A SIMPLE NATURAL DEDUCTION SYSTEM

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A natural deduction system **ND** is introduced and proofs in it illustrated. This system is much simpler and more flexible than familiar natural deduction systems: it avoids completely such complicating devices as subscripting, flagging, ordering of variables, and distinctions of several sorts of variables or parameters. Furthermore proofs are valid line by line.

ND is presented as a formal system. It still needs emphasizing that natural deduction systems can be made just as formal as Hilbert-type systems. The precise conditions on substitution — quite important for natural deduction but often suppressed — are made explicit in the rules of **ND**.

The correctness of **ND** is shown by comparing **ND** with a Hilbert-type system **H**. **H** has as formal theorems effectively the theorems of restricted predicate logic and of Hilbert's ε-system. Both **ND** and **H** have as primitive symbol the term-forming indefinite description operator 'ε'. 'ε' is formally similar to Hilbert's ε-symbol, (1) but it is intended to have a quite different interpretation, '$\varepsilon x f(x)$' is read 'any x which is f', *not* 'an x which is f provided there exists an x which is f'. An item x may not exist or even be possible. Likewise the quantifiers 'A' and 'S', though formally similar to '∀' and '∃', have different intended interpretations. The non-ontological quantifier 'S' reads 'for some', and not 'there exists a'.

A well-formed formula A of **ND** or of **H** is *ε-free* if no ε-term does occur in A. It is shown that the ε-free theorems of **ND** coincide formally with the theorems of restricted predicate logic.

In the final section **ND** and a simpler system $\mathbf{ND}_1$ are investigated independently of classical predicate logics. Interpretations

(1) For details see D. HILBERT and P. BERNAYS, *Grundlagen der Mathematik*, vol. II, Berlin (1939), § I.

and validity are defined for these systems; consistency, completeness and Skolem-Löwenheim theorems are easily reached.

I

The primitive frame of **ND**

Vocabulary

$\supset$	$\sim$	A	S	ε	(	)	
x	y	z	x'	y'	z'		
f	g	h	f'	g'	h'	f'' ..	
p	q	r	p'	....			

Formation rules

1. A sentential variable p is a wff.
2. If f is an n-ary predicate and $z_1 \dots z_n$ are n individual variables or wf terms, then $f(z_1 \dots z_n)$ is a wff.
3. If A is a wff, $\sim$A is a wff.
4. If A and B are wff, then $(A \supset B)$ is a wff.
5. If A is a wff and x an individual variable, then εxA is a wf term.
6. If A is a wff and x an individual variable, then (Ax)A and (Sx)A are wff.

Transformation rules

1. *Hypothesis introduction*: $\dfrac{\cdot}{A} \,.\, (A)$

2. *Hypothesis elimination*: $\dfrac{A_1\ ,\ A_2}{A_1 \supset A_2} \,.\, \sim(A_1)$, where A_1 is the last hypothesis of A_2.

3. *A-introduction*: $\dfrac{B}{(Ax)\,B}$

provided variable x is not free in an hypothesis of B.

4. *A-elimination*: $\dfrac{(\mathrm{A}x)\,\mathrm{B}}{\mathrm{S}^{x}_{z}\,\mathrm{B}\,|}$

where z an individual variable or wf term.

5. *S-introduction*: $\dfrac{\mathrm{S}^{x}_{z}\,\mathrm{B}|}{(\mathrm{S}x)\,\mathrm{B}|}$

where z an individual variable or wf term.

6. *S-elimination*: $\dfrac{(\mathrm{S}x)\,\mathrm{B}}{\mathrm{B}_{x}\,(\varepsilon_{y}\mathrm{B})}$.

7. *Sentential derivation*: $\dfrac{\mathrm{A}_1, \mathrm{A}_2 \ldots \mathrm{A}_n}{\mathrm{B}}$

provided $\mathrm{A}_1 \supset .\ \mathrm{A}_2 \supset \ldots \supset .\ \mathrm{A}_n \supset \mathrm{B}$ is a substitution-instance of a tautology of sentential logic.
(By admitting the case where $n = 0$ some trivial steps could be removed from later derivations.)

ND has no axioms.

A *deduction in* **ND** is a finite sequence of wff each of which is derived from preceding wff in the sequence (including null wff) according to the transformation rules of **ND**. A deduction is a *completed deduction* if all hypotheses (introduced by hypothesis introduction) have been eliminated (by hypothesis elimination) by the end of the sequence.

A wff A is a *theorem* of **ND** iff A is the last member of a completed deduction.

The substitution notation is explained as follows:

1. $\mathrm{S}^{u}_{\mathrm{N}}\mathrm{A}\,|$ is the wff resulting from substituting variable, wf term or wff N for all free occurrences of variable u in A, provided that no free occurrence of u in A is in a wf part $(\mathrm{A}w)\mathrm{C}$ or $(\mathrm{S}w)\mathrm{C}$ or $\varepsilon w\mathrm{C}$ of A, where w is a free variable of N (or is N if N is a variable); and otherwise is A.

2. $\mathrm{S}'^{u}_{v}\mathrm{A}\,|$ is the wff resulting from substituting variable v for all (free) occurrences of variable u in wff A, provided v is free in the resulting wff where and only where u is free in A; and otherwise is A.

3. $Bx(\varepsilon yC) = \mathrm{\check{S}}^{x}_{\varepsilon y S'^{x}_{y} C} \; B \,|$

4. A is a *substitution-instance* of wff B of sentential logic iff A results from B by simultaneous substitution of ε-variants $C_1^x, \ldots, C_n^x$ of wff $C_1, \ldots, C_n$ for distinct variables $p_1, \ldots, p_n$ in B, where $\mathrm{\check{S}}^{x}_{\varepsilon y S'^{z}_{y} C} \, A \,|$ is an *ε-variant* of $\mathrm{\check{S}}^{x}_{\varepsilon z C} \, A \,|$. For example, $f(\varepsilon y g(y)) \equiv f(\varepsilon z g(z))$ is a substitution-instance of $p \equiv p$.

