ON NATURAL DEDUCTION IN MODAL LOGIC WITH TWO PRIMITIVES (*)

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1. Modal logic is, of course, strongly analogous to quantificational logic. Now in intuitionistic quantificational logic, one takes both quantifiers, the existential and the universal, as primitive notions, instead of defining existence in terms of universality and negation, or universality in terms of existence and negation, as one usually does in classical logic.

Suppose we take also both modal operators, necessary and possible, as primitive. It then becomes necessary to state introduction and elimination rules for these two operators, necessary and possible, separately. Professor Fitch does state rules also for possibility (1), while most other writers do not formulate natural deduction rules for possibility since they take it to be a defined notion (2).

In Beth's and Nieland's treatment of modal semantic tableaus which are based upon Kripke's investigations, you get closed semantic tableaus which cannot directly be turned into a Jaskowskian lineair natural deduction, by a sheer graphical rearrangement. In order to obtain such a lineair deduction, one would first have to convert the closed semantic modal tableau into a closed deductive modal tableau. The deductive tableau

* This paper was read during the *3rd International Congress for Logic, Methodology and Philosophy of Science,* held in Amsterdam, 1967.

(1) F. B. FITCH, *Symbolic Logic,* New York, 1952, p. 71 f. Fitch discussed intuitionistic modal logic in: Intuitionistic Modal Logic with Quantifiers, *Portugaliae Mathematica,* vol. 7, Fasc. 2, 1948, where he takes a rule which corresponds to the "Barcan formula" as primitive. The rules for strict reiteration are found in: Natural Deduction Rules for Obligation, *American Philosophical Quarterly,* vol. 3 (1966), n° 1.

(2) "Evidently, if negation is present, one could define a possible statement as one whose negation is not necessary. But one should be able to define possibility directly." (H. B. CURRY, in *Foundations of Mathematical Logic,* New York etc., 1963, p. 360).

can then very easily be turned into a lineair, vertical natural deduction.

Beth and Nieland wanted to axiomatize the systems of semantic tableau rules based on Kripke's models, by showing that one can eliminate the applications of the modal semantic tableau rules one by one, inside the tableau, by adding applications of certain modal axiom schemes to the premises, allowing only one modal inference rule

$$\vdash U \rightarrow \vdash \Box U,$$

plus of course the usual non-modal principles for sentential logic. In order to show this, they used some rather special axioms which are not found elsewhere.

I believe that this was again because they did not make use of the intermediate method of *deductive* modal tableaus; (4) once the rules for making deductive tableaus are made clear, one sees almost at a glance which axiom schemes one needs, one reads them off the deductive tableau, so to speak.

The axioms one gets then are, as far as necessity is concerned, of course the same axioms which are listed for instance by

(3) E. W. BETH and J. J. F. NIELAND, Semantic Construction of Lewis's Systems S4 and S5, in: ADDISON, HENKIN, TARSKI (eds.), *The Theory of Models*, Amsterdam 1965. — See also: Compte-rendus des travaux effectués par l'université d'Amsterdam dans le cadre du Contrat Euratom; *Rapport CETIS N° 26*, 1961, reports n° 2, 6, and 7.

(4) E. M. BARTH, On the transformation of closed semantic tableaus into natural and axiomatic deductions, *Logique et Analyse*, vol. 9 (1966). — The recommendation given there in the last paragraph on p. 163, that the concludendum Z always be "treated" first, is too general, as is easily seen in the case of the sequents Av(BvC) / (AvB) v (AvC) and (Ex)[A(x)&B] / (Ey) A (y). For these sequents there are intuitionistically valid closed tableaus only if we deviate from the recommended rule. This can be explained in the following manner. Those tableau-rules which are justified by existential generalization (*eg*) and disjunction (disjunctive weakening, *dw*) imply that we make a choice as to how we want to derive (Ex)U(v) or U v V, while if the concludendum Z has any other form than one of these two, then there is only one manner in which one can exploit the logical form of Z in this system. A strategically wise choice can only be made with a maximum of information; therefore the exploitation of Z should in these cases be postponed until all those tableau-rules have been applied to the formulas in K which do not introduce a new concludendum.

