COMPACTNESS AND LÖWENHEIM-SKOLEM PROOFS IN MODAL LOGIC (¹)

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1. *Introduction*

In this article we prove a number of theorems in the semantics of the modal systems M (von Wright), B (Becker's "Brouwersche" system), S_4, and S_5 (Lewis), and their extension to quantification theory. In section 2. we describe the topological method used (employed elsewhere to prove similar theorems for free quantification and identity theory) in a general way. In section 3. the method is applied to propositional modal logic, and in section 4. (via a theorem on substitution in infinite sets) to modal quantification theory.

2. *Semantic entailment, compactness, and ultrafilters*

The present section concerns the semantics of arbitrary formal languages, and not just the languages of modal logic. By a *language* L we mean a mathematical structure comprising at least a set of *sentences* W_L, and a set of *admissible valuations* V_L, each member of V_L mapping some subset of W_L into the set $\{T, F\}$. The following definitions are standard.

Definition 1. A sentence A of L is *valid* ($\vdash$ A) in L iff v(A) = T for every admissible valuation v of L.

Definition 2. A set of sentences X of L *semantically entails* a sentence A of L (X $\vdash$ A) *in* L iff every admissible valuation v of L such that v(B) = T for all B in X is such that v(A) = T.

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Henceforth we shall adopt the abbreviations

$$H_L(A) = \{v \varepsilon V_L : v(A) = T\},$$

$$H_L = \langle V_L, \{H_L(A) : A \varepsilon W_L\}\rangle;$$

we shall use "member [subset] of H_L "for" member [subset] of V_L", omit the subscript L when convenient, and call the sets H_L (A) *elementary classes* and H_L the *valuation space of L.*

The following definitions and theorem introduce our basic topic of concern.

Definition 3. The valuation space H of L is *compact* iff any set of sentences X of L such that $\bigcap_{B \varepsilon X} H(B) = \Lambda$ has a finite subset Y such that $\bigcap_{B \varepsilon Y} H(B) = \Lambda$.

Definition 4. A language L has *finitary semantic entailment* iff for any sentence A of L, any set of sentences X such that $X \vdash A$ in L has a finite subset Y such that $Y \vdash A$ in L.

Definition 5. A language L has *exclusion negation* iff for every sentence A of L there is a sentence $\sim A$ of L such that $H(\sim A) = H - H(A)$.

Theorem 1. If L has exclusion negation then L has finitary semantic entailment iff the valuation space of L is compact.

The proof is immediate from the consideration that in a language with exclusion negation, $X \vdash A$ if and only if $\bigcap_{B \varepsilon X} H(B) \cap H(\sim A) = \Lambda$. This theorem shows that in the usual context one may concentrate on compactness, although for applications in completeness proofs, one needs in general a proof that the language has finitary entailment. For this reason, we turn now to a general method for proving this.

Definition 6. A *filter* F *on* a set X is a non-empty family of non-empty subsets of X such that if Y, Z are in F, so is $Y \cap Z$ and any superset of Y.

Definition 7. An *ultrafilter on* a set X is a filter on X not properly contained in any other filter on X.

Definition 8. A filter F on a valuation space H *converges to* a member v of H iff for every elementary class H(A), $v \varepsilon H(A)$ iff $H(A) \varepsilon F$.

We say that a filter *converges* iff there is some valuation to which it converges (2). We note that any filter is contained in an ultrafilter on the same set (e.g. Gaal, p. 265, Th. 1); and if F is an ultrafilter on X and Y, Z subsets of X, then $Z \cap Y \varepsilon F$ iff $Z \varepsilon F$ and $Y \varepsilon F$, and either X-Y or Y is in F (e.g. Gaal, p. 265, Th. 2).

Theorem 2. If every ultrafilter on the valuation space H of language L converges, then L has finitary semantic entailment.