Sample theorems of **ND**

1. $(Sy)(Ax)f(x,y) \supset (Ax)(Sy)f(x,y)$

Proof:	$(Sy)(Ax)f(x,y)$	$((Sy)(Ax)f(x,y))$
	$(Ax)f(x, \varepsilon y(Ax)f(x,y))$	
	$f(x, \varepsilon y(Ax)f(x,y))$	
	$(Sy)f(x,y)$	
	$(Ax)(Sy)f(x,y)$	
	$(Sy)(Ax)f(x,y) \supset (Ax)(Sy)f(x,y)$	$\sim((Sy)(Ax)f(x,y))$

2. $(Ax)g(x) \supset (Sx)g(x)$

Proof:	$(Ax)g(x)$	$((Ax)g(x))$
	$g(x)$	
	$(Sx)g(x)$	
	$(Ax)g(x) \supset (Sx)g(x)$	$\sim((Ax)g(x))$.

Proofs can be presented more perspicuously by using a lining technique to show the extent of hypotheses. In place of expressions written to the right of a (vertical) proof in hypothesis introduction and elimination, a line '⌈' will be introduced, when an hypothesis A is introduced, to the left of the proof but to the right of all lines already appearing to the left of the proof sequence. This line will be continued down the proof until the last member of the sequence before hypothesis A is eliminated at which stage the line will terminate. Because of the condition on hypothesis elimination, lines will never cross. The lining technique has a similar status to familiar conventions for simplifying primitive bracketing notation. Derivations can also be clarified by writing in the margin to the left of a sequence element indications as to how that element is obtained. These techniques will be illustrated in succeeding proofs.

3. $(\mathrm{S}x)(p \vee f(x)) \supset . \, p \vee (\mathrm{S}x)f(x)$

Proof:

$(\mathrm{S}x)(p \vee f(x))$	H-introduction
$p \vee f(\varepsilon y(p \vee f(y)))$	S-elimination
$f(\varepsilon y(p \vee f(y)))$	H-introduction
$(\mathrm{S}x)f(x)$	S-introduction
$f(\varepsilon y(p \vee f(y))) \supset (\mathrm{S}x)f(x)$	H-elimination
$p \vee (\mathrm{S}x)f(x)$	S-derivation
$(\mathrm{S}x)(p \vee f(x)) \supset . \, p \vee (\mathrm{S}x)f(x)$	H-elimination

4. $\sim (\mathrm{A}x) \sim f(x) \equiv . \, (\mathrm{S}x)f(x)$

Proof:

$f(x)$	
$(\mathrm{S}x)f(x)$	S-introduction
$f(x) \supset (\mathrm{S}x)f(x)$	
$\sim (\mathrm{S}x)f(x) \supset \sim f(x)$	S-derivation
$\sim (\mathrm{S}x)f(x)$	
$\sim f(x)$	S-derivation
$(\mathrm{A}x) \sim f(x)$	A-introduction
$\sim (\mathrm{S}x)f(x) \supset (\mathrm{A}x) \sim f(x)$	
$\sim (\mathrm{A}x) \sim f(x) \supset (\mathrm{S}x)f(x)$	S-derivation
$(\mathrm{S}x)f(x) \,\&\, (\mathrm{A}x) \sim f(x)$	
$(\mathrm{S}x)f(x)$	
$f(\varepsilon y f(y))$	S-elimination
$(\mathrm{A}x) \sim f(x)$	S-derivation
$\sim f(\varepsilon y f(y))$	A-elimination
$f(\varepsilon y f(y)) \,\&\, \sim f(\varepsilon y f(y))$	S-derivation
$(\mathrm{S}x)f(x) \,\&\, (\mathrm{A}x) \sim f(x) \supset . \, f(\varepsilon y f(y)) \,\&\, \sim f(\varepsilon y f(y))$	
$f(\varepsilon y f(y)) \supset f(\varepsilon y f(y)) \supset . \, (\mathrm{S}x)f(x) \supset \sim (\mathrm{A}x) \sim f(x)$	S-derivation
$f(\varepsilon y f(y))$	
$f(\varepsilon y f(y))$	S-derivation
$f(\varepsilon y f(y)) \supset f(\varepsilon y f(y))$	
$(\mathrm{S}x)f(x) \supset \sim (\mathrm{A}x) \sim f(x)$	S-derivation
$\sim (\mathrm{A}x) \sim f(x) \equiv . \, (\mathrm{S}x)f(x)$	S-derivation

These proofs illustrate several of the techniques used in **ND** proofs. In practice the proofs would be written in a much abbreviated form.

Two *derived rules* of **ND**.

1. *A quantifier conversion rule*: $\dfrac{\sim(\mathrm{A}x)\sim\mathrm{B}}{(\mathrm{S}x)\mathrm{B}}$.

The proof directly generalises upon 4. above.

2. Extended *A-introduction*: $\dfrac{\mathrm{S}'^{x}_{y}\,\mathrm{B}|}{(\mathrm{A}x)\,\mathrm{B}}$,

provided variable y is not free in an hypothesis of the premiss.

Proof: Case 1. $\mathrm{S}'^{x}_{y}\mathrm{B}|$ is B. Then the result is immediate from A-introduction.

Case 2. $\mathrm{S}'^{x}_{y}\mathrm{B}|$ is not B.

⌈ $(\mathrm{A}y)\mathrm{S}'^{x}_{y}\mathrm{B}|$	
$\mathrm{S}'^{y}_{x}\,\mathrm{S}'^{x}_{y}\,\mathrm{B}\,||$	A-elimination
B	$\mathrm{S}'^{y}_{x}\,\mathrm{S}'^{x}_{y}\,\mathrm{B}\,||$ is B
$(\mathrm{A}x)\mathrm{B}$	A introduction; in this case x is not free in the hypothesis
$(\mathrm{A}y)\mathrm{S}'^{x}_{y}\,\mathrm{B}| \supset (\mathrm{A}x)\mathrm{B}$	

Now (1) $\mathrm{S}'^{x}_{y}\,\mathrm{B}|$

$(\mathrm{A}y)\mathrm{S}'^{x}_{y}\,\mathrm{B}|$,	provided y is not free in an hypothesis of (1)
$(\mathrm{A}x)\mathrm{B}$	S-derivation

Pathological examples which illustrate typical violations of flagging, ordering and related restrictions in familiar natural deduction systems.

1. $(\mathrm{S}x)g(x) \supset (\mathrm{A}x)g(x)$

Attempted proof:

$(\mathrm{S}x)y(x)$
$g(\varepsilon y g(y))$
$(\mathrm{A}x)g(x)$

This last step is illegal: no rule permits generalisation on a wf term.