Kripke (5). The introduction and elimination axioms for the modal operators are quite analogous to those for the quantifiers, with the addition of the Brouwer axioms and Becker's axioms.

I shall use Θ and K for classes of formulas and U, V and Z for single formulas. Z is the "concludendum" of the deduction problem which is to be solved.

Professor Fitch's rule for possibility elimination (6) can be stated as follows in our notation:

Θ_1
◇U
Θ_2
┌...................... ← *introduction of a new possible world*
—————┐ ← *introduction of a hypothesis*
U
Θ_3
V
—————┘ ← *withdrawal of hypothesis*
└...................... ← *leaving the possible world*
◇V **pe**-Fitch

In the first place, one may for systematic reasons split this rule into two, a possibility elimination **pe** which is modelled upon the rule of existential instantiation or exposition, and a rule **cr1** of contingent reiteration, by which you can "get out of" the possible world again and back into the world you just were in. The three rules for *contingent* reiteration (see below) are, I think, a necessary complementation of Fitch's recent rules for *strict* reiteration; (7) so we shall at least need the following 10 rules:

(5) S. A. KRIPKE, Semantical analysis of modal logic I, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik,* vol. 9 (1963).
(6) *Symbolic Logic,* p. 71.
(7) In: Natural Deduction Rules for Obligation, *American Philosophical Quarterly,* vol. 3, N° 1, p. 32.

Θ_1	Θ_1	Θ_1	Θ_1	Θ_1
$\Diamond$U	$\Box$U	$\Box$U	U	$\Box$U
Θ_2	Θ_2	Θ_2	Θ_2	Θ_2
⁂┌.	┌.		┌.	┌. . . . .
U **pe**	Θ_3	⁂ U **ne**	Θ_3	Θ_3
	⁂ U **sr1**		⁂$\Diamond$ U **sr2**	⁂$\Box$ U **sr3**
(all R)	(all R)	(R refl.)	(R symm.)	(R refl. & trans.)

	Θ_1	Θ_1	Θ	Θ_1	Θ_1
	┌.	┌.		┌.	┌. . . . .
	Θ_2	Θ_2	V	Θ_2	Θ_2
	V	V	⁂$\Diamond$ V **pi**	$\Box$Z	$\Diamond$V
(empty) $\rightarrow$					
	⁂└.	└.		└.	└. . . . .
	$\Box$V **ni**	⁂$\Diamond$ V **cr1**		⁂Z **cr2**	⁂$\Diamond$V **cr3**

2. Now it is clear that Fitch's rule for possibility elimination, or our possibility elimination **pe** together with **cr1**, does not (8) permit us to attack every deduction problem K_1, $\Diamond$U, K_2/Z, but only such problems where Z is $\Diamond$V for some V. We need a link between Z and $\Diamond$V for some V. As such, it suffices to assume $\Diamond\wedge\rightarrow\wedge$, where $\wedge$ is "the absurd" or "contradiction". This is obviously a logical identity or tautology whatever the relation R of one possible world to another, if we assume two-valuedness. The corresponding principle of natural deduction I shall call "the missing link".

(8) Unless R is symmetric; see below.

Then the deduction problem

Prem	Concl
K_1	
$\Diamond U$	
K_2	

can be attacked as follows:

Prem	Concl
K_1	Z $\wedge$-**elim**
$\Diamond U$	$\wedge$ **ml** (missing link)
K_2	$\Diamond\wedge$ **cr1**
⁂	....
Upe	$\wedge$

$\wedge \rightarrow Z$, which I shall call $\wedge$-elimination, is the (intuitionistic) principle which is rejected in Johansson's Minimal Logic (9).

In this way the premiss $\Diamond U$ can be exploited tactically for an arbitrary concludendum Z.

Another question is the following. What do we have to add, if anything, in order to be able to derive $\sim\Box\sim U\rightarrow\Diamond U$ (the one half of the definition of possibility in terms of necessity and negation). It is clearly a logical identity or tautology whatever the properties of the relation R; this can be shown by setting up a semantic tableau.

