MODAL REDUCTION AXIOMS IN EXTENSIONS OF S1

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Modal logics are commonly formulated with a primitive necessity or possibility operator, though often they may equally well have non-contingency or contingency as the sole modal primitive, necessity and possibility being introduced by definition (1). Examples of normal modal logics with such bases are given in [2], and similar foundations can easily be constructed for the weaker classical systems such as **S1**, **S2** and **S3**. One attraction of these formulations is that non-contingency extension axioms often provide very simple and illuminating relationships between different modal systems, and another is that iterated modalities are seen from a different aspect (2). For example, in contrast with the possibility or necessity based versions, the Trivial Δ system has proper irreducible modalities, McKinsey's $\mathbf{S4.1}_{\Delta}$ has fewer irreducible modalities than $\mathbf{S5}_{\Delta}$, and $\mathbf{S4}_{\Delta}$ and $\mathbf{S5}_{\Delta}$ have the same irreducible modalities.

Theorems 4, 10, 14 and 23 of this paper answer some of the questions left open at the end of [3].

The notation follows that of [3]: '$\rightarrow$' symbolises strict implication, '$=$' strict equivalence, and 'SSE' refers to the rule of substitutivity of strict equivalents. 'Δ^n' is used to denote n iterations of the operator 'Δ'. A subscript 'Δ' is attached to system labels when it is wanted to emphasise that the modal primitive is 'Δ'. Such systems are said to be *non-contingency based.*

The system **S1** is formulated as in Feys [1] but augmented by the definition: $\Delta A =_{df} \Box A \vee \Box \sim A$.

(1) Sufficient conditions for this to be possible are that SSE is derivable and that $p.(\Box p \vee \Box \sim p) = \Box p$ is provable. Example of systems for which these conditions fail are **SO.5**, **S1°**, **S2°**.

(2) By an 'iterated modality' we mean a sequence of zero or more symbols each of which is either a negation symbol or a primitive monadic modal symbol.

A non-contingency based version **S1**$_\Delta$ results by taking 'Δ', '$\sim$' and '.' as primitive, using the postulates of **S1** as in [1], adding the axiom

30.16 $\Delta p = \Delta \sim p$ (*or*

30.17 $\Delta p \rightarrow \Delta \sim p$)

and replacing the definitions F30.3 by the following set:

$A \vee B =_{df} \sim(\sim A \,.\, \sim B)$
$A \supset B =_{df} \sim(A \,.\, \sim B)$
$A \equiv B =_{df} (A \supset B) \,.\, (B \supset A)$
$\Box A =_{df} A \,.\, \Delta A$
$\Diamond A =_{df} \sim \Box \sim A$
$A \rightarrow B =_{df} \Box(A \supset B)$
$A = B =_{df} (A \rightarrow B) \,.\, (B \rightarrow A)$

Theorem 1. **S1** and **S1**$_\Delta$ are deductively equivalent.

Proof: The strict equivalences corresponding to the definitions of '$\Box$' and 'Δ' are derived in the respective systems, the remainder of the proof presents no difficulty.

(1) $\Box p = p \,.\, \Delta p$ is a theorem of **S1**

1.	$\Delta p \rightarrow p$	F37.12.
2.	$\Box p \rightarrow \Box p \vee \Box \sim p$	SL, F34.1.
3.	$\Box p \rightarrow p \,.\, \Delta p$	1, 2, F35.22, F35.11, Df Δ.
4.	$\Box \sim (p \,.\, \Box \sim p)$	F36.0, F33.23, SSE.
5.	$p \,.\, \Box \sim p \rightarrow \Box p$	4, F35.42, F35.11.
6.	$p \,.\, \Box p \rightarrow \Box p$	SL, F34.1.
7.	$p \,.\, \Box p \vee p \,.\, \Box \sim p \rightarrow \Box p$	5, 6, F35.23, F35.11.
8.	$p \,.\, (\Box p \vee \Box \sim p) \rightarrow p \,.\, \Box p \vee p \,.\, \Box \sim p$	SL, F34.1.
9.	$p \,.\, \Delta p \rightarrow \Box p$	8, 7, F31.021, Df Δ.
10.	$\Box p = p \,.\, \Delta p$	3, 9, SL.

(2) $\Delta p = \Box p \vee \Box \sim p$ is a theorem of **S1**$_\Delta$. (The derivability of the postulates of **S1** in **S1**$_\Delta$ is assumed.)

1.	$p \,.\, \Delta p \vee \sim p \,.\, \Delta \sim p = \Box p \vee \Box \sim p$	SL, F34.1, Df $\Box$.
2.	$p \,.\, \Delta p \vee \sim p \,.\, \Delta p = \Box p \vee \Box \sim p$	1, 30.16, SSE.
3.	$p \,.\, \Delta p \vee \sim p \,.\, \Delta p = \Delta p$	SL, F34.1.
4.	$\Delta p = \Box p \vee \Box \sim p$	2, 3, SSE.

Theorem 2. The systems (**S1** + $\Box p \supset \Box\Box p$) and (**S1** + $\Delta p \supset \Delta\Delta p$) are deductively equivalent.