2. $f(y) \supset (\mathrm{A}x)f(x)$

Attempted proof:

$f(y)$
$(\mathrm{A}x)f(x)$

This step is invalid; for generalisation has been made on a variable free in the hypothesis.

3. $(\mathrm{S}x)f(x)\ \&\ (\mathrm{S}x)g(x) \supset .\ (\mathrm{S}x)(f(x)\ \&\ g(x))$.

Attempted proof:

$(\mathrm{S}x)f(x)\ \&\ (\mathrm{S}x)g(x)$
$(\mathrm{S}x)f(x)$
$f(\varepsilon y f(y))$
$(\mathrm{S}x)g(x)$
$g(\varepsilon y g(y))$
$f(\varepsilon y f(y))\ \&\ g(\varepsilon y g(y))$
$(\mathrm{S}x)(f(x)\ \&\ g(x))$.

But no rule permits S-introduction on different ε-terms.

4. $(\mathrm{A}x)(\mathrm{S}y)f(x,y) \supset (\mathrm{S}y)(\mathrm{A}x)f(x,y)$.

Attempted proof:

$(\mathrm{A}x)(\mathrm{S}y)f(x,y)$
$(\mathrm{S}y)f(x,y)$
$f(x,\varepsilon y f(x,y))$
$(\mathrm{A}x)f(x, \varepsilon y f(x,y))$
$(\mathrm{S}y)(\mathrm{A}x)f(x,y)$

This step is invalid: the introduction of 'S' here does not accord with the S-introduction rule. $(Ax)f(x, \varepsilon y f(x, y))$ is not $S^{y}_{\varepsilon y f(x, y)}\ (Ax)f(x, y) \mid$, since x would be bound under the substitution. This kind of failure can be foreseen: it is preceded by the binding up of a variable free in an ε-term, in particular by A-introduction. Note that:

$(Ax)(Sy)f(x,y) \supset (Ax)f(x,\varepsilon y f(x,y))$ is an ε-theorem.

In 1955 Wang ([2]) suggested combining use of Hilbert's ε-symbol with the method of natural deduction as developed by Quine. But the system **Lg** which Wang proposed is inadequate: for either it permits the derivation of pathological examples like 2. and 4., or, interpreted differently as suggested by the assertion signs, it sanctions very few theorems.

II

The system **H**. The vocabulary and formation rules of **H** are like those of **ND** except that quantifiers 'A' and 'S' are omitted throughout.

Axiom schemata

H1: A, provided A is a substitution instance of a tautology of sentential logic.

H2: $\check{S}^{x}_{z}\ B \mid \supset Bx(\varepsilon y B)$, z an individual variable or wf term (weak ε-scheme).

Transformation rules

RH1: $A, A \supset B \rightarrow B$. (modus ponens)

RH2: $B \rightarrow Bx(\varepsilon y \sim B)$. (generalisation)

H1 could be extended to include RH2.

([2]) See Hao Wang, *A Survey of Mathematical Logic*, Science Press - Peking and North-Holland Publishing Company - Amsterdam (1963), 315-316. A related idea was introduced by Hailperin: see T. Hailperin, 'A Theory of Restricted Quantification II', *The Journal of Symbolic Logic*, Volume 22 (1957), 118-126.

Definitions

1. $(Sx)B =_{Df} Bx(\varepsilon yB)$.
2. $(Ax)B =_{Df} Bx(\varepsilon y \sim B)$.

Theorem 1: Every theorem of **H** is a theorem of **ND**.

Proof: The axioms, rules and definitional equivalences of **H** are derived for **ND**.

H1 ⌈B
| B
B ⊃ B
A , provided A is a substitution instance of a tautology of sentential logic: by S-derivation.

H2 ⌈$\overset{\circ}{S}{}^x_z\, B \mid$
| $(Sx)B$ S-introduction
| $Bx(\varepsilon yB)$ S-elimination
$\overset{\circ}{S}{}^x_z\, B \mid \supset Bx(\varepsilon yB)$.

RH1. If A and A ⊃B are theorems of **ND** then B is a theorem of **ND**. For the completed deductions of A and A ⊃B can be combined. Then, using S-derivation, B can be obtained as the conclusion of a completed deduction.

RH2. If B is a theorem of **ND** then B is the conclusion of a completed deduction. To this deduction the following elements are added:

	B	
	$(Ax)B$	A-introduction, since B has no hypotheses.
(2)	$Bx(\varepsilon y \sim B)$	A-elimination.

Since (2) is the conclusion of a completed deduction, (2) is also a theorem of **ND**.

Definition 1. ⌈$(Sx)B$
| $Bx(\varepsilon yB)$
$(Sx)B \supset Bx(\varepsilon yB)$
⌈$Bx(\varepsilon yB)$
| $(Sx)B$ S-introduction

$Bx(\varepsilon yB) \supset (Sx)B$
$(Sx)B \equiv . Bx(\varepsilon yB)$ S-derivation

Definition 2.

$(Ax)B$
$Bx(\varepsilon y \sim B)$
$(Ax)B \supset Bx(\varepsilon y \sim B)$ A-elimination
$\sim B$
$(Sx) \sim B$
$\sim B \supset (Sx) \sim B$
$\sim (Sx) \sim B \supset B$
$\sim (Sx) \sim B$
B
$(Ax)B$
$\sim (Sx) \sim B \supset (Ax)B$

$\sim (Ax)B \supset (Sx) \sim B$
$\sim (Ax)B$
$(Sx) \sim B$
$\sim Bx(\varepsilon y \sim B)$
$\sim (Ax)B \supset \sim Bx(\varepsilon y \sim B)$
$Bx(\varepsilon y \sim B) \supset (Ax)B$
$(Ax)B \equiv Bx(\varepsilon y \sim B)$

Thus every proof in **H** can be replaced by a completed deduction in **ND**.

An ε-free formula (sequence) A of **H** is a *relettered wff* (*theorem, proof*) of Church's $\mathbf{F}^{1P}$ if A is obtained from a wff (theorem, proof) A′ of $\mathbf{F}^{1P}$ by replacing each symbol in A′ by its corresponding symbol in the vocabulary of **H**, e.g. 'F', by 'f', 'G' by 'g', '∀' by 'A' [3].

Theorem 2: If A is a relettered wff of Church's $\mathbf{F}^{1P}$, then A is a wff of **H**.

