A SYNTHESIS OF TRUTH-FUNCTION DIAGRAMS

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1. In a paper entitled "A Lattice Diagram for the Propositional Calculus" (1) Evenden showed how features of that calculus (logic) could be displayed on a diagram based on the Boolean lattice, in a manner akin to the use of the square of opposition in traditional logic. In a paper entitled "A Diagram-method In Propositional Logic" (2) Hubbeling showed how diagrams akin to the Venn-Euler and Gonseth diagrams could be used to solve problems in that logic (calculus) in a manner that displayed many of it's distinctive features. In the present paper we combine both kinds of diagram and add an interpretation of the result that enables it to be used in a manner akin to the Wittgenstein-Post method of truth tables. It is hoped that this synthesis is sufficiently rich in allusions and insights to at least serve as a valuable adjunct in teaching.

2. Figure 1a represents the universe of discourse for two propositional variables, p and q (the squares are merely numbered for convenience of reference). In the "language" of the Venn-Euler diagrams for propositional functions, the upper half of 1a is shaded if p is true, giving 1b, the lower half if p is false (1c), the left half if q is true (1d), and the right half if q is false (1e). Then any truth function of p and q can be uniquely represented by further diagrams in the manner exemplified by (1f) and (1g). The symbolism of these diagrams can be changed by replacing shaded squares by the sign "+" and unshaded squares by the sign "—", as shown in the corresponding sequence of diagrams 2a/2g.

One can also, if one wishes, interpret these diagrams as a compact form of truth table. To do so, interpret square 1 as "the case where p and q are true", square 2 as "the case where p is true and q is false", etc, and interpret "+" as "the function is true" and "—" as "the function is false". Then, for example, the interpretation of 2f becomes "where p and q are true the function is true, where p is true and q false the function is true, where p is false and q true the function is true and where both are false the function is false".

Other semantic interpretations can, of course, be given to the "+" and "—" signs, for example: "is a valid move" and "is an invalid move" (in chess) and "is an open pair of contacts" and "is a closed pair of contacts" (in an electrical circuit). Such interpretations do not invalidate the following considerations.

3. There are exactly sixteen diagrams of the type exemplified in Figure 2, corresponding to the sixteen possible truth functions of p with q. These sixteen diagrams can be combined on a single diagram constructed as a Boolean lattice, as shown in Figure 3. On this diagram, the function at the top is the contradictory function and that at the bottom the tautology. The lines on the diagram represent formal implications and also display how the functions can be formed from successive disjunctions of the functions higher in the diagram. For example, q is formally implied by p.q and by $\sim$p.q and is equivalent to the disjunction of these functions, as the arrows in both cases show. All two-variable functions are, of course, equivalent to one or other of those shown on the diagram and all theorems to the lowermost function.

A surprising number of insights into propositional logic can be obtained from study of the diagram and many of these have already been noted in the earlier paper on lattice diagrams. The system of digits of that earlier paper can be translated into the present symbolism, by interpreting "1", "2", "3", "4", respectively, as a "+" in the squares of those numbers in figure 2a. The points made about the earlier diagram are then valid for the present one.

The present diagram is an advance on the earlier one, in that it immediately makes evident the symmetries underlying propositional logic and which are by no means always evident in presentations of that logic in the usual symbolism. Also, the diagram very clearly displays the truth status of the functions: for example, it can be seen at once that for $p \equiv q$ to be true, both variables must be true or both false and it can be seen that the only failing case of $p \supset q$ is p true q false.

4. In the earlier paper on diagram-methods it was remarked that diagrams of the type given in figure 2 become useful when associated with relevant operation assignments. Suppose, for example, that one has the diagram "P" for a function "p" and the diagram "Q" for a function "q" and that one wishes to find the diagram "D" for "p⊃q". Then the following operation assignment specifies D:-

/P+ /Q+ //D+
/P+ /Q— //D—
/P— /Q+ //D+
/P— /Q— //D+

In this assignment, the first line is to be interpreted "wherever "+" occurs in a square in P and "+" occurs in the corresponding square in Q, write "+" in the corresponding square in D" and similarly for the other lines of the assignment.

We now remark that the diagrams of functions shown on figure 3 can be re-interpreted to serve as operation assignments for the corresponding operations. To do this, interpret square 1 (see 1a, 2a) as the square in which one enters the sign (Viz. "+" or "—") to be placed in D in *any* square that has "+" in both P and Q; interpret square 2 as that in which one enters the sign in D corresponding to a "+" in P and a "—" in Q and similarly for squares 3 and 4. If, for example, this interpretation is applied to the diagram labelled "p⊃q" in figure 3, it will be seen to be the same assignment as that noted above. For the philosophy of logic it is, of course, important that the same diagram can be interpreted both as a propositional function and as an operation assignment. This parallels the fact that logical laws can be translated into logical operation rules ([3]).

Mention must be made of the diagrams labelled "p", "q", "∼p" and "∼q" on figure 3. They form no exception to the foregoing remarks, as will be seen when it is recollected that they can be regarded as degenerate two variable functions. For example, ∼p is ∼p.q :V: ∼p.∼q and it is instructive to check through the validity of the operation assignment for ∼p by substituting figure 4, where w, x, y, and z, can indifferently be "+" or "—".

5. Slightly re-expressed, the position is that figure 3 tabulates all possible truth functions of (one or) two variables and operation assignments corresponding to all theoretically possible binary connectives. There are, of course, sixteen such connectives, not withstanding that only conjunction, disjunction, implication, equivalence and the Scheffer and Nicod connectives are in general use. Indeed, a complete system of symbols for two variable logic, indicating all the sixteen binary connectives (and all the sixteen possible operations) has been developed by the German schools of mathematical logic of Munster and Munchen (H. Scholz et al.).