Proof. Let X be a non-empty set of L; $\{X_i\}$, $i \varepsilon I$ the finite non-empty subsets of X; A a sentence of L; and let $X_i \vdash A$ be false for all $i \varepsilon I$. Then define

$\mathfrak{J}_m = \{v \varepsilon H : v(A) \neq T \text{ and } v(B) = T \text{ for all } B \varepsilon X_m\}$ for $m \varepsilon I$; let $\mathfrak{J} = \{\mathfrak{J}_m\}$, $m \varepsilon I$. By hypothesis, $\mathfrak{J}_m \neq \wedge$ for any $m \varepsilon I$. Moreover $\mathfrak{J}$ is closed under finite intersection: $\mathfrak{J}_{m_1} \cap \ldots \cap \mathfrak{J}_{m_n} = \mathfrak{J}_n$ where $X_n = X_{n_1} \cup \ldots \cup X_{m_n}$. It is easily checked therefore

$$F = \{Y \subset H ; Y \supset \mathfrak{J}_m \text{ for some } m \varepsilon I\}$$

is a filter on H. By a well-known theorem on filters noted above, there is an ultrafilter F′ on H such that $F \subset F'$.

Let us assume that all ultrafilters on H converge. Then there is a valuation v such that $v \varepsilon H(B)$ iff $H(B) \varepsilon F'$, for all sentences B of L.

(a) Let $B \varepsilon X$; then $\{v \varepsilon H : v(B) = T \text{ and } v(A) \neq T\} \varepsilon \mathfrak{J}$, and

(2) Our notion of convergence is not the topological notion, though similar to it.

this is a subset of H(B), so H(B) $\varepsilon F \subset F'$. Hence $v \varepsilon H(B)$ that is, v(B) = T, for all B in X.

(b) Let $B \varepsilon X$; then $\{v \varepsilon H : v(B) = T \text{ and } v(A) \neq T\} \varepsilon \mathfrak{J}$; and this set has an empty intersection with H(A). Hence if H(A) were in F′, F′ would contain $\wedge$; so H(A) $\notin$ F′. Hence $v \notin H(A)$: $v(A) \neq T$.

Therefore, X does not semantically entail A.

QED.

3. *Application to propositional modal logic*

The languages corresponding to the modal systems M, B, S_4, S_5 (see e.g. Kripke) shall be called L_m, L_b, L_4, L_5; we use τ to range over the index set {m, b, 4, 5}.

Definition 9. A *τ-model structure* (τ-ms) is a couple M = <K, R>, where K is a non-empty set and R a dyadic reflexive relation, and such that R is symmetric if τ = b, transive if τ = 4, and both transitive and symmetric if τ = 5.

Definition 10. The syntactic system *Synt* is a triple <*A, S, W*> where *A* is a denumerable set (*atomic sentences*); *S* is the set {$\wedge$, ~, □,), (}
W is the least set containing *A* and such that if A, B are in W, so are (A $\wedge$ B), ~(A), □A).

Definition 11. A *valuation over* a τ-ms M = <K, R> is a mapping v of K x W into {T, F} subject to the conditions that for all α in K and A, B in W:
$v_\alpha(\sim A) = T$ iff $v_\alpha(A) = F$,
$v_\alpha(A \wedge B) = T$ iff $v_\alpha(A) = v(B) = T$,
$v_\alpha(\Box A) = T$ iff $v_\beta(A) = T$ for all β in K such that $\alpha R \beta$, where we designate v relativized to α in K as v_α, and omit parentheses where convenient.

If there is a τ-ms M = <K, R> and member α of K such that $v' = v_\alpha$, we call v′ a *τ-valuation*.

Definition 12. The *language* L_τ is the couple $<Synt, V_\tau>$ where V is the set of all τ-valuations.

We call W the set of sentences of L_τ and V_τ the set of admissible valuations of L_τ, and write H_τ for $H_{L\tau}$.

Theorem 3. Every ultrafilter on H_τ converges.