Proof: By induction over the formation rules of $\mathbf{F}^{1P}$ and **H** once the correlation between symbols is set up explicitly.

Theorem 3: Every relettered theorem of Church's $\mathbf{F}^{1P}$ is a theorem of **H**.

Proof: The propositional postulates of $\mathbf{F}^{1P}$ follow directly from H1;the rules of $\mathbf{F}^{1P}$ are immediate. It remains to establish relettered quantificational schemata of $\mathbf{F}^{1P}$:

(3) For details of $\mathbf{F}^{1P}$ see A. Church, *Introduction to Mathematical Logic,* vol. I, Princeton (1956), 169-173.

(1) $(Ax)A \supset \check{S}^x_z A|$, z a variable or wf term;
(2) $(Ax)A \supset B) \supset . A \supset (Ax) B$, x not free in A;
and the scheme
(3) $(Sx)A \equiv \sim(Ax)\sim A$.
ad (1) $\check{S}^x_z \sim A \supset . \sim Ax(\varepsilon y \sim A)$. H2.

Result by contraposition and definition 2.

ad (2) $\check{S}^x_{\varepsilon y \sim B} \sim (A \supset B) | \supset . \sim(A \supset Bx(\varepsilon y \sim(A \supset B)))$, x not free in A. H2.

⁂ $\sim(A \supset Bx(\varepsilon y \sim B)) \supset . \sim(A \supset Bx(\varepsilon y \sim (A \supset B)))$, x not free in A.

⁂ $A \supset Bx(\varepsilon y \sim B(A \supset B)) \supset . A \supset Bx(\varepsilon y \sim B)$, x not free in A.

⁂ $(Ax)(A \supset B) \supset .A \supset (Ax)B$, x not free in A. Definition 2.

ad (3). 1.	$\sim(Ax)\sim B \equiv \sim \sim Bx(\varepsilon y \sim \sim B)$	Definition 2.
2.	$\equiv (Sx)\sim \sim B$	Definition 1.
3.	$Bx(\varepsilon y \sim \sim B) \supset . Bx(\varepsilon y \sim \sim B)$	H2.
4.	$\sim \sim Bx(\varepsilon y B) \supset . \sim \sim Bx(\varepsilon y \sim \sim B)$	H2.
5.	$\sim \sim Bx(\varepsilon y \sim \sim B) \equiv . Bx(\varepsilon y B)$	3, 4.
6.	$(Sx)\sim \sim B \equiv . (Sx)B$	5, Definition 1.
7.	$(Sx)B \equiv \sim(Ax)\sim B$	2, 6.

Theorem 4: Every relettered theorem of Church's $\mathbf{F}^{1P}$ is a theorem of **ND**. By theorems 1 and 3.

A deduction theorem for **H**

A finite sequence of wff $B_1, B_2, \ldots B_m$ of **H** is a *proof from hypotheses* $A_1, A_2 \ldots A_n$ *in* **H** if for each i either

(1) B_i is one of $A_1, A_2 \ldots A_n$,

or

(2) B_i is an axiom,

or

(3) B_i is inferred by modus ponens from major premiss B_j and minor premiss B_k where $j<i$, $k<i$,

or

(4) B_i is inferred by A-elimination from B_j, where $j<i$,

or

(5) B_i is inferred by S-introduction from B_j, where $j<i$,

or

(6) B_i is inferred by A-introduction from B_j, where $j<i$,

or

(7) B_i is inferred by S-elimination from B_j, where $j<i$.

'$A_1 \ldots A_n \vdash B_m$' symbolises 'there is a proof of B_m from hypotheses $A_1 \ldots A_n$'.

Theorem 5 (Deduction theorem): If $A_1, \ldots A \vdash B_n$, then $A_1, \ldots A_{n-1}, \vdash A_n \supset B$.

Proof: Let $B_1, B_2, \ldots B_m$ be a proof of $B(=B_m)$ from hypotheses $A_1, \ldots A_n$. Construct the finite sequence of wff $A_n \supset B_1, A_n \supset B_2$, $A_n \supset B_m$. It is shown how to insert a finite number of additional wff into this sequence so that the resulting sequence is a proof of $A_n \supset B$ from hypotheses $A_1, \ldots A_n$. The following cases are exhaustive:

Cases 1a, 1b, 2, 3: as in Church [4] apart from minor modifications.

Case 4: B_i is $\check{S}^x_z B \mid$ and B_j is $(Ax)B$. Insert before $A_n \supset B_i$ the theorem $(Ax)B \supset \check{S}^x_z B \mid$ and its consequence $A_n \supset (Ax)B \supset . A_n \supset \check{S}^x_z B \mid$.

$A_n \supset B_i$ then results by RH1.

Case 5. B_i is $(Sx)B$ and B_j is $\check{S}^x_z B \mid$, for some B. Insert before $A_n \supset B_i$ the theorem $\check{S}^x_z B \mid \supset (Sx)B$, an immediate consequence of H2, and its consequence $A_n \supset \check{S}^x_z B \mid \supset . A_x \supset (Sx)B$. $A_n \supset B_i$ results by RH1.

Case 6: B_i is $(Ax)B_j$, where x is not free in $A_1, A_2 \ldots A_n$. Insert before $A_n \supset B_i$ the theorem $(Ax)(A_n \supset B_j) \supset . A_n \supset B_i$, and $(Ax)(A_n \supset B_j)$ which results by RH2. $A_n \supset B_i$ then follows by RH1.

(4) A. CHURCH, *op. cit.*, 197-198.

Case 7: B_i is $Bx(\varepsilon yB)$ and B_j is $(Sx)B$. Insert $B_j \supset B_i$ (from definition 1) and $A_n \supset B_j \supset . A_n \supset B_i$ before $A_n \supset B_i$. Then apply RH1.

Theorem 6: If every wff in the set $A_1, A_2, \ldots A_n$ also occurs in the set $C_1, C_2, \ldots C_r$ and if $A_1, A_2, \ldots A_n \vdash B$, then $C_1, C_2, \ldots C_r \vdash B$. The proof is an adaption of Church's *362 (5). Only generalisations upon individual variables need however be considered.

Theorem 7: A is a theorem of **H** iff there is a proof of A from null hypotheses in **H**.