Proof: An examination of the proof sequence of theorem 1 of [3] shows it is a valid deduction of $\Box p \supset \Box\Box p$ in (**S1** + $\Delta p \supset \Delta\Delta p$). Conversely, $\Delta p \supset \Delta\Delta p$ is provable in (**S1** + $\Box p \supset \Box\Box p$) by the proof sequence of theorem 8 of [2]. However here the justification of line 3 must be altered. $\Box\Box p \vee \Box\Box \sim p \supset \Box(\Box p \vee \Box \sim p)$ is provable in **S1** by F34.1, F33.311, SD and SL, and this with line 2 and SL gives line 3.

Theorem 3. The systems (**S1** + $\Box p \supset \Box\Box p$) and (**S1** + $\Delta p \supset \Delta\Box p$) are deductively equivalent.

Proof: The proof sequences of theorem 3 of [3] give the necessary results as they are also valid proofs in these weaker systems.

Theorem 4. The deductively equivalent systems mentioned in theorems 1 and 2, and now labelled system **S4**⁻ properly include **T** and are properly included in **S4**.

Proof: By a result of Yonemitsu [5] it is immediate that $\Box p \supset \Box\Box p$ added to **S1** gives a system that deductively includes **T**. However it is well-known that $\Box p \supset \Box\Box p$ is not provable in **T**. Hence **S4**⁻ properly includes **T**.

It is also immediate that **S4**⁻ is deductively included in **S4**, and it remains to show that the inclusion is proper. The following set of matrices

.	1	2	3	4	5	6	7	8	~	□
*1	1	2	3	4	5	6	7	8	8	1
*2	2	2	4	4	6	6	8	8	7	8
3	3	4	3	4	7	8	7	8	6	7
4	4	4	4	4	8	8	8	8	5	8
5	5	6	7	8	5	6	7	8	4	8
6	6	6	8	8	6	6	8	8	3	8
7	7	8	7	8	7	8	7	8	2	8
8	8	8	8	8	8	8	8	8	1	8

with values 1 and 2 designated, a modification of Parry's F50.1, satisfy **S4**$^-$ and show $\Box p \rightarrow \Box\Box p$ provable in **S4**, to be independent of **S4**$^-$ (fails for $p = 3$).

Theorem 5. The systems (**S1** + $\Delta^n p$) all deductively include the system **T** (where $n \geqslant 1$).

Proof:		
1.	$\Delta^n p$	Axiom
2.	$\Box \Delta^{n-1} p \vee \Box \sim \Delta^{n-1} p$	1, Df Δ
3.	$\Box \Delta^{n-1} p \vee \sim \Delta^{n-1} p$	2, F37.12, SL.
4.	$\Delta^{n-1} p \supset \Box \Delta^{n-1} p$	3, SL.
5.	$\Delta^n p \supset \Box \Delta^n p$	4.
6.	$\Box \Delta^n p$	5, 1, MD.
7.	$\Box(\Box \Delta^{n-1} p \vee \Box \sim \Delta^{n-1} p)$	6, Df Δ.
8.	$\Diamond \Delta^{n-1} p \rightarrow \Box \Delta^{n-1} p$	7, F33.23, SL, F34.1, SSE.
9.	$\Delta^{n-1} p \rightarrow \Diamond \Delta^{n-1} p$	F36.0.
10.	$\Delta^n p \rightarrow \Box \Delta^n p$	9, 8, F31.021.
11.	$\Delta^n p = \Box \Delta^n p$	10, F37.12, SL.
12.	$\Box \Delta^n p = \Box\Box \Delta^n p$	11, F31.19.
13.	$\Box\Box \Delta^n p$	12, SL, 6, SD.
14.	$p \supset p \rightarrow \Delta^n p$	F35.41, 6, F35.11.
15.	$\Delta^n p \rightarrow . p \supset p$	F35.41, F31.11, F35.11.
16.	$\Delta^n p = . p \supset p$	14, 15, SL.
17.	$\Box\Box \Delta^n p = \Box(p \rightarrow p)$	16, F31.19.
18.	$\Box(p \rightarrow p)$	17, 13, SL, SD.

The theorem now follows from a result of Yonemitsu [5].

Theorem 6. The system (**S1** + $\Delta^n p$) each contain a corresponding reduction theorem, $\Delta^n p = \Delta^{n+1} p$ ($n \geqslant 1$).

Proof:		
1.	$\Box \Delta^n p$	As in theorem 5, line 6.
2.	$\Box \Delta^{n+1} p$	1.
3.	$\Delta^{n+1} p \rightarrow \Delta^n p$	F35.41, 1, F35.11.
4.	$\Delta^n p \rightarrow \Delta^{n+1} p$	F34.41, 2, F35.11.
5.	$\Delta^n p = \Delta^{n+1} p$	3, 4, SL.

Theorem 7. For each $n \geqslant 1$, the system (**S1** + $\Box\Diamond\Delta^n p$) is deductively equivalent to the system (**S1** + $\Delta^{n+1} p \rightarrow \Delta^n p$).