1	0
	$\sim\Box\sim U \rightarrow \Diamond U$
$\sim\Box\sim U$	$\Diamond U$ [*ind. proof*]
[$\sim\Diamond U$ *hyp*]	[$\Diamond U$]
	$\Box\sim U$
....	....
	$\sim U$
U	
	U

(9) I. Johansson, *Der Minimalkalkül,* ein reduzierter intuitionistischer Formalismus, *Compositio Math.,* vol. *4* (1936).

A deductive tableau does not close, however, with the rules we have adopted so far, for $\Box \sim U$ replaces $\Diamond U$ as "concludendum" as soon as it enters the tableau, so we cannot get hold of the undermost formula U on the right.

In non-modal logic one can of course give a classical natural deduction of the corresponding quantificational formula, $\sim(v) \sim U(v) \rightarrow (Ev)U(v)$. The device, as far as the tableau-technique is concerned, is there to conserve $(Ev)U(v)$ as $\sim(Ev)U(v)$ on the left, by applying the intuitionistically unacceptable rule of indirect proof, and afterwards using the *ex falso quodlibet* when this formula $(Ev)U(v)$ is needed again on the right (see formulas in brackets).

Prem	Concl
	$\sim(v)\sim U(v) \rightarrow (Ev)U(v)$
$\sim(v)\sim U(v)$	$(Ev)U(v)$ *ind. proof*
[$\sim(Ev)U(v)$]	$(Ev)U(v)$ *ex falso quodlibet*
	$(v)\sim U(v)$ *univ. generalization*
	$\sim U(p)$ *red. ad abs.*
$U(p)$	$\sim U(p)$ *ex falso quodlibet*
	[$(Ev)U(v)$ *e.g.*]
	$U(p)$ *triv.*

However, this does not suffice in the case of modal operators, for the occurrence of $\Box \sim U$ on the right leads to the introduction of a new world, and we need $\Diamond U$ or U on the right in that new world. This can be achieved by adopting a natural deduction rule which may be called the *ex impossibili necessitate quodlibet,*

Θ_1
$\sim\Diamond U$
Θ_2
⌈. . . . .
Θ_3
U
⁂ Z **einq**

the axiomatic version of which is

(1) $\sim\Diamond U \rightarrow \Box(U \rightarrow Z)$

or:

(2) $\sim\Diamond U \rightarrow \Box(U \rightarrow \wedge)$ in minimal logic.

One can of course use a stronger rule or formula

(3) $\sim\Diamond U \rightarrow \Box(\Diamond U \rightarrow Z)$

but the first one is sufficient to prove completeness.

Because of its similarity to the elimination rule for negation, corresponding to the axiom scheme

(4) $\sim U \rightarrow (U \rightarrow Z)$ (*ex falso quodlibet*),

the formula (1) reflects more clearly what one does in a modal deductive tableau from a purely formal or graphical point of view, than does the formula $\sim\Box\sim U \rightarrow \Diamond U$, which is a part of the definition of possibility in terms of necessity and negation.

However, when R is symmetric, the *ex impossibili* as well as the missing link can be missed:

	can be replaced by	
.		.
.		.
.		.
.		.
$\wedge$		$\wedge$
⌊.		$\Box\wedge$ ($\wedge$-**elim.**)
$\Diamond\wedge$ **cr1**		⌊. . . . .
⁂ $\wedge$ **ml**		⁂ $\wedge$ **cr2**

(This is not a derivation of the missing link-principle, but shows that this principle is not needed. But the principle itself is also derivable when we have access to **cr2**). Further, any application of the *ex impossibili*:

	can be replaced by	
Θ_1		Θ_1
$\sim\Diamond U$		$\sim\Diamond U$
Θ_2		Θ_2
⌈		⌈ . . . .
Θ_3		Θ_3
U		U
⁂ Z **einq**		$\Diamond\sim\Diamond U$ **sr2**
		‖⌈ . . . . .
		.
		.
		$\sim\Diamond U$ **pe**
		$\Diamond U$ **sr2**
		$\Box Z$ **efq**
		‖⌊ . . . . .
		⁂ Z **cr2**

Instead of the *ex impossibili,* which is modelled upon the *ex falso,* one might take as the new modal rule one which is modelled on the *reductio ad absurdum* in the formulation in which it furnishes an introduction rule for negation: $(U \rightarrow \sim U) \rightarrow \sim U$, and in which form it is intuitionistically valid. The modal version (in axiomatic form).