Proof: If A is a theorem, then the proof of A provides a proof of A from null hypotheses. Conversely if there is a proof of A from null hypotheses, then the proof of the deduction theorem specifies effective methods for converting this proof sequence in a proof of A. For each derived rule, used in cases (4)-(7), is there vindicated by derivation from rules and theorems of **H**. The result then follows using the definition of 'theorem of **H**'.

Theorem 8: Every theorem of **ND** is a theorem of **H**.

Proof: It is shown that a completed deduction with A as last member in **ND** can be effectively replaced by a proof of A in **H**. The argument is by induction over the length of deductions in **ND**. The theorem then follows by generalisation.

Suppose, for induction hypothesis, that a deduction $B_1, B_2, \ldots B_{m_k}$ in **ND** of B_{m_k} from hypotheses $A_1, A_2, \ldots A_{n_k}$ can be replaced for **H** by the set of proofs from hypotheses:

$$A_1, \quad A_{n_1} \vdash B_{m_1}, \qquad (1_1)$$

$$\vdots$$

$$A_1, \ldots A_{n_k} \vdash B_{m_k} \qquad (1_{2k}),$$

where the set of hypotheses $A_1, A_2, \ldots A_{n_i}$ of (1_i) is included (properly or improperly) in the set of hypotheses of (1_{i+1}) for

(5) A. Church, *op. cit.*, 199-200.

each i, and $B_{m_1}, \ldots B_{m_k}$ belong to the proof sequence $B_1, \ldots, B_{m_k}$.

On this induction hypothesis it is shown that a deduction sequence in **ND** to $B_{(m+1)_k}$, the successor, in the deduction of A to B_{m_k} can be replaced for **H** either (i) by the set of proofs from hypotheses

$$A_1, \ldots A_{n_1} \vdash B_{m_1}, \qquad (1_1)$$

$$\vdots$$

$$A_1, \ldots A_{n_k} \vdash B_{(m+1)_k}, \qquad (1_k')$$

where $A_1, \ldots A_{n_k}$ and $B_{m_1}, \ldots B_{mk-1}$ are as before:

or (ii), just in the case that an hypothesis is eliminated in the **ND** proof sequence, by the set of proofs from hypotheses

$$A_1, \ldots A_{n_1} \vdash B_{m_1} \qquad (1_1)$$

$$\vdots$$

$$A_1, \ldots A_{n_j} \vdash B_{(m+1)_k}, \qquad (1_j),$$

where $B_{(m+1)_k}$ is $A_{n_k} \supset B_{m_k}$ and where A_{n_j} is the last hypothes-is introduced in the **ND** sequence before A_{n_k};

or (iii), just in the case that a new hypothesis is introduced in the **ND** proof sequence, by the set of proofs from hypotheses:

$$A_1, \quad A_{n_1} \vdash B_{m_1}, \qquad (1_1)$$

$$\vdots$$

$$A_1, \ldots A_{n_k} \vdash B_{m_k}' \qquad (1_k)$$

$$A_1, \ldots A_{n_k}, B_{(m+1)_k} \vdash B_{(m+1)_k} \qquad (1_{k+1})$$

The complication can be attributed to the occurrence of sub proofs. There are three cases to consider:-

(a) $B_{(m+1)_k}$ is joined to the deduction sequence in **ND** of A by sentential derivation, A-elimination, S-introduction, A-introduction or S-elimination. Then case (i) obtains, and $(1'^k)$ is obtained from (1_k) and/or (1_1)—(1_{k-1}) by corresponding steps permitted in a proof of $B_{(m+1)_k}$ from hypotheses in **H** and by application of theorem 6. For example, if B_{m_j} is (Ax)C and $B_{(m+1)_k}$is $\dot{S}^x_z C \mid$ and $B_{(m+1)_k}$ is obtained from B_{m_j} in **ND** by A-elimination, then (l_k') is obtained from (l_j) by case (5) given under the characterization of a proof from hypotheses and theorem 6. Note that the conditions on A-introduction are guaranteed by the **ND** proof sequence.

(b) $B_{(m+1)_k}$ is $A_{n_k} \supset B_{n_k}$, and is joined to the deduction of A in **ND** by elimination of hypothesis A_{n_k}. Then case (ii) obtains, and (l_j') is derived from (l_k) by the deduction theorem for **H**. In case (ii) exhausted proofs from hypotheses are discarded.

(c) $B_{(m+1)_k}$ is joined to the proof sequence of A in **ND** by hypothesis introduction. Case (iii) obtains. Since (l_{k+1}) is a proof from hypotheses for **H**, it can be added to the set of proofs from hypotheses.
Case (a) also takes care of the induction basis.

It follows by induction that every stage in the proof sequence of A in **ND** can be effectively replaced for **H** by a set of proofs from hypotheses. Therefore the last stage, where A is obtained as the conclusion of a completed deduction, can be effectively replaced for **H** by a proof of A from zero hypotheses. For an hypothesis is introduced and eliminated in the representing sets of proofs from hypotheses for **H**, when and only when an hypothesis is introduced or eliminated in the **ND** proof sequence. But, by the proof of theorem 7, a proof of A from zero hypotheses effectively yields a proof of A in **H**.

Theorem 9: **ND** and **H** are deductively equivalent, i.e. their theorem sets coincide. By theorem 1 and 7.

By eliminating quantifiers as in **H**, the primitive rule structure of **ND** could be much reduced. Consider the natural deduction system ND_1 which has the same primitive and defined symbols as **H** and the following rules: Hypothesis introduction and elimination, sentential derivation, and

3′. $$\frac{B}{Bx(\varepsilon y \sim B)}$$ provided x is not free in an hypothesis of B.

6′. $$\frac{S^{x}_{z} B \mid}{Bx(\varepsilon y B)}$$

Rules 3′ and 6′ can be combined in one artificial rule scheme. 'Deduction', 'completed deduction' and 'theorem' are defined as for **ND**.

Theorem 10: $\mathbf{ND}_1$, **ND** and **H** are deductively equivalent.

Theorem 11 (6): If B and $A_1, A_2, \dots A_n$ are ε-free and there is a proof of B from hypotheses $A_1, \dots A_n$ in **H**, then there is a relettered (ε-free) proof of B from hypotheses $A_1, A_2, \dots A_n$ in $\mathbf{F}^{1P}$.