Proof: It is sufficient to derive $\triangle^{n+1}p \rightarrow \triangle^n p$ in the first set of systems, and $\Box\Diamond\triangle^n p$ in the second set.

ad $\triangle^{n+1}p \rightarrow \triangle^n p$

1. $\Box\Diamond\triangle^n p$	Axiom
2. $\Box\sim\triangle^n p \rightarrow \triangle^n p$	F35.42, 1, Df$\Diamond$, F35.11.
3. $\Box\triangle^n p \rightarrow \triangle^n p$	F37.12.
4. $\Box\triangle^n p \vee \Box\sim\triangle^n p \rightarrow \triangle^n p$	F35.23, 2, 3, SL, F35.11.
5. $\triangle^{n+1}p \rightarrow \triangle^n p$	4, Df$\triangle$.

ad $\Box\Diamond\triangle^n p$

1. $\triangle^{n+1}p = \triangle^n p$	Axiom.
2. $\Box\triangle^n p \vee \Box\sim\triangle^n p \rightarrow \triangle^n p$	1, SL, Df$\triangle$.
3. $\Box\sim\triangle^n p \rightarrow \triangle^n p$	F35.231, 2, F35.11, SL.
4. $\Box\sim\triangle^n p \rightarrow \sim\triangle^n p$	F37.12.
5. $\Box\sim\triangle^n p \rightarrow \triangle^n p \,.\, \sim\triangle^n p$	F35.22, 3, 4, F35.11.
6. $\sim(\triangle^n p \,.\, \sim\triangle^n p) \rightarrow \Diamond\triangle^n p$	F31.34, 5, SL, SD.
7. $\Box\sim(\triangle^n p \,.\, \sim\triangle^n p) \supset \Box\Diamond\triangle^n p$	F33.311, 6, SD.
8. $\Box\sim(\triangle^n p \,.\, \sim\triangle^n p)$	SL, F34.1.
9. $\Box\Diamond\triangle^n p$	7, 8, MD.

Theorem 8. For each $n \geqslant 1$, the system (**S1** + $\triangle^n p = \triangle^{n+1}p$) deductively includes the system (**S1** + $\Box\Diamond\triangle^n p$) and is deductively included in the system (**S1** + $\triangle^n p$).

Proof: The proof is immediate from theorem 6 and 7.

Theorem 9. (**S1** + $\triangle p$) is deductively equivalent to the Trivial system.

Proof: 1. $\Box\triangle p$	By theorem 5.
2. $\Diamond p \rightarrow \Box p$	1, Df$\triangle$, F33.23, SSE, SL.
3. $p \rightarrow \Diamond p$	F36.0.
4. $p \rightarrow \Box p$	2, 3, F31.021.
5. $\Box p = p$	4, F37.12, SL.

Conversely, 1 follows from 5 by SL, SSE, and Df$\triangle$.

Theorem 10. (**S1** + $\triangle\triangle p$) is deductively equivalent to **S5**.

Proof: (**S1** + $\triangle\triangle p$) deductively includes **T** by Theorem 5 above, and (**T** + $\triangle\triangle p$) is deductively equivalent to **S5** by theorem 13 of [2]. No problem arises over the necessitation rule of **T**, since (**S1** + $\Delta\Delta p$) is deductively equivalent to (**S1** + $\Box\Delta\Delta p$) by theorem 5 and hence all axioms may be considered to be of the form $\Box A$, and Yonemitsu's induction [6] holds.

Theorem 11. (**S1** + $\Delta\Delta\Delta p$)$\subset$ (**S1** + $\Delta\Delta p$)$\subset$(**S1** + Δp).

Proof. That the inclusions hold is immediate. It remains to show they are proper.

(1) (**S1** + Δp) properly includes (**S1** + $\Delta\Delta p$) since Δp is not provable in (**S1** + $\Delta\Delta$p) by the matrix F56.3 (Lewis group III).

(2) (**S1** + $\Delta\Delta p$) properly includes (**S1** + $\Delta\Delta\Delta p$) since $\Delta\Delta p$ is not provable in (**S1** + $\Delta\Delta\Delta p$) by the matrix of theorem 4.

Theorem 12. (**S1** + $\Delta\Delta\Delta p$) neither includes nor is included in either of **S4** or **S4.1**.

Proof: $\Delta\Delta\Delta p$ is not provable in **S4** or **S4.1** by matrix F56.2 (Lewis group II), and $\Delta p \rightarrow \Delta\Delta p$ is not provable in (**S1** + $\Delta\Delta\Delta p$) by the matrix of theorem 4.

Theorem 13. (**S3** + $\Delta\Delta\Delta p$) properly includes **S4** and is included in **S5**.

Proof: By line 13 of the proof of theorem 5, $\Box\Box\Delta\Delta\Delta p$ is provable in (**S3** + $\Delta\Delta\Delta p$), hence by F63.51 (**S3** + $\Delta\Delta\Delta p$) includes **S4**. The inclusion is proper by the proof of theorem 12 and it is immediate from theorem 8 that the system is included in **S5**.

Theorem 14. The systems (**S1** + $\Delta p \rightarrow \Delta\Delta p$) and (**S1