper classes are not considered to be objects.

One commonly seen version of GBB is to allow starting with certain atoms which are urelements but not allow adding new atoms at future stages (cf. [13; p. 233]). One gets what might be called *regular set theory* if the collection of all atoms is the empty set. Then the only objects are sets and GBB reduces to a vague wording of the simple "big-bang" axiom V=H, which is equivalent to REG. Thus we can conclude that REG is true for the concept of set embodied in GBB if we can show that we can assume without loss of generality that there are no atoms.

In GBB we said that atoms would be physical or conceptual objects. We will also consider sets as conceptual objects. To say that object X is a conceptual object is not meant to imply that we are assuming X exists only in the mind of the conceiver and has no existence independent of thought. In this paper we do not take sides between conceptualism and platonism (or realism). Rather we mean that a conceptual object is a non-material object whose existence has been or can be discovered by the use of pure reason.

4. *Evidence for REG*

One problem with using physical objects such as elementary particles or hamburgers as atoms is that it gives a time dependent set theory, with sets changing as objects are created or destroyed. It is better to think of a fixed universe consisting only of conceptual objects; then if desired, one can set up a time dependent informal correspondence between physical and conceptual objects. What can be done in a set theory with physical atoms can be duplicated up to "conceptual isomorphism" in a set theory without physical atoms. Hence without loss of generality we can assume that there are no physical atoms.

But what about atoms which are conceptual objects; e.g., what about natural numbers ? Conceptually a natural number isn't necessarily a collection and isn't thought of as having any internal structure. The natural numbers are in fact good examples of atoms which are introduced conceptually at an early stage in GBB and are defined formally by the properties they satisfy. Thus instead of bothering with such foundational questions as "What is the number 1 ?", the mathematician will usually just assume that he is given a collection of objects (which we are calling atoms) satisfying the second order axioms of number theory and will proceed from there. Formally this corresponds to extending the formal theory NBG by adding new symbols "N", "O", "S", "+", "." and at least the following new proper axioms.

(i) $N \in V$ (N is the set of natural numbers)

(ii) $0 \in N$

(iii) $\forall x\, [x \in N \rightarrow Sx \in N]$

(iv) $(\forall x, y)_N\, [Sx = Sy \rightarrow x = y]$

(v) $(\forall x)_N\, Sx \neq 0$

(vi) $(\forall Z[0 \in Z \,\&\, (\forall x)_N[x \in Z \rightarrow Sx \in Z] \rightarrow N \subseteq Z]$

(vii) $x+y$ and $x.y \in N$ for all $x, y \in N$

(viii) $x+0 = x$ and $x+Sy = S(x+y)$ for all $x, y \in N$

(ix) $x.0 = 0$ and $x.Sy = x.y + x$ for all $x, y \in N$.

Let us call this extended theory NBG+N.

Thus the mathematician just leaves the natural numbers $0, 1 = S0$, $2 = SS0$, ... as unspecified objects or atoms satisfying certain properties. Whether these atoms are also sets is usually left unspecified. If we are working in ordinary NBG where there are no urelements and $X \in V$ implies X is a set, then all the integers are sets, which sets however being left open. If we are working in a modified NBG where we allow urelements as well as sets and classes, (cf. [12]), then we don't even know whether the integers are urelements or sets.

Logicians have discovered several ways of defining an NBG set ω and NBG set functions f_S, f_+, $f_.$ such that if N, 0, S, +, . are interpreted as ω, $\emptyset$, f_S, f_+ and $f_.$ respectively, then (i) - (ix) are theorems of (a definitional extension of) NBG (cf. [9; pp.

175-8] and [10; Ch. IV]; (when referring to [10] we mean the results of Ch. IV translated into NBG). In fact for such definitions as given in [9] and [10] it is provable in NBG that ω itself, the elements of ω, and the sets f_S, f_+ and $f_.$ are regular sets (i.e., elements of H). Put simply, N, O, S, +, . can be defined in terms of regular sets. Thus in summary, what can be done in a set theory with the natural numbers as atoms can be duplicated (in fact in many ways) in some definitional extension of NBG.

The average mathematician when studying number theory still probably works conceptually in what would be formalized as NBG+N rather than some definitional extension of NBG. He would (or should) look at the work of [9] and [10] as providing models for NBG+N in NBG, hence demonstrating that he can work in NBG+N free of worries of contradiction if he already accepts NBG. It is true that anything that can be done in NBG+N can be done in a definitional extension of NBG, but the converse is false; e.g., one can prove such "accidental" properties of the natural numbers as $1 \in 3$ in [9] or $1 \in 2$ and $2 \in 3$ but $1 \notin 3$ in [10].

Let us give another example of a mathematical object which conceptually isn't necessarily a collection. Specifically, given any two sets x and y, we have in mind the ordered pair $<x, y>$. Even if we wanted to, we couldn't start off with all the ordered pairs taken as atoms at some fixed stage α: it makes sense to talk about the ordered pair $<x, y>$ only during some stage after the conceptual construction of the sets x and y themselves. Thus if we were to treat ordered pairs as atoms, we would need to allow the addition of new atoms at every stage.

Instead of bothering with the foundational question "What is the ordered pair $<x, y>$?", what the mathematician conceptually does corresponds formally to extending NBG by adding a new symbol "$<, >$" and the following proper axioms:

(i) $(\forall x_1, x_2, y_1, y_2)[<x_1, x_2> = <y_1, y_2> \longleftrightarrow x_1 = y_1 \ \& \ x_2 = y_2]$
(ii) $(\forall x, y)[<x, y> \in V]$

In 1904 N. Wiener discovered that the ordered pair of two pure sets could be defined in simple set theory. Kuratowski's improved definition reads

$$<x, y> = \{ \{x\}, \{x, y\}\}.$$

For another definition which though equally artificial does have many advantages over Kuratowski's, consider

$$<x, y> = \{\{\emptyset, x\}, \{y\}\}.$$

(cf. [6]). For these definitions it is easily shown that an ordered pair of regular sets is a regular set: in symbols,

$$x, y \in H \Rightarrow <x, y> \in H.$$

In summary: what can be done in a set theory allowing the addition of ordered pairs as atoms can be duplicated (in many different ways) in regular set theory.