Proof: Let B be ε-free and let the proof of B from ε-free hypotheses $A_1, A_2, \dots A_n$ consist of the sequence

$$B_1, B_2, \dots B_m, \qquad (\alpha)$$

where $B_m = B$. In this proof sequence make the following replacements:-

(i) Replace each ε-term introduced by S-elimination systematically, in an order determined by the first appearance of the ε-term in the proof-sequence, by the first variable alphabetically later than any variable already occurring in the proof sequence (α). Terms $\varepsilon x A$ and $\varepsilon y B$ are replaced by the same variable iff A is $S'^{y}_{y} B \mid$.

(ii) Do the same for remaining ε-terms introduced by A-elimination. Then a new sequence

$$B_1', B'_2, \dots B_m' \qquad (\beta)$$

is obtained.

(6) This theorem embodies the main part of Hilbert's second ε-theorem: see HILBERT and BERNAYS, *op. cit.*, 18.

With the exception of finitely many, say k, applications of S-elimination of the form $\frac{(Sy)C_i}{\check{S}_w C_i}$ $(1 \leqslant i \leqslant k)$, the sequence (β), after supplementation by finitely many insertions, satisfies Church's requirements on a proof from hypotheses in $\mathbf{F}^{1P}$. First, occurrences of cases (1), (2), (3) and (6) under a proof from hypotheses in **H** in the sequence (α) are guaranteed in the sequence (β) by cases (1), (2) and (7), (3) and (4), respectively, of a proof from hypotheses in $\mathbf{F}^{1P}$. The substitutions made in going from (α) to (β) provide no obstacle, because of the choice of new substituted variables. It remains to consider cases (4) and (5) under a proof from hypotheses in **H**.

ad (4): Suppose B_j in sequence (α) is obtained from B_i (where $i<j$) by A-elimination; so B_j and B_i are of the form $\check{S}^x_a B |$, and $(Ax)B$ respectively. Then B_j' is subst $\check{S}^x_u B |$, where the changes made under Subst. in B are those, specified in (i) and (ii) above, and where u is z if z is a variable and an appropriately chosen new variable if z is a term. Thus B_j' is $\check{S}^x_u$ Subst $B |$. Also B_i' is Subst $(Ax)B$, i.e. (Ax) Subst B. Consequently B_j' follows from B_i' by A-elimination. Now insert before B_j' into (β) $B_i' \supset B_j'$ as permitted by (2) in the requirements on a proof from hypotheses in $\mathbf{F}^{1P}$. Then B_j' results from B_i' in the supplemented sequence (β) by proof steps permitted in a proof from hypotheses in $\mathbf{F}^{1P}$.

ad (5): Suppose B_j' in (α) is derived from B_i by S-introduction. The case is similar to (4) except that further insertions must be made in (β), namely a proof of $\check{S}^x_u \supset (Sx) B |$ before Bj′.

Consequently

$$A_1, \dots A_n, \check{S}^y_w C_1 |, \dots \check{S}^{y_k}_{w_k} C^k | \vdash B$$

holds for $\mathbf{F}^{1P}$, where a proof of B from these hypotheses is provided by the supplemented sequence (β), and where w_i is the i^{th} variable alphabetically later than any in (α), for each i — $(1<i\leqslant k)$ — except variables free in hypothesis $\check{S}^{y_i}_{w_i} C_i |$

for some i. Suppose there are m cases of generalisation upon variables free in hypotheses of the proof sequence, where the j^{th} $(1 \leqslant j \leqslant m)$ is of the form:

$$\frac{D_j}{(Az_j)\, D_j,}$$ and z_j is not free in $A_1 \ldots A_n$.

Then the supplemented sequence (β) guarantees for $\mathbf{F}^{1P}$ the following proof of B from hyoptheses:

$$A_1 \ldots A_n, (Az_m)D_m, \ldots (Az_1)D_1,$$
$$\check{S}^{y_1}_{w_1} C_1 \mid, \ldots \check{S}^{y_k}_{w_k}\ C_k \mid \vdash B. \qquad (\alpha)$$

Now each of the hypotheses $\check{S}^{y_k}_{w_k}\ C_k \mid, \ldots$

$\check{S}^{y_1}_{w_1} C_1 \mid$ of (α) can be eliminated in turn. Suppose for induction basis and step the elimination has been completed up to $\check{S}^{y_{i+1}}_{w_{i+1}}$ $C_{i+1} \mid$, so that $A_1, \ldots A_n, (Az_m)b_m \ldots, \check{S}^{y_1}_{w_1}C_1 \mid \ldots, \check{S}^{y_{i+1}}_{w_{i+1}}\ C_{i+1} \mid \vdash B$

It follows by the deduction theorem for $\mathbf{F}^{1P}$.

$$A_1 \ldots \check{S}^{y_i}_{w} C_i \mid \vdash \check{S}^{y_{i+1}}_{w_{i+1}}\ C_{i+1} \mid \supset B.$$

Since w_{i+1}, by its choice, does not occur in the premisses $\check{S}^{y_1}_{w}$ $C_1 \mid, \ldots \check{S}^{y_i}_{w.}\ C_i \mid$ of this proof from hypotheses, it follows by generalisation (7)).

$$A_1 \ldots \check{S}^{y_i}_{w_i} C_i \mid \vdash\ (A^{w}{}_{i+1})(\check{S}^{y_{i+1}}_{w_{i+1}}\ C_{i+1} \mid \supset B).$$

Since, further, w_{i+1} has no free occurrences in B, by relettered *364,

$$A_1, \ldots, \check{S}^{y_i}_{w_i} C_i \mid\ \vdash (S^{w}{}_{i+1})\ \check{S}^{y_{i+1}}_{w_{i+1}} \mid \supset B.$$

(7) This step is not permitted in a proof from hypotheses for $\mathbf{F}^{1P}$ since $(Az_j)D_j$, for some j, may contain w_{i+1} free. However the step is permissible since hypotheses $(Az_m)D_m$, $(Az_1)D_1$ are not eliminated by the deduction theorem. For the more general deduction theorem used, see S. C. Kleene, *Introduction to Metamathematics*, New York (1952), p. 97.