(5) $(\Diamond U \rightarrow \sim\Diamond U) \rightarrow \Box\sim U$ (*modal red. ad abs.*)
can be used instead of (1) to bring a deductive tableau for $\sim\Box\sim U \rightarrow \Diamond U$ to a closure.

However, both (1) and (5) can be derived from

(6) $(\Diamond U \rightarrow \wedge) \rightarrow \Box(U \rightarrow \wedge)$
if negation, $\sim U$, is defined by: $(U \rightarrow \wedge) \rightarrow \sim U$ and $\sim U \rightarrow (U \rightarrow \wedge)$, and if we accept $\wedge$-elimination, $\wedge \rightarrow Z$. (There is, then, no introduction rule for $\wedge$, as there is in Gentzen's

N-system; in this sense, $\wedge$ is undefined. Heyting takes contradiction to be an undefined notion ([10])).

3. In all systems built on a reflexive and symmetric R, i.e., in the systems B and S5, we can give simple constructive proofs of the so-called Barcan formula

$$(v)\Box U(v) \rightarrow \Box(v)U(v)$$

and the related formula

$$\Diamond(Ev)U(v) \rightarrow (Ev)\Diamond U(v).$$

Prem	Concl
Ø	$(v)\Box U(v) \rightarrow \Box(v)U(v)$ **cond**
	$\Box(v)U(v)$ **ni**
	$(v)U(v)$ **ug**
	$U(p)$ **cr2**
$\Diamond(v)\Box U(v)$ **sr2**	
$(v)\Box U(v)$ **pe**	
	$\Box U(p)$ **triv**
$\Box U(p)$ **uinst**	

Prem	Concl
Ø	$\Diamond(Ev)U(v) \rightarrow (Ev)\Diamond U(v)$ **cond**
$\Diamond(Ev)U(v)$	$(Ev)\Diamond U(v)$ **cr2**
$(Ev)U(v)$ **pe**	
	$\Box(Ev)\Diamond U(v)$ **ni**
$U(p)$ **einst**	
	$(Ev)\Diamond U(v)$ **eg**
$\Diamond U(p)$ **sr2**	
	$\Diamond U(p)$ **triv**.

([10]) A. HEYTING, *Intuitionism*, Amsterdam 1956, p. 98.

Prior [11] derived the Barcan formula in 1956 in S5 as superimposed upon a quantificational logic which is, I think, for these purposes equivalent to the one used here. Hintikka [12] and Föllesdal showed independently of each other in 1961 that for deriving the Barcan formula you only need the reflexivity and symmetry of R, not the transitivity.

Lately, some logicians, notably W. H. Hanson and L. Åqvist, have taken up semantic tableau techniques for the study of philosophically interesting topics like deontic logic and tense logic. Åqvist studied a deontic tense logic he chose to call DDT, but said that he had not yet checked all details in the completeness proof of his axiom system for this logic [13]. This is however easily done along the lines suggested here, that is by intercalating a deductive tableau method for DDT, and showing that every closed *semantic* DDT-tableau can be transformed automatically into a closed *deductive* DDT-tableau, and that the steps in the deductive tableau can be removed by applications of his axioms.

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(11) A. N. PRIOR, Modality and quantification in S5, *Journal of Symbolic Logic,* vol. *21,* 1956.

(12) J. HINTIKKA, Modality and quantification, *Theoria,* vol. *27* (1961); D. Föllesdal, *Referential Opacity and Modal Logic* (1966). See also D. PRAWITZ, *Natural Deduction* (1965), p. 78.

(13) W. H. HANSON, Semantics for deontic logic, *Logique et analyse,* vol. *8* (1965); L. ÅQVIST, "Next" and "Ought". Alternative foundations for Von Wright's tense-logic, with an application to deontic logic, *Logique et analyse,* vol. *9* (1966).