The same has been shown to be true for other classical conceptual objects such as rational, real, or complex numbers, geometrical surfaces in R^3, n-tuples and functions of real or complex numbers, etc.: what can be done in a set theory with such atoms can be duplicated in some definitional extension of NBG without atoms. It is even provable in NBG that the sets usually chosen to represent such classical objects are regular sets and the usual set theoretic operations applied to regular sets give regular sets. Thus for classical mathematics it seems to be true that we can work in a set theory with no atoms without any loss of generality.

5. *Evidence against REG*

General set theory and modern mathematics, however, are not limited to the study of classical structures and objects. Are there any non-classical conceptual objects which can't be expressed in terms of a set theory with no atoms ? Good candidates for such objects seem to be objects $x_0, x_1, x_2, \ldots$ such that $x_n = \{x_{n+1}\}$ for all n. This gives us

$$\ldots \in x_2 \in x_1 \in x_0 .$$

One of the main goals of this paper is to describe a general construction procedure which can be used to justify the intuitive existence of infinitely descending $\in$-chains of conceptual objects.

Let us start by discussing $\in$-descending chains in general. If we are not excluding the possibility of infinitely descending $\in$-chains, it seems that we must allow the possibility that for any ordinal α we can find an $\in$-descending chain of length α. But what would we mean by a chain of length even $\omega+1$, denoted by

$$x_\omega \in \ldots \in x_n \in \ldots \in x_1 \in x_0\,?$$

The only reasonable condition seems to be to require (in addition to the obvious condition that $x_{n+1} \in x_n$ for all $n<\omega$) that x_ω be in x_n for all n beyond some m: in symbols,

$$(\exists m)(\forall n)\,[n \geqslant m \rightarrow x_\omega \in x_n].$$

This suggests the following generalization. An *$\in$-descending chain of length* δ is defined as a sequence $\{x_\alpha\}_{\alpha<\delta}$ such that

(i) $x_{\alpha+1} \in x_\alpha$ for all $\alpha+1<\delta$
(ii) for every limit ordinal $\lambda<\delta$, there exists a $\beta<\lambda$ such that $x_\lambda \in x_\alpha$ for all $\beta \leqslant \alpha < \lambda$.

We say that z is at the bottom of an $\in$-descending chain $\{x_\alpha\}_{\alpha\leqslant\delta}$ of length $\delta+1$ starting at x if $x_0 = x$ and $x_\delta = z$.

Even if infinitely descending $\in$-chains exist the situation isn't too messy since, as shown in the next theorem, any two sets connected by such an $\in$-chain are also connected by a finite $\in$-chain.

THEOREM 1. If $\{x_\alpha\}_{\alpha\leqslant\delta}$ is an $\in$-descending chain starting at x and ending at x_δ, then there exists a finite sequence of ordinals
$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \delta$$

such that

$$x_\delta = x_{\alpha_n} \in x_{\alpha_{n-1}} \in \ldots \in x_{\alpha_1} \in x_{\alpha_0} = x.$$

PROOF. Assume proposition is true for all $\in$-descending chains of length less than δ and consider any $\in$-descending chain of length δ. If δ is a successor ordinal, say $\delta = \beta+1$, we have

$$x_\delta = x_{\beta+1} \in x_\beta \in \ldots \in x_1 \in x_0.$$

If δ is a limit ordinal, then for some $\beta<\delta$ we have

$$x_\delta \in x_\alpha \text{ for all } \beta \leqslant \alpha < \lambda$$

hence

$$x_\delta \in \ldots \in x_1 \in x_0.$$

In either case by induction hypothesis the chain $x_\beta \in \ldots \in x_1 \in x_0$ can be replaced by a finite subchain, giving us the finite $\in$-chain

$$x_\delta \in x_\beta = x_{\alpha n} \in \ldots \in x_{\alpha 1} \in x_{\alpha 0} = x_0. \quad \text{QED.}$$

The notion of an $\in$-ascending chain can be similarly defined and we can easily show that any $\in$-ascending chain of length $\delta+1$ starting at x and ending at z can be replaced by a finite $\in$-chain. In summary, any two sets connected by an $\in$-ascending or $\in$-descending chain can already be connected by a finite $\in$-chain.

If there are $\in$-chains of infinite length descending from a set x, theorem 2 below shows that there is an upper bound (depending on x) to the length of such $\in$-chains.

Let us first introduce some terminology which we will use frequently. We say that *z is an $\in$-ancestor of* x iff there exists an $\in$-descending chain $\{x_\alpha\}_{\alpha\leqslant\delta}$ such that $x_0 = x$, $x_\delta = z$ and $\delta>0$. By theorem 1 we can always choose $\{x_\alpha\}_{\alpha<\delta}$ such that δ is some finite ordinal n.

Let $\bigcup(Z)$ or $\bigcup Z$ denote the union of the sets in the class Z. We then define $\bigcup^n Z$ by induction as follows:

$$\bigcup^0 Z = Z.$$
$$\bigcup^{n+1} Z = \bigcup(\bigcup^n Z).$$

The union axiom of NBG just says that if Z is a set, then so is $\bigcup Z$ hence also $\bigcup^n Z$ for all $n \geqslant 0$.