As w_{i+1} does not occur free in C_{i+1}, by choice of w_{i+1}, it follows by relettered *378 and by substitutivity of equivalents,

$$A_1, \dots \check{S}^{y_i}_{w_i}\ C_i \mid\vdash (S^{y}{}_{i+1})\ C_{i+1} \supset B.$$

Since, however, $(\check{S}y_{i+1})\ C_{i+1}$ has already been derived at an earlier stage in the deduction sequence (β), it follows by *362,

$$A_1, \dots \check{S}^{y_i}_{w_i}\ C_i \mid\vdash (\check{S}y_{i+1})\ C_{i+1}.$$

Thus, by modus ponens

$$A_1 \dots \check{S}^{y_i}_{w_i}\ C_i \mid\vdash B\ ,$$

and by iteration

$$A_1, \dots, A_n, (Az_m)D_m, \dots (Az_1)D_1 \vdash B. \qquad (\delta)$$

But the proof from hypotheses (δ) at the same time supplies premises $(Az_m)D_m, \dots (Az_1)D_1$, by generalization from the already available $D_m, \dots D_1$, respectively, since variables $z_m, \dots z_1$ are not free in hypotheses $A_1, \dots A_n$. Thus the proof sequence which guarantees (δ) at the same time guarantees

$$A_1, \dots A_n \vdash B. \qquad (\varepsilon)$$

Finally, (ε) satisfies the requirements of a proof of B from hypotheses $A_1, \dots A_n$ for $\mathbf{F}^{1P}$, since *its* proof sequence does satisfy requirements for a proof from hypotheses for $\mathbf{F}^{1P}$.

Theorem 12: If A is an ε-free theorem of **H** then A is a relettered theorem of $\mathbf{F}^{1P}$.

Proof: If A is an ε-free theorem of **H** then, by theorem 7, there is a proof of A from null hypotheses. Therefore, by theorem 11, there is an ε-free proof of A in $\mathbf{F}^{1P}$ from null hypotheses. Therefore there is a proof, necessarily ε-free of relettered A in $\mathbf{F}^{1P}$; so A is a relettered theorem of $\mathbf{F}^{1P}$.

Theorem 13: A is an ε-free theorem of **ND** and of $\mathbf{ND}_1$ iff A is a relettered theorem of $\mathbf{F}^{1P}$.

Proof from theorems 12, 10 and 4.

Theorem 14: **ND**, $\mathbf{ND}_1$ and **H** are consistent.

Proof: As $\mathbf{F}^{1P}$ is consistent, relettered $\mathbf{F}^{1P}$ is consistent. Therefore, by theorems 10 and 13, **ND**, $\mathbf{ND}_1$ and **H** are consistent.

III

Consistency and completeness of **ND** are now studied directly. Proofs are truncated by proving results for $\mathbf{ND}_1$.

From any wff of $\mathbf{ND}_1$ (or of **H**) an *associated ε-free wff* is obtained by deleting (or eliminating definitionally) all quantifiers and replacing every occurrence of the form εzA, where A is a wff, by an individual variable. From any wff an *associated wff of sentential logic* (an *aws*) is obtained by first forming the associated ε-free wff and then, in the latter, replacing every wff part $f(z_1, \ldots z_n)$ by a sentential variable not previously occurring in the wff, in accordance with the requirement that two wf parts $f(z_1, \ldots z_n)$ and $g(y_1, \ldots y_n)$ are replaced by the same sentential variable iff f and g are the same predicate.

Theorem 1: Every theorem of $\mathbf{ND}_1$ (and of **H**) has a theorem of classical sentential logic as aws.

Theorem 2: **ND** and $\mathbf{ND}_1$ are simply and absolutely consistent.

Proof: If $\mathbf{ND}_1$ is simply inconsistent, then for some wff B both B and ∼B are theorems of $\mathbf{ND}_1$. Therefore by theorem 1 their aws B* and ∼B* are theorems of classical sentential logic. As this logic is however simply consistent $\mathbf{ND}_1$ is simply consistent.

The semantics for $\mathbf{ND}_1$ now introduced presupposes some non-ontological set theory.

A *semantical basis* for $\mathbf{ND}_1$ (or **H**) is a triple S = $\langle$d, e, c$\rangle$, where (i) the individual domain d is a set (possibly null) of items, (ii) e is a class of relation-in-extension over d, and (iii) c is a choice function over d, such that, for each non-null subset d′ of d, c(d′) is an item of d′, and such that otherwise c(Λ) = c(d), and if d = Λ, c(d) = Λ. A basis S is *usual* if e is the class of all relations over d. The null domain is not excluded: over this domain f(x), (Sx)f(x) and (Ax)f(x) have truth-value F, since (Aw)(Λ∉w), while ~f(x), (Sx)~f(x) and (Ax)~f(x) have value T.

An *interpretation I of* $\mathbf{ND}_1$ and its applications over basis S is an assignment I which assigns to each sentential constant one of the truth-values T or F, to each individual constant an element of d, and to each n-place predicate constant an n-place relation of e. Thus sentential variables are variables having T and F as their range; individual variables are variables having items of d as their range; and n-place predicate variables are variables having n-place relations of e as their range. For each constant and variable u, I(u) is the assignment-value of u under I. 'A has truth-value T' is abbreviated 'TA'. Truth-value assignments are relative to a given interpretation. Now, given I, truth-values and further assignment-values are defined inductively, relative to given assignments to free variables, as follows:

(i) A wff consisting of a sentential variable p alone has value T iff I(p) is T;

(ii) A wff $f(s_1, \ldots s_n)$ consisting of n individual subject terms $s_1, \ldots s_n$ and of n-place predicate f has value T iff $<I(s_1), I(s_2), \ldots I(s_n)> \varepsilon\ I(f)$, i.e. $Tf(s_1, \ldots s_n) \equiv I(f)(I(s_1), \ldots I(s_n))$;

(iii) $I(\varepsilon x A) = c(\{I(x){:}\ TA\})$;

(iv) $T{\sim}A \equiv {\sim}TA$;

(v) $T(A \supset B) \equiv {\sim}TA \vee TB$.

A wff is *valid for* I iff it has truth value T under interpretation I for every assignment of values to its free variables; *satisfiable for I* if it has value T for some assignment of values to its free variables. A wff of $\mathbf{ND}_1$ is *valid* iff it is valid for every interpretation over usual bases. An interpretation I over S is a *model* for a set Γ of wff iff every wff in Γ has value T under interpretation I over S for some simultaneous assignment of values to free variables of wff of Γ. A model is *denumerable* iff the individual domain of the semantical basis is denumerable.