Proof. Let $F(\tau)$ be the family of all ultrafilters on H_τ. We define the relation $R(\tau)$ on $F(\tau)$ as follows:

if $F, F' \varepsilon F(\tau)$ then $FR(\tau) F'$ iff for all $A \varepsilon W$ such that $H_\tau (\Box A) \varepsilon F$, $H_\tau(A) \varepsilon F'$.

Lemma 1. $M(\tau) = <F(\tau), R(\tau)>$ is a τ-ms.

Lemma 2. The mapping v of $F(\tau) \times W$ into $\{T, F\}$ such that $v_F(A) = T$ iff $H_\tau(A) \,\varepsilon\, F$, for all $A \,\varepsilon\, W$, for all F in $F(\tau)$, is a valuation over $M(\tau)$.

It is clear that each ultrafilter F on H_τ converges to the τ-valuation v_F. Hence it remains only to prove the lemmas.

Proof of lemma 1. In L_τ, $\Box A \vdash A$; hence $H_\tau(\Box A) \subset H_\tau(A)$; hence $R(\tau)$ is reflexive. In L_b, $A \vdash \Box \Diamond$ (cf. Kripke); let $FR(b)F'$ and $H_b(\Box B) \varepsilon F'$. If $H_b(B) \notin F$ then $H_b(\sim B) \,\varepsilon\, F$ (because an ultrafilter on K contains either $H_b(B)$ or $H_b - H_b(B)$), so then F' would contain $H_b(\Diamond \sim B) = H_b(\sim \Box B)$. But $H_b(\Box B) \cap H_b(\sim \Box B) = \Lambda$, so this is impossible. Hence R(b) is also symmetric.

In L_4, $\Box A \vdash \Box \Box A$; hence $H_4(\Box A) \leqslant H_4(\Box \Box A)$. Therefore if $FR(4)F'$, and $H_4(\Box A) \,\varepsilon\, F$, then $H(\Box A) \,\varepsilon\, F'$. Hence R(4) is also transitive. In L_5 we prove similarly that R(5) is both symmetric and transitive. QED.

Proof of lemma 2. Because an ultrafilter F must contain either $H_\tau(B)$ or H_τ-$H_\tau(B)$ for any sentence B, $v_F(\sim B) = T$ iff $v_F(B) \neq T$. Because an ultrafilter contains $H_\tau(B \wedge C) = H_\tau(B) \cap H_\tau(C)$ iff it contains both $H_\tau(B)$ and $H_\tau(C)$, $v_F(B \wedge C) = T$ iff $v_F(B) = V_F(C) = T$.

That $v_F(\Box B) = T$ iff $v_{F'}(B) = T$ for all F' such that $FR(\tau)F'$, we prove in two steps.

(i) If $H_\tau(\Box A \,\varepsilon\, F$ then $H_\tau(A) \,\varepsilon\, F'$ for all F' such that $FR(\tau)F'$; follows by the deminition of $R(\tau)$.

(ii) If $H_\tau(\Box A) \notin F$ then $F^* = \{H_\tau(\sim A)\} \cup \{H_\tau(B) : H_\tau(\Box B)$

ε F} is a family of sets such that each of its finite subfamilies has a non-empty intersection. For if it were not so, then there would be sentences $B_1, \ldots, B_n$ such that $B_1, \ldots, B_n \vdash A$ holds in L_τ and $H_\tau(\Box B_1), \ldots, H\ (\Box B_n)\ \varepsilon$ F. But then $\Box B_1, \ldots, \Box B_n \vdash \Box A$ would hold in L_τ (let $v_\alpha(\Box A) = F$ and $v_\alpha(\Box B_i) = T$ for i = 1, i, ..., n; then there is a β such that $\alpha R \beta$ in the relevant model structure and $v_\beta(B_i) = T$ for i = 1, ..., n but $v_\beta(A) = F$); so then H $(\Box A)$ would be in F. We conclude that the family of supersets of members of F* is a filterbase on H_τ, included in an ultrafilter F′. Clearly FR(τ)F′ and $H_\tau(A) \notin$ F′.

QED.