For all $n \geqslant 0$ we can easily show by induction that there exists sets $x_0, x_1, \ldots, x_{n+1}$ such that

$$x_{n+1} \in x_n \in \ldots \in x_1 \in x_0 \Leftrightarrow x_{n+1} \in \bigcup{}^n x$$

For if true for case $n = k$, then for case $n = k+1$ we have

$$\begin{aligned} & x_{k+2} \in x_{k+1} \in \ldots x_1 \in x_0 \\ \Leftrightarrow & x_{k+2} \in_{k+1} \in \bigcup{}^k x \\ \Leftrightarrow & x_{k+2} \in \bigcup{}^{k+1} x. \end{aligned}$$

Thus clearly z is an $\in$-ancestor of a set x iff $z \in \bigcup{}^n x$ for some $n \geqslant 0$. We can now even define the *ancestor function* A where for any set x A(x) equals the set of all $\in$-ancestors of x: in symbols,

$$A(x) = \bigcup_{n=0} (\bigcup{}^n x) = \bigcup \{ \bigcup{}^n x \mid n < \omega \}.$$

THEOREM 2. Assume that there are no finite $\in$-cycles. Then for each set x there is an ordinal δ_x such that all $\in$-descending chains starting at x are of length less than δ_x.

PROOF. For any set x let δ_x be the first cardinal number greater than the cardinality of A(x) — in the standard development of NBG+AC cardinal numbers are ordinal numbers. Now consider any $\in$-descending chain $\{x_\alpha\}_{\alpha \leqslant \delta}$ with $x_0 = x$. All the x_α are different: for otherwise the $\in$-descending chain connecting x_α and x_β where $x_\alpha = x_\beta$ and $\alpha \neq \beta$ could be replaced by a finite $\in$-chain, giving us a finite $\in$-cycle. Also each x_α is an $\in$-ancestor of $x_0 = x$. Hence there can't be any more x_α than there are elements of A(x), hence $\delta < \delta_x$, as claimed. QED.

Now that we have worked a little with $\in$-descending chains in general, we show how to conceptually construct such chains of arbitrary length δ. For comparison we first review how one constructs $\in$-ascending chains.

Given any set x and any ordinal δ we can easily justify in NBG the existence of an $\in$-ascending chain $\{x_\alpha\}_{\alpha<\delta}$ of length δ starting at x. In particular we define $f : \delta \rightarrow V$ by

$$\begin{aligned} f(0) &= x \\ f(\alpha+1) &= f(\alpha) \cup \{ f(\alpha) \} \text{ for } \alpha+1 < \delta \\ f(\lambda) &= \bigcup_{\alpha<\lambda} f(\alpha) \quad \text{for limit ordinals } \lambda < \delta. \end{aligned}$$

Then if we denote $f(\alpha)$ by x_α we have $x_0 \in x_1 \in \ldots$ and even $x_0 \subset x_1 \subset \ldots$, assuming $x = x_0$ is a regular set.

Figuratively speaking we see that in constructing $\in$-ascending chains we conceptually construct each x_β in terms of previously completely constructed $x_\alpha (\alpha < \beta)$. We call this a *recursive process*. In contrast, to conceptually construct $\in$-descending chains $\{x_\alpha\}_{\alpha<\delta}$ for $\delta \geqslant \omega$ we will use a new process, called a *simultaneous process*, in which infinitely many sets are simultaneously constructed.

The construction process for simultaneously building the sets x_α for $\alpha < \delta$ will take several stages, numbered $0, 1, \ldots, \delta'$ say. (These stages have nothing to do with the stages mentioned in the GBB definition of a set given in § 3.) At each stage β for $\beta < \delta'$ we will work a little on the construction of various of the x_α's by adding some "elements" to such x_α's. At stage δ' we will complete our constructing activities by conceptually sealing off all the x_α's. At any stage $\beta < \delta'$ we have that each x_α is only an incomplete or potential set: it is entirely possible that more elements may be added to x_α at stage $\beta + 1$. It is only at stage δ' that each x_α becomes a completed collection, hence a set; and it is the case that conceptually speaking all he x_α's become sets simultaneously.

In the rest of this section by an integer n or an ordinal δ we mean the von-Neumann representation of n or δ, say as developed in [9]. In particular we need the facts that all von-Neumann integers are von-Neumann ordinals which in turn are all regular sets satisfying $\alpha < \beta$ iff $\alpha \in \beta$, $\alpha + 1 = \alpha \cup \{\alpha\}$ and $\beta = \{\alpha \mid \alpha < \beta\}$.

Let us illustrate our new conceptual process by constructing some regular sets, first by means of the recursion process and then again by means of the simultaneous process.

EXAMPLE 1. Consider any integer $N \geqslant 0$. Then we claim that there exists sets $x_0, \ldots, x_N$ such that for $0 \leqslant n \leqslant N$ we have

$$x_n = \{n, x_{n+1}, \ldots, x_N\}.$$

To construct such x_n by the recursion process we proceed as follows. Let $y_0 = \{N\}$ and for any $k < N$ let

$$y_{k+1} = \{N - (k+1), y_0, \ldots, y_k\}.$$

Finally, let $x_n = y_{N-n}$ for $0 \leqslant n \leqslant N$.

Clearly the y_n are regular sets since they are built up by recursion starting with the regular set $\{N\}$. Hence the x_n are regular.

In our construction of the same x_n by the simultaneous process we will use stages $0, \ldots, N, N+1$. We first use suggestive wording to aid the conceptual visualization of the process.