Theorem 3: Every theorem of $\mathbf{ND}_1$ is valid.

Theorem 4: (Skolem-Löwenheim): Every consistent set of wff of **ND** and of $\mathbf{ND}_1$ (and of **H**) and of applied $\mathbf{ND}_1$ has a denumerable model.

Proof: Let Δ be a consistent set of wff of $\mathbf{ND}_1$ or of an applied $\mathbf{ND}_1$ by adding constants. Let Γ be obtained from Δ by substituting for free variables of wff of Δ constant ε-terms distinct from any occurring in wff of Δ, in such a way that a different constant ε-term is substituted for the free occurrences of each different individual variable and no variables are bound by the substitution. Then, if Γ has a denumerable model, Δ also does. Because Γ consists of closed wff, since constant ε-terms are those that contain no free variables, it suffices to prove the theorem for closed consistent sets of wff.

Let Γ be a consistent set of closed wff of $\mathbf{ND}_1$ or of an applied $\mathbf{ND}_1$. By Lindenbaum's lemma ([8]), Γ has a maximal consistent extension, say J.

A semantical basis $S_1 = \langle d_1, e_1, c_1 \rangle$ for a model I_1 is specified thus: d_1 is the domain of closed terms (ε-terms and individual constants) of $\mathbf{ND}_1$, e_1 is the class of all relations-in-extension over d_1, and c_1 is so specified in terms of the interpretation I_1 over s_1 that $c_1(\{I_1(x)\colon TA\}) = \varepsilon x A$. I_1 is as follows: A sentential constant p is assigned T iff $\vdash_J p$; an individual constant is assigned itself under I_1; and an n-place predicate constant f is assigned an n-place relation $I_1(f)$ of e_1 such that for closed terms $s_1, s_2, \ldots s_n$:

$$I_1(f)(I_1 s_1), \ldots I_1(s_n)) \text{ has value T iff } \vdash_J f(s_1, \ldots s_n).$$

Hence (i) $I_1(\varepsilon x A) = c_1\ (\{I_1(x)\colon TA\}) = \varepsilon x A$; and

(ii) $TI_1(f)(s_1, s_2, \ldots s_n) \equiv \vdash_J f(s_1, s_2, s_n)$.

The specification of c_1 is consistent: for if $T(Sx)A$ then $TA(\varepsilon x A)$, while if $\sim T(Sx)A$ then, as $\{I_1(x)\colon TA\}$ is null, $\varepsilon x A$ can be selected from d_1. Now

([8]) See A. Church, *op. cit.*, **452. The simplification of Henkin's proof of the Skolem-Löwenheim theorem used here is suggested by Wang's work: see H. Wang, *op. cit.*, p. 318.

(*) A closed wff A of Γ has truth-value T for I_1 iff $\vdash_J A$. The proof is by induction over the number n of occurrences of 'ε', '∼' and '⊃' in A. For induction basis, let A be a wff where n = 1. Since A is closed A must be of the form $f(\varepsilon x g(x))$. Then

$$\begin{aligned} Tf(\varepsilon x g(x)) &\equiv I_1(f)(I_1 \varepsilon x g(x))) \\ &\equiv I_1(f)(\varepsilon x g(x)) && \text{by (i)} \\ &\equiv \vdash_J f(\varepsilon x\, g(x)) && \text{by (ii)} \end{aligned}$$

Otherwise, in the case of certain applied $\mathbf{ND}_1$ systems, the basis for n = 0 is immediate from (ii). To complete the induction there are three uses to consider:

Case 1. A is of the form $f(s_1, \ldots s_{i-1}, \varepsilon x B, \ldots s_n)$, where B contains at most x free. Then

$$\begin{aligned} Tf(s_1, \ldots s_{i-1}, \varepsilon x B, \ldots s_n &\equiv I_1(f)(I_1(s_1), \ldots I_1(\varepsilon x B), \ldots I_1(s_n)) \\ &\equiv \vdash_J f(s_1, \ldots S_{i-1}, \varepsilon x B, \ldots s_n) \text{ by (ii)} \end{aligned}$$

Case 2. A is the form $\sim B$. Then

$$\begin{aligned} TA &\equiv T \sim B && \text{since B is closed} \\ &\equiv \sim TB \equiv \sim \vdash_J B, && \text{by induction hypothesis} \\ &\equiv \vdash_J \sim B && \text{since J is maximal consistent.} \\ &\equiv \vdash_J A \end{aligned}$$

Case 3. A is of the form $(B \supset C)$. Then

$$\begin{aligned} TA &\equiv T(B \supset C) \\ &\equiv \sim TB \vee TC \equiv \sim \vdash_J B \vee \vdash_J C && \text{by induction hypothesis} \\ &\equiv \vdash_J \sim B \vee \vdash_J C && \text{since J is maximal consistent} \\ &\equiv \vdash_J \sim B \vee C && \text{since J is maximal consistent and by sentential logic} \\ &\equiv \vdash_J A \end{aligned}$$

As (*) holds, I_1 is a model for Γ. For every wff A of Γ is provable in J, and therefore has truth-value T for I_1. Finally I_1 is a denumerable model since the constant terms are denumerable. Since **ND** is deductively equivalent to $\mathbf{ND}_1$ the theorem also follows for **ND**.

Corollary: The Skolem-Löwenheim theorem holds for $\mathbf{F}^{1P}$.

Theorem 5 (Completeness theorem): Every valid wff of $\mathbf{ND}_1$ is a theorem of $\mathbf{ND}_1$ (and of **ND**) (9).

Theorem 6 Every valid wff of $\mathbf{F}^{1P}$ is a theorem of $\mathbf{F}^{1P}$.

Proof: Similar to the proof of theorem 5, applying the corollary of theorem 4.

Certain alternative completeness proofs can also be shortened in $\mathbf{ND}_1$ using ε-terms (10).

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(9) For a proof, using theorem 5, see A. CHURCH, *op. cit.*, p. 245. Note, however, that a different definition of validity is adopted.

(10) For instance the Quine-Dreben proof and R. Smullyan's simplification of this: see W. QUINE, *Methods of Logic* revised edition, New York (1959), 253-259, B. DREBEN 'On the completeness of quantification theory', *Proceedings of the National Academy of Science*, vol. 38 (1952), 1047-1052.