Finitary entailment and compactness theorems for M, B, S_4, and S_5 now follow by theorems 1. and 2.

4. *Application to quantificational modal logic*

Compactness proofs can be extended to quantification theory via a theorem on variable substitution in infinite sets of sentences and a device due to Beth and Hasenjaeger (Beth, pp. 264-265); van Fraassen sections III and IV). We begin by extending the language of modal logic to quantification theory along the lines of Thomason's semantics for the system Q_1 (Thomason, section 5), omitting the theory of identity, names, and definite descriptions. This is probably the simplest way in which modal logics can be extended to quantification theory, but the application of a general method may appropriately be shown in a simple case.

Definition 13. The syntactic system *QSynt* is a quadruple $<V, P, S_q, W_q>$ where:
V is a denumerable set (the *variables*);
P is a non-empty set, at most denumerable (the *predicates*) of which each member has associated with it an integral degree n>o;
S_q is the set $\{\sim, \wedge, \Box,), (\}$;
W_q is the least set such that: if P^n is a predicate of degree n and $x_1, \ldots, x_n$ are variables, then $(P^n x, \ldots, x_n)\ \varepsilon\ W_q$ and if A, B are in W_q, so are $\sim(A)$, $(A \wedge B)$, $\Box(A)$, and $(x)(A)$.

Definition 14. A *τq-model structure* (τq-ms) is a quadruplet M = <K, R, D, f> where <K, R> is a τ-ms, D a non-empty set (the *domain*), and f a function which assigns to each n-ary predicate P^n of QSynt a set $f_\alpha(P^n)$ of n-tuples of members of D, for each member α of K.

Definition 15. A *valuation* over a τq-ms M = <K, R, D, f> is a mapping v of the variables of QSynt into D, and of K x W_q into {T, F} such that v is a valuation over <K, R> and for all α in K, $v_\alpha(P^n x_1 \ldots x_n)$ = T iff $<v(x_1), \ldots, v(x_n)> \varepsilon f_\alpha(P^n)$, $v_\alpha((x)A)$ = T iff $v'_\alpha(A)$ = T for every valuation v′ over M which is like v_α except perhaps with respect to x.

If there is a τq-ms M = <K, R, D, f>, member α of K, and valuation v over M, we call the restriction v of v to α a *τq-valuation* (*over* M).

Definition 16. The *language* $L_{\tau q}$ is the couple <Qsynt, $V_{\tau q}$> where $V_{\tau q}$ is the set of all τq-valuations.

We turn now to substitution (cf. van Fraassen, section I). A *substitution function* is a one-to-one mapping of the set of variables into itself. When E is the expression $e_1 e_2 \ldots e_n$, we define $f(E) = e_1^* e_2^* \ldots e_n^*$ where $e_i^* = f(e_i)$ if e_i is a variable, and $e_i^* = e_i$ otherwise.

It is easy to see that such substitution cannot result in confusion of bound variables, and that $f(\sim A) = \sim f(A)$, $f(A \wedge B) = (f(A) \wedge f(B))$, $f(\Box A) = \Box f(A)$, $f((x)A) = (f(x))f(A)$. If X is a subset of W_q, we define

$$f(X) = \{f(A) : A \varepsilon X\}.$$

Convention: when f is the only substitution function being discussed, we write E* for f(E). We shall say that v_α satisfies a set $X \subseteq W_q$ iff $v_\alpha(B)$ = T for *all* B ε X.

Theorem 4. For any substitution function f, any τq-ms M, and any set of sentences X is satisfied by a τq-valuation over M if and only if f(X) is satisfied by a τq-valuation over M.

Proof. Let v be valuation over M; and $k \varepsilon D_M$; then we define the valuations v^* and v^k to be those valuations over M such that for all $x \varepsilon V$,

$v^*(x) = v(x^*)$;

$v^k(x) = v(y)$ when x is y^* and $v^k(x) = k$ when x is not y^* for any $y \varepsilon V$.