A remarkably instructive and serviceable decision procedure for two-variable functions can be used on figure 3, as indicated in the following brief, informal, sketch. The procedure is contingent upon all functions of the type xRy being entered as labels on figure 3, where x and y are p or q or $\sim p$ or $\sim q$ and R is any binary connective in the symbolism in use. Figure 3 has been made up for the connectives of Principia Mathematica, with the omission, for simplicity, of the minor variants q.p (against p.q), $\sim q \supset \sim p$ (against $p \supset q$) etc.

The expression representing the function to be studied will always be made up of "first order" expressions that are either p, $\sim p$, q, $\sim q$, or of the type xRy. The diagrams corresponding to these expressions are all found from figure 3, on which they are all entered. These expressions form part of further expressions of the type $\sim \Phi$ or of the type $\Phi R \Psi$, where Φ, Ψ, are first order expressions and R is a binary connective. Various possibilities must now be considered.

Case 1. $\Phi R \Psi$, and the diagrams for Φ and Ψ are identical. If R is "." or "v" the resulting diagram, D, is either diagram. If R is "$\equiv$" or "$\supset$", D is the tautology diagram. Case 2. $\Phi \supset \Psi$ and the diagrams for Φ and Ψ are not identical, but on figure 3 they are joined by a sequence of arrows from Φ to Ψ, D is the tautology diagram. Case 3. $\sim \Phi$. D is formed by reversing all signs on the diagram. Case 4. Any other case. D is formed by interpreting figure 3 as a table of operation assignments.

By repetition of these operations, the status of the function can always be decided.

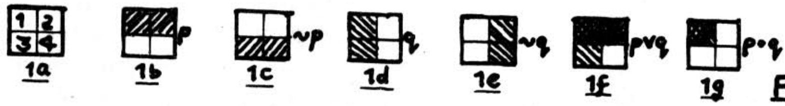

FIG. 1.

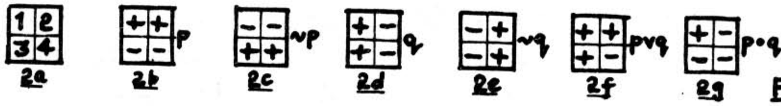

FIG. 2.

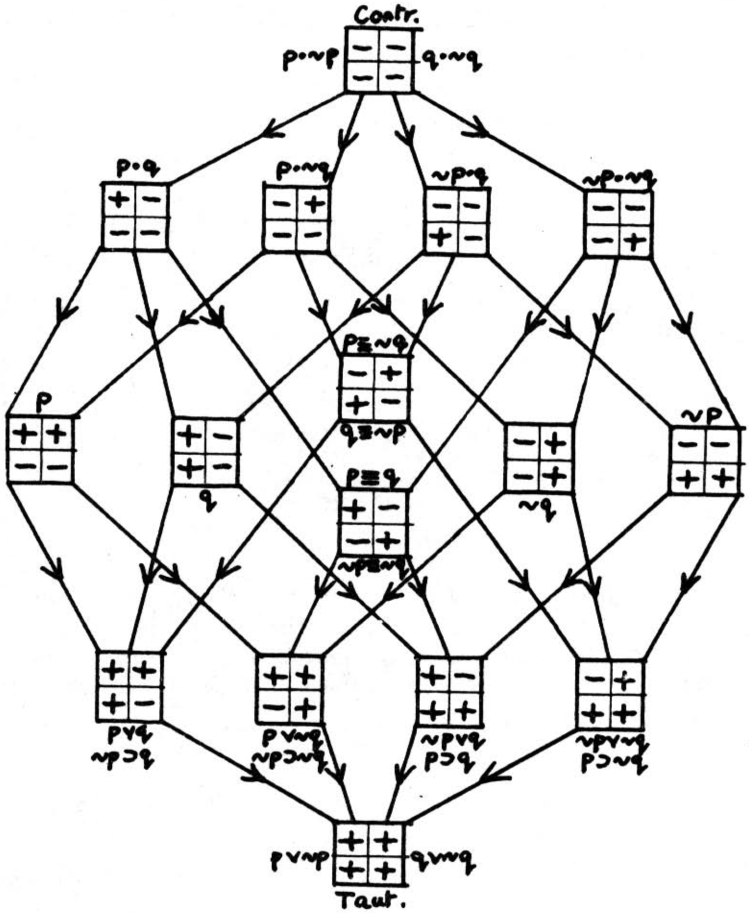

FIG. 3.

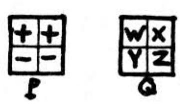

FIG. 4.

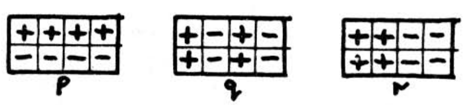

FIG. 5.

6. In the earlier papers it was pointed out that the diagrams for functions of two variables can in principle be extended to functions of any number of variables. The lattice diagram for more than two variables is, however, intolerably complicated. Basic function diagrams for three variables are shown in figure 5.

Now it can easily be seen that when the diagrams on figure 3 are interpreted as operation assignments, they are valid for manipulating the diagrams in figure 5 and the corresponding diagrams for any number of variables. Since, moreover, the connectives in general use are invariably binary and figure 3 lists assignments for all binary connectives, that figure still provides a very serviceable proceedure for deciding functions of more than two variables.

7. In conclusion, the diagram shown in figure 3 has an unusually rich mnemonic content, displays a remarkable number of features of propositional logic and is a useful starting point for further discussion.

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[3] Vide I. M. BOCHENSKI and A. MENNE, *Grundriss der Logistik,* Paderborn, 1954, section 9, p. 43ff. English translation: D. Reidel Pub. Co., Dordrecht, Holland.