At stage 0 we make an open necked (conceptual) balloon and label it x_0. We then put (one copy of) the integer 0 inside x_0 by passing it through the open neck of x_0. This condition insures that we will have $0 \in x_0$ when we finish all our stages. Assume we have completed stages $0, 1, \ldots, n-1$ where $n \leqslant N$. At stage n we introduce and start to work with x_n and continue with our construction activities on the x_i for $i < n$. In particular we make another conceptual balloon, label it x_n, and put one copy of x_n inside each copy of each x_i for $i < n$. We then put (one copy of) the integer n inside each copy of x_n. These conditions insure that we will have (representations of) $x_n \in x_i$ for $i < n$ and $n \in x_n$ when we finish all our stages. At stage $N+1$ we (conceptually) simultaneously seal off the necks of all copies of all the balloons labeled x_n for $0 \leqslant n \leqslant N$. This sealing off insures that what is in the balloons representing the set x_n is only that which has been placed there during one of the stages $0, 1, \ldots, N$; e.g., we will have $x_N = \{N\}$ and $x_{N-1} = \{N-1, x_N\}$. Visually, what is in the sealed balloon x_n is what one sees upon bursting x_n. E.g., upon breaking x_0 one sees a copy of the integer 0 and balloons labeled $x_1, x_2, \ldots, x_N$; one does not see a copy of any of the integers $1, 2, \ldots, N$. Thus the balloon x_n being inside the balloon x_0 will be interpreted as $x_n \in x_0$, not as $x_n \subseteq x_0$.

Another advantage of the balloon interpretation is that we can picture the x_n's as stretchable: we can always put more objects inside x_n as long as the neck of the balloon x_n isn't sealed off.

A different less heuristic description of the procedure for simultaneously constructing the desired sets can be stated in terms of conditions imposed on potential sets $x_0, \ldots, x_N$ during the various stages, with no mention of balloons.

Stage $0 \leqslant n \leqslant N$: require $n \in x_n$ and $x_n \in x_i$ for all $i<n$. Stage $N+1$ (closure stage): require that for any n the object or potential set x_n contains only those elements forced into x_n by the conditions imposed during the stages $0, 1, \ldots, N$.

The "proof" that $x_n = \{n, x_{n+1}, \ldots, x_N\}$ is now trivial. We have that $n, x_{n+1}, \ldots, x_N$ are elements of x_n by stages $n, n+1, \ldots, N$ respectively. No other elements were added to x_n in any of the other stages. By stage $N+1$ these are the only elements in x_n, as desired.

One obviously needs to place some restrictions upon what conditions can be introduced during various stages of a simultaneous process. E.g., if we required $x_0 \in x_1$, $x_1 \in x_2$ and $x_2 \in x_0$ during stages 0, 1 and 2 respectively and applied closure in stage 3 we would get

$$x_0 \in x_2 \in x_1 \in x_0,$$

an $\in$-cycle, which we don't want.

Let us say that set x is *normal* iff x is not an $\in$-ancestor of x: in symbols, $x \notin A(x)$. Let us say that x is *semi-regular* iff x and every $\in$-ancestor of x is normal. In other words x is semi-regular iff for any $\in$-descending chain $\{x_\alpha\}_{\alpha<\delta}$ with $x = x_0$ we have $\alpha<\beta<\delta$ implies $x_\alpha \neq x_\beta$.

Since it is intuitively true that there are no finite $\in$-cycles, we expect all sets to be semi-regular. Thus whenever we construct some non-regular sets we will prove or at least discuss their semi-regularity.

That we need to place some more restrictions on the simultaneous construction process is shown by the following example and the discussion that follows.

EXAMPLE 2. Consider any non-empty index class I. Then for each $i \in I$ we can construct an $\in$-descending chain $\{x^i_n\}_{n<\omega}$ satisfying

$$x^i_n = \{x^i_{n+1}\} \text{ for } n \geqslant 0.$$

A description of a simultaneous construction of such x^i_n, stated

in terms of conditions imposed on the x^i_n during various stages, is as follows:
Stage $n \geqslant 0$: require $x^i_{n+1} \in x^i_n$ for each $i \in I$.
Stage ω: closure step.
That $x^i_n = \{x^i_{n+1}\}$ follows trivially from the definition of stages n and ω. Clearly all the x^i_n are non-regular. But there are obvious problems. What is the difference between say x^i_0 and x^j_0 if $i \neq j$? These two objects appear merely to be copies of each other, but we can't prove that they are equal. For that matter, x^i_0 even seems to be a copy of x^i_n for any $n > 0$.

For any set x let $A^*(x) = \{x\} \cup A(x)$, often called the *transitive closure of* x. We say that two sets x and y have the *same* $\in$*-structure* iff $A^*(x)$ and $A^*(y)$ are $\in$*-isomorphic,* i.e., there exists a 1:1 function f from $A^*(x)$ onto $A^*(y)$ such that

$$u \in v \Leftrightarrow f(u) \in f(v) \text{ for all } u, v \in A^*(x).$$

We abbreviate this by "$A^*(x) \simeq A^*(y)$ under f".

Intuitively one feels that two sets with the same $\in$-structure are equal. Strong support for his feeling is given by the following result, provable in NBG without REG.

THEOREM 3. Let x and y be regular sets. Then x and y have the same $\in$-structure iff $x = y$.
PROOF. Use the proof of the Shepherdson-Mostowski theorem, cf. [3; p. 73]. Since x and y are assumed to be regular and since REG restricted to the class H of all regular sets is provable in NBG, the proof can be carried out without assuming REG. QED.

Now we don't know what part of our intuition or knowledge of regular sets should carry over to non-regular sets, if the latter exist in any reasonable sense. But it does seem falacious to recognize the existence of two or more distinct sets which can't be conceptually distinguished by just looking at their $\in$-structures. And we do want to construct as reasonable non-regular sets as possible. So we propose the following axiom.

STRUCTURE AXIOM: If two sets have the same $\in$-structure, then they are equal: in symbols

$$A^*(x) \simeq A^*(y) \Rightarrow x = y.$$

THEOREM 4. There is no sequence $\{x_n\}_{n=0}^{\infty}$ such that

$$x_n = \{x_{n+1}\} \text{ for } n \geqslant 0.$$

PROOF (in NBG + Structure + No $\in$-cycles): Assume there is such a sequence of sets. Then $A^*(x_i)$ and $A^*(x_j)$ have the same $\in$-structure: the $\in$-isomorphism f is given by

$$f(x_{i+k}) = x_{j+k} \text{ for } k \geqslant 0.$$

Hence according to the Structure Axiom $x_i = x_j$. Thus we have $x_0 \in x_0$, contradicting the no $\in$-cycles axiom. QED.