The theorem now follows from the following two lemmas.

Lemma 3. For any τq-valuation v_α and any sentence A, $v_\alpha(A^*) = v^*(A)$.

Lemma 4. For any τq-valuation v_α and member k of D_M, v a valuation over M, and any sentence A, $v_\alpha(A) = v_\alpha{}^k(A^*)$.

These lemmas are proved by strong induction on the length of A exactly as are lemmas 1. and 2. in (van Fraassen, section I) except for the clauses concerning necessity which we prove below.

Proof of lemma 3. Hypothesis of induction: for all sentences B of length less than A, and all valuations v_β over M, $v_\beta(B^*) = v^*_\beta(B)$.

Case □: From the hypothesis of induction it follows that for all β such that αRβ, $v_\beta(B^*) = T$ if and only $v_\beta{}^*(B) = T$. Therefore, $v_\alpha((\Box B)^*) = v_\alpha(\Box(B^*)) = T$ iff $v^*_\alpha((\Box B)) = T$.

Proof of lemma 4. Hypothesis of induction: for all sentences B of length less than A, and all valuations v_β over M, $v_\beta(B) = v_\beta{}^k(B^*)$.

Case □: From he hypothesis of induction it follows that for all β such that αRβ, $v_\beta(B) = T$ iff $v_\beta{}^k(B^*) = T$. Hence $v_\alpha(\Box B) = T$ iff $v_\alpha{}^k(\Box(B^*)) = v_\alpha{}^k((\Box B)^*) = T$.

QED.

From here on we shall need to consider only one substitution function f: let us designate alphabetically the i^{th} variable as x_i and let $f(x_i) = x_{2i}$.

A set $X \subseteq W_q$ in which no odd variables appear (bound *or* free) we call a *regular set*; clearly f(Y) is a regular set for any $Y \subseteq W_q$. We call a τq-valuation v_α a *regular* valuation iff v satisfies the following conditions (3) (cf. Beth, pp. 264-265; van Fraassen, section III):

> With each natural number m we associate a variable y_m ; y_1 is th first *odd* variable which does not appear in alphabetically the first sentence of the form (x)A; y_{n+1} is the first *odd* variable after y_n which does not appear in alphabetically the first (n + 1) sentences of the form (x)A.
>
> Then v_α is *regular* iff $v_\alpha((x)A) = F$ only if $v_\alpha((y_k/x)A) = F$, where (x)A is alphabetically the k^{th} sentence which begins with a universal quantifier.

Here (y/x)A is the result of replacing all free occurrences of x in A by occurrences of y, after rewriting bound variables if necessary to avoid confusion of bound variables. We assume without proof the familiar result that $v'(y/x)A = v(A)$ if v′ is like v except that $v'(y) = v(x)$, for any sentence A of $L_{\tau q}$ and any τq-valuations v′ and v.

The first result we need is that to the satisfaction of regular sets, only regular valuations are relevant.

Theorem 5. A regular set is satisfied by a valuation v_α if and only if it is satisfied by a regular valuation v'_α over the same τq-ms.

This is immediate from the following lemma:

Lemma 5: For any sentence A, if v_α and v'_α are valuations over the same τq-ms and alike with respect to all variables which occur in A, then $v_\alpha(A) = v'_\alpha(A)$.

This is proved by an easy induction on the length of A. We are now in a position to prove that wtih respect to the questions of

(3) The present formulation is slightly different from earlier formulations; the device is due independently to Beth and Hasenjaeger (Beth, loc. cit.).

finitary entailment (and *mutatis mutandis* for compactness) we need only consider the space of regular valuations.

Theorem 6. $X \vdash A$ in $L_{\tau q}$ if and only if all regular valuations which satisfy f(X) also assign T to f(A).

Proof. If $X \vdash A$ then $XU\{\sim A\}$ is not satisfiable, so $f(XU\{\sim A\})$ is not satisfiable (Theorem 4). *A fortiori,* $f(XU\{\sim A\})$ is not satisfied by any regular valuation; hence every regular valuation which satisfies f(X) also satisfies f(A).

If $X \vdash A$ does not hold, then $XU\{\sim A\}$ is satisfiable, hence $f(XU\{\sim A\})$ is satisfiable (Theorem 4). But $f(XU\{\sim A\})$ is a regular set, and must then also be satisfied by some regular valuation (Theorem 5). But then not every regular valuation which satisfies f(X) also satisfies f(A).

QED.

Let R_τ be the set of regular τq-valuations and $R_\tau(A)$ those members of R_τ which assign T to A. Then $H_{R\tau} = \langle R_\tau, \{R_\tau(A) : A \varepsilon W_q\}\rangle$ will be called the regular valuation space of $L_{\tau q}$. That $L_{\tau q}$ has finitary entailment now follows from Theorems 2, 6, and 7.

Theorem 7. Every ultrafilter on $H_{R\tau}$ converges to a valuation over a τq-ms with denumerable domain.

Proof. Let F(τ) now be the set of all ultrafilters on $H_{R\tau}$ and define the relation R(τ), set D(τ), and function f(τ) as follows:

R(τ) : as in Theorem 3.

D(τ) = *V* (the set of variables).

f(τ) the function assigning to each n-ary predicate P^n the set $f(\tau)_F(P^n) = \{\langle x_1, \ldots, x_n\rangle : R_\tau(P^n x_1 \ldots x_n)\varepsilon F\}$ for each F in F(τ).

Lemma 6. $M(\tau) = \langle F(\tau), R(\tau), D(\tau), f(\tau)\rangle$ is a τq-ms with denumerable domain.

Lemma 7. The mapping v such that v(x) = x for all x in *V*, and of $F(\tau) \times W_q$ into {T, F} such that $V_F(A) = T$ iff $R_\tau(A)\varepsilon F$, for all $A\varepsilon W_q$ for all F in F(τ), is a regular valuation over M(τ).

It is clear that each ultrafilter F on $H_{R\tau}$ converges to the τq-valuation v_F, a regular valuation over a model with a denumerable domain D(τ) = V. Hence it remains only to prove the lemmas.

Proof of lemma 6. That <F(τ), R(τ)> is a τ-ms is proved as for lemma 1. That D(τ) is a denumerable set, follows from the definition of QSynt. That the function f(τ) is as required by definition 14. also follows immediately from its definition.

QED.

Proof of lemma 7. That v is a valuation over <F(τ), R(τ)> is proved as in lemma 2. In addition $v_F(P^n x_1 \ldots x_n)$ = T iff R $(P^n x_1 \ldots x_n) \varepsilon F$, iff $<v(x_1), \ldots, v(x_n)> \varepsilon f(\tau)_F(P^n)$ by the definition of f(τ). Finally we prove that $v_F((x)A)$ = T iff $v'_F(A)$ = T for all v′ like v except perhaps at x. It is clear that such a v′ must assign a variable y, which is possibly not x, to the variable x, so that we need only prove that $R_\tau((x)A)\varepsilon F$ iff $R_\tau((y/x)A)\varepsilon F$ for all variables y. But $R_\tau((x)A) \subseteq R_\tau((y/x)A)$ and $(R_\tau\text{-}R_\tau((x)A)) \subseteq (R_\tau\text{-}R_\tau((y_k/x)A))$ where (x)A is alphabetically the k^{th} sentence to begin with a universal quantifier, by the definition of regular valuations. This ends the proof.

QED.

By Theorem 6, this implies that $L_{\tau q}$ has finitary entailment; in addition the following Löwenheim-Skolem theorem is a corollary to theorem 7.

Theorem 8. A set of sentences of $L_{\tau q}$ is satisfiable only if it is satisfiable by a valuation over a τq-ms with denumerable domain.

It is easy to see that this theorem can be generalized to the cardinality of the set of variables, which need not to kept denumerable. The extension to identity theory can also be carried out in the usual manner.

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