We say that x is a *non-regular* set if we can find sets $x_1, x_2,$ ... such that

$$\ldots \in x_2 \in x_1 \in x.$$

Whenever we construct a sequence $\{x_\alpha\}_{\alpha<\delta}$ of what we consider to be non-regular sets we will prove that

$$A^*(x_\alpha) \simeq A^*(x_\beta) \Rightarrow x_\alpha = x_\beta$$

And clearly no non-regular set can be $\in$-isomorphic to a regular set.

Even though the simultaneous process needs restrictions placed on it, we still find it a useful conceptual tool for constructing non-regular sets. This will now be illustrated in the next three theorems.

Our first example is a simple one.

THEOREM 5. By the simultaneous construction process there exists an infinite $\in$-descending chain $\{x_n\}_{n<\omega}$ given by

$$x_n = \{n, x_{n+1}\} \text{ for } n \geqslant 0.$$

Given such sets, it is provable in NBG that they have the following properties:

(i) all x_n are non-regular but still finite
(ii) $m \neq n \Rightarrow x_m \neq x_n$ and not: $A^*(x_m) \simeq A^*(x_n)$
(iii) x_n is semi-regular for each n.

PROOF. A description of a simultaneous construction of the desired x_n, stated in terms of conditions imposed on the x_n during various stages, is as follows:

Stage $n \geqslant 0$: require $n \in x_n$ and $x_{n+1} \in x_n$.
Stage: ω: closure.
For any n the only stage at which any potential element is added to x_n is stage n, and then we add the element n and the potential element x_{n+1}. Thus, by stage ω we have

$$x_n = \{n, x_{n+1}\}.$$

The proofs of properties (i) - (iii) are similar to but simpler than the proofs of similar properties for the next theorem which we will prove in detail. QED.

Our next theorem shows one way to construct an $\in$-descending chain of length δ, given any $\delta > 0$.

THEOREM 6. By the simultaneous process, for any ordinal $\delta > 0$ there exists an $\in$-descending chain $\{x_\alpha\}_{\alpha<\delta}$ such that

$$x_\alpha = \{x_\mu \mid \alpha < \mu < \delta\} \cup \{\alpha\} \text{ for all } \alpha < \delta.$$

It is provable in NBG that such sets, if they exist, have the following properties:

(i) if $\delta < \omega$, then all x_α are regular;
(ii) if $\delta \geqslant \omega$, say $\delta = \lambda + n$ where λ is a limit ordinal and $n \geqslant 0$, then x_α is non-regular for $\alpha < \lambda$ and regular for $\lambda \leqslant \alpha < \delta$.
(iii) $\alpha < \beta < \delta \Rightarrow x_\alpha \neq x_\beta$ and not: $A^*(x_\alpha) \simeq A^*(x_\beta)$.
(iv) x_α is semi-regular for any $\alpha < \delta$.

PROOF. A description of a simultaneous construction of the desired x_α, stated in terms of conditions imposed on the x_α during various stages, is as follows:
Stage $\beta (0 \leqslant \beta < \delta)$: require $\beta \in x_\beta$ and $x_\beta \in x_\alpha$ for all $\alpha < \beta$.
Stage δ: closure.

Now consider what potential elements we put in x_α for some fixed $\alpha < \delta$. At any stage $\mu < \alpha$ we put nothing in; at stage $\mu = \alpha$ we only put α in; at any stage μ where $\alpha < \mu < \delta$ we only put x_μ in. According to stage δ there are no other elements in x_α, hence as desired we have

$(1)_\alpha$ $\quad x_\alpha = \{x_\mu \mid \alpha < \mu < \delta\} \cup \{\alpha\}$.

Now to discuss the regularity of the x_α for $\alpha<\delta$. If δ isn't a limit ordinal, then

$$\delta = \lambda+N+1$$

where $\lambda = 0$ or a limit ordinal depending on whether $\delta<\omega$ or $\delta>\omega$. One then shows that

$$x_\lambda, x_{\lambda+1}, \ldots, x_{\lambda+N}$$

are regular sets as in example 1. If $\delta\geqslant\omega$, say $\delta=\lambda+n$ where λ is a limit ordinal, then for any $\alpha<\lambda$ we have

$$\alpha+k<\lambda \text{ for all } k\geqslant 0,$$

hence

$$\ldots \in x_{\alpha+2} \in x_{\alpha+1} \in x_\alpha$$

hence x_α is non-regular.

Next we show that for any $\alpha<\delta$ we have

$$(2)_\alpha \quad A(x_\alpha) = A_\alpha =_{Df} \{x_\mu \mid \alpha<\mu<\delta\} \cup \delta$$

hence

$$(3)_\alpha \quad A^*(x_\alpha) = \{x_\mu \mid \alpha\leqslant\mu<\delta\} \cup \delta.$$

If $\mu<\alpha$ or $\mu = \alpha$, then $\mu \in\alpha \in x_\alpha$ or $\mu\in x_\alpha$ respectively, hence $\mu \in A(x_\alpha)$. If $\alpha<\mu<\delta$, then $\alpha \in\mu \in x_\mu \in x_\alpha$ hence μ and x_μ are in $A(x_\alpha)$. Thus $A_\alpha \subseteq A(x_\alpha)$.
To show $A(x_\alpha) \subseteq A_\alpha$, assume $y \in A(x_\alpha)$, hence

$$y = y_n \in \ldots \in y_1 \in x_\alpha.$$

Clearly $y_1 \in A_\alpha$ by $(1)_\alpha$. Assume $y_k \in A_\alpha$ and $y_{k+1} \in y_k$. Then $y_{k+1} \in A_\alpha$. For if $y_k \in\delta$, then so is $y_{k+1} \in\delta \subseteq A_\alpha$; and if

$$y_k = x_\mu \text{ for some } \alpha<\mu<\delta$$

then $y_{k+1} = \mu$ or $y_{k+1} = x_\nu$ for $\alpha<\mu<\nu<\delta$, hence $y_{k+1} \in A_\alpha$.

Thus by induction $y = y_n \in A_\alpha$, as desired.

We can easily show that

$$x_\alpha \neq x_\beta \text{ if } \alpha<\beta<\delta.$$

We have $\alpha \in x_\alpha$. But $a \notin x_\beta$ for otherwise

$$\alpha = x_\mu \text{ for some } \beta<\mu<\delta \text{ by } (1)_\beta;$$

but since $\mu \in x_\mu$ we would then have the contradiction

$$\mu \in \alpha \text{ where } \alpha<\beta<\mu.$$

Next we show that x_α is normal for all $\alpha<\delta$. For if not then

$$x_\alpha \in A(x_\alpha) = \{x_\mu \mid \alpha<\mu<\delta\} \cup \delta.$$

Since $x_\alpha \neq x_\mu$ for $\alpha\neq\mu$ we must have $x_\alpha \in \delta$, say $x_\alpha = \mu$. But we also have $\alpha \in x_\alpha$, hence $\alpha<\mu<\delta$, hence the contradiction

$$\mu \in x_\mu \in x_\alpha = \mu.$$

Each x_α is semi-regular since x_α and all its $\in$-ancestors are normal.

We are at last ready to prove that if $A^*(x_\beta) \simeq A^*(x_\beta)$, say under the $\in$-isomorphism f, then $x_\alpha = x_\beta$. First we show by induction that $f(\mu) = \mu$ for all $\mu<\delta$. Assume

(4) $f(\mu) = \mu$ for $\mu<\nu<\delta$.

Then $\mu \in \nu \Rightarrow = f(\mu) \in f(\nu)$, hence $\nu\subseteq f(\nu)$. Conversely,

$$\begin{aligned} & x\in f(\nu) \\ \Rightarrow\ & f^{-1}(x) \in f^{-1}(f(\nu)) = \nu \\ \Rightarrow\ & f(f^{-1}(x)) \in \nu \text{ by } (4) \\ \Rightarrow\ & x\in \nu. \end{aligned}$$

Thus $f(\nu) = \nu$, as desired.

Next we have $f(x_\alpha) = x_\beta$ because otherwise

$$\begin{aligned} & f(x_\alpha \neq x_\beta \\ \Rightarrow\ & f(x_\alpha) \in A(x_\beta) \text{ since } f(x_\alpha) \in A^*(x_\beta) = A(x_\beta) \cup \{x_\beta\} \\ \Rightarrow\ & f(x_\alpha) \in y_n \in \ldots \in y_0 = x_\beta \text{ for some } y_0, y_1, \ldots, y_n \\ \Rightarrow\ & x_\alpha \in f^{-1}(y_n) \in \ldots \in f^{-1}(y_0) = f^{-1}(x_\beta) \in A^*(x_\alpha) \\ \Rightarrow\ & x_\alpha \in A(x_\alpha) \end{aligned}$$

contracting normality of x_α.

Finally we have $\alpha = \beta$, hence $x_\alpha = x_\beta$ as desired. For if not, then without any loss of generality we can assume $\alpha<\beta<\delta$. Then

$$\begin{aligned}&\alpha \in x_\alpha \\ &\Rightarrow \alpha = f(\alpha) \in f(x_\alpha) = x_\beta = \{x_\mu \mid \beta<\mu<\delta\} \cup \{\beta\} \\ &\Rightarrow \alpha = x_\mu \text{ for some } \beta<\mu<\delta \\ &\Rightarrow \mu \in x_\mu = \alpha \text{ where } \alpha<\beta<\mu,\end{aligned}$$

a contradiction. QED.

Our last example is also the most sophisticated.

THEOREM 7. Consider any ordinal $\delta>0$ and any semi-regular set x. Then we can find an $\in$-descending chain $\{x_\alpha\}_{\alpha\leqslant\delta}$ such that $x_\delta = x$. In fact using the simultaneous construction process we can build such a sequence so that it is provable in NBG that these sets, if they exist, have the following properties:

(i) $\alpha<\beta\leqslant\delta \Rightarrow x_\beta \in x_\alpha,\ x_\beta \subset x_\alpha$, not: $A^*(x_\beta) \simeq A^*(x_\alpha)$
(ii) x_α is semi-regular for any $\alpha\leqslant\delta$.

PROOF. Let γ be the first ordinal larger than all ordinals in x or any of the $\in$-ancestors of x. We now give a description of a simultaneous construction of sets x_α for $\alpha\leqslant\delta$, stated in terms of conditions imposed on the x_α during various stages.
Stage $\mu(0\leqslant\mu\leqslant\delta)$: require $x_\mu \in x_\alpha$ and $\gamma+\mu \in x_\alpha$ for all $\alpha<\mu$. Also require $x \subset x_\mu$.
Stage $\delta+1$: closure.
One easily shows that for any $\alpha\leqslant\delta$ we have

$$(5)_\alpha \quad x_\alpha = x \cup \{x_\mu \mid \alpha<\mu\leqslant\delta\} \cup \{\gamma+\mu \mid \alpha<\mu\leqslant\delta\}$$

Note that this will give $x_\delta = x$.

That $x_\beta \in x_\alpha$ and $x_\beta \subseteq x_\alpha$ for any $\alpha<\beta\leqslant\delta$ is obvious from $(5)_\alpha$ and $(5)_\beta$. To prove $x_\beta \subset x_\alpha$ we obviously have

$$\gamma+\alpha+1 \in x_\alpha,$$

but using $(5)_\beta$ we can show that

$$(6) \qquad \gamma+\alpha+1 \notin x_\beta.$$

In detail we have $\gamma+\alpha+1 \notin x$ by definition of γ. Also

$$\gamma+\alpha+1 \notin \{x_\mu \mid \beta<\mu\leqslant\delta\}.$$

For if $\beta<\mu<\delta$, then $x_\mu \neq \gamma+\alpha+1$ since $\gamma+\mu+1 \in x_\mu$ by $(5)_\mu$ but $\gamma+\mu+1 \notin \gamma+\alpha+1$ since $\alpha<\mu$; and if $\mu = \delta$ then $x_\delta \neq \gamma+\alpha+1$

because $\gamma+\alpha \in \gamma+\alpha+1$ whereas γ hence $\gamma+\alpha$ is larger than any ordinal in $x_\delta = x$. Finally since $\alpha<\beta$ we have

$$\gamma+\alpha+1 \notin \{\gamma+\mu \mid \beta<\mu\leqslant\delta\}.$$

Next one verifies that

$$(7)_\alpha \quad A(x_\alpha) = A_\alpha =_{\mathrm{Df}} A(x) \cup \{x_\mu \mid \alpha<\mu\leqslant\delta\} \cup \{\mu \mid \mu\leqslant\gamma+\delta\}.$$

To show that each x_α is normal, assume otherwise.
Then for some α we have $x_\alpha \in A(x_\alpha)$. Using $(7)_\alpha$ we show that this leads to a contradiction. In particular $x_\alpha \notin A(x)$ and $x_\alpha \neq \mu$ for $\mu\leqslant\gamma+\delta$ because x_α isn't normal by assumption whereas any element of $A(x)$ or any ordinal μ is.
Also $x_\mu \neq x_\alpha$ for any $\alpha<\mu\leqslant\delta$.

Each x_α is semi-regular since x_α and all its $\in$-ancestors are normal.

Finally we show that if $A^*(x_\alpha) \simeq A^*(x_\beta)$, say under the $\in$-isomorphism f, then $x_\alpha = x_\beta$. As in proof of similar part in theorem 6 we show that $f(\mu)=\mu$ for all $\mu \in A^*(x_\alpha)$ and $f(x_\alpha) = x_\beta$. Then if $\alpha\neq\beta$ without any loss of generality we can assume $\alpha<\beta\leqslant\delta$.

Then

$$\gamma+\alpha+1 \in x_\alpha \text{ by } (5)_\alpha$$
$$\Rightarrow \gamma+\alpha+1 = f(\gamma+\alpha+1) \in f(x_\alpha) = x_\beta,$$

contradicting (6). QED.

We have given some examples of non-standard conceptual objects. That these objects exist conceptually seems almost as unquestionable as the conceptual existence of infinite sets. What is questionable is whether they are sets according to a reasonable conception of what a set is.

In terms of the GBB definition of a set given in § 3 simultaneously constructed non-regular sets would be introduced as atoms at various stages. There could be no upper bound on such stages since regular sets of any rank can be used in the construction of non-regular sets.

If our non-regular sets (or at least some of them) are accepted as "true" sets, then REG is false. If they aren't sets, then one still must show that what can be done in a theory with such atoms can essentially be done without them.

Non-regular sets have been used by some authors. "Sets" x such that $x = \{x\}$, hence $x \in x$, have been used so as to handle individuals (urelements); cf. [10; pp. 30-32]. But such usage has been to improve the formal presentation with no suggestion of really thinking of an individual as an object which is an element of itself. Mostly non-regular sets have been used to show independence results; e.g., [8]. Even then non-regular sets have only formally existed: they constituted elements of a non-standard model in which some "true" axiom of set theory was shown to be falsifiable (hence independent).

That REG is independent from NBG+AC was first proven in [1] and more recently in much stronger forms in [2] and [5]. The results in [2] and [5] seem more than strong enough to show the relative consistency (with respect to NBG+ AC) of any reasonable formal axiom which would justify the existence of our simultaneously constructed sets. That these results are too strong follows from the fact that they allow such pathologies as $x \in x$.

6. *A neutral viewpoint*

One main point of this paper is not to prove that REG is false but to show that REG is a questionable rather than an obviously true axiom of set theory. Having made an attempt to do this in § 5 we will now take a neutral viewpoint neither accepting REG nor its negation. However, some limitation type axiom is necessary to eliminate such pathologies as $\in$-cycles. REG is a strong version of such a limitation axiom. We now introduce the weakest possible version which is equivalent to but simpler in form and more usable than the "no (finite)" $\in$-cycles" hypothesis.

AXIOM OF LIMITATON (abbrev.: LIM). There exists a function ϱ with domain V and a strict partial ordering $<$ on $\mathscr{R}(\varrho)$, the range of ϱ, such that

$$x \in y \Rightarrow \varrho(x) < \varrho(y) \text{ for all } x, y \in V.$$

All the following theorems are provable in NBG.

THEOREM 1. LIM $\Rightarrow$ no finite $\in$-cycles.
PROOF. If there were sets $x_1, \ldots, x_n$ such that

$$x_1 \in x_2 \in \ldots \in x_n \in x_1$$

then

$$\varrho(x_1) < \varrho(x_2) < \ldots < \varrho(x_n) < \varrho(x_1),$$

hence $\varrho(x_1) < \varrho(x_1)$ by transitivity of $<$, yet not: $\varrho(x_1) < \varrho(x_1)$ by assymmetry of $<$, a contradiction. QED.

THEOREM 2. No finite $\in$-cycles $\Rightarrow$ LIM. In particular, if there are no finite $\in$-cycles, then we can find a $\varrho : V \rightarrow V$ such that

(i) $x \in y \rightarrow \varrho(x) \subset \varrho(y)$
(ii) $x \subseteq y \rightarrow \varrho(x) \subseteq \varrho(y)$

PROOF: Define $\varrho : V \rightarrow V$ by $\varrho(x) = A(x)$ where $A(x)$ denotes the set of $\in$-ancestors of x. Then (ii) just says that $x \subseteq y$ implies $A(x) \subseteq A(y)$ which is obviously true. To prove (i), assume $x \in y$. Clearly $A(x) \subseteq A(y)$ and $x \in A(y)$. If $x \in A(x)$, then there would be $x_1, \ldots x_n$ such that $x = x_1 \in \ldots \in x_n \in x$, contradicting hypothesis of no $\in$-cycles. Thus $x \in A(y) - A(x)$, hence $A(x) \subset A(y)$ as desired. QED.

Theorems 1 and 2 tell us that an equivalent form of LIM would be
LIM*: There exists a function $\varrho : V \rightarrow V$ such that

$$x \in y \Rightarrow \varrho(x) \subset \varrho(y) \text{ fo all } x, y.$$

Note that theorems 1 and 2 imply that, besides eliminating $\in$-cycles, LIM also eliminates other pathologies such as $x \subseteq y \in x$.

We conjecture that LIM and the Structure Axiom of § 6 are mutually independent relative to NBG+AC. However LIM and the Structure Axiom are related to REG, as we now prove in the next two theorems.

THEOREM 3. REG $\Rightarrow$ LIM and the Structure Axiom.
PROOF. REG implies there are no $\in$-cycles, hence LIM. Also REG implies the Structure Axiom by theorem 5.3. QED.

THEOREM 4. If we can choose ϱ and $<$ in LIM such that $\mathscr{R}(\varrho)$ is strictly well-ordered by $<$, then REG holds.
PROOF. We want to show that

$$X \neq \emptyset \Rightarrow \exists x[x \in X \text{ and } x \cap X = \emptyset].$$

Let $A = \{\varrho(x) \mid x \in X\}$. Then A is a non-empty subclass of the class $\mathscr{R}(\varrho)$ strictly well-ordered by $<$, hence A has a $<$-first element, say α_0. Choose one element x_0 from $\{x \in X \mid \varrho(x) = \alpha_0\}$. We don't need AC since we are making only one choice.) Then $x_0 \in X$ and $x_0 \cap X = \emptyset$: for if $x_0 \cap X \neq \emptyset$, say $u \in x_0 \cap X$, then $\varrho(u) \subset \varrho(x_0) = \alpha_0$ yet $u \in X$, contradicting the definition of α_0. QED.
Comment: The rank function ϱ defined in § 2 satisfied

(i) $x \in y \Rightarrow \varrho(x) \subset \varrho(y)$
(ii) $\mathscr{R}(\varrho)$ is strictly well-ordered by "$\subset$".

What we can't show in NBG and need REG for is to show that the domain H of the rank function is all of V.

Say the ϱ of LIM is interpreted as a function which relates each set to a stage of some kind, may be a stage measuring the complexity of sets. Then LIM says that if $x \in y$ then $\varrho(x) < \varrho(y)$, hence x belongs to a lower, earlier, or simpler stage than does y. LIM thus says that the stages are partially ordered. REG says that the stages are well-ordered.

Having rejected REG as an axiom and chosen the weaker axiom LIM, we are now back to where we started in § 1: how do we justify the choosing of one element from each class in an intuitive collection $\mathscr{X}$ of mutually disjoint (possibly proper) classes ? Given any such $\mathscr{X}$ we can define an intuitive equivalence relation R given by

$$R(x, y) \leftrightarrow \exists X[X \in \mathscr{X} \;\&\; x \in X \;\&\; y \in X].$$

Conversely, given any intuitive equivalence relation R, we have that R breaks up its field into mutually disjoint classes, viz., the R-equivalence classes. Thus we have a correspondence between intuitive collections $\mathscr{X}$ of mutually disjoint classes and intuitive equivalence relations. In general we can't work with such collections of classes in NBG, but we can handle many

of the corresponding equivalence relations. For any NBG equivalence relation R let $[x]_R$ denote the R-equivalence class determined by x. We suggest the following replacement for AC.
AXIOM AC*: For each equivalence relation R, there exists a class C such that C intersects every non-empty R-equivalence class in exactly one point: in symbols,

$$(\exists C)\,(V_x)[\,[x]_R \neq 0 \rightarrow \exists u[C \cap [x_R = \{u\}]].$$

THEOREM 5. AC* ⇒ AC

PROOF. Consider any class X of pairwise disjoint non-empty sets. Define $R \subset V \times V$ by

$$<x, y> \in R \leftrightarrow (\exists w)x[x \in w \; \& \; y \in W].$$

Then R is an equivalence relation on its field. (The field of a relation is the union of its domain and range.) Clearly the R-equivalence classes are just the sets $w \in X$. QED.

In summary, we have weakened the limitation type axiom from REG to LIM which in turn necessitated strengthening the choice axiom from AC to AC* in order to handle the problem raised in § 1.

It seems that AC* is adequate for our purposes, resolves the difficulties of § 1, avoids need of hypothesizing REG, and yet is still intuitively true. There are some stronger (candidates for) axioms of class-set theory which we now give.

STRONG WELL-ORDERING AXIOM (abbrev.: WO*): The class V of all sets can be well-ordered.

WEAK BIG BANG AXIOM (abbrev.: WBB): There is a function F with domain On such that

(i) $\alpha < \beta \Rightarrow F(\alpha) \subseteq F(\beta)$, all $\alpha, \beta \in 0n$
(ii) $V = \bigcup_{\alpha \in 0n} F(\alpha)$

The relation between WBB and GBB of § 3 is clearly that GBB implies WBB.

It is easy to show that W0* implies AC* and WBB. Conversely one can show that AC+WBB implies W0*, hence AC*, by using the idea of the proof that AC+REG implies W0* as given in [11; pp. 84-7]. Thus if one accepts the GBB characterization of a set, then one must accept WBB and hence doesn't need AC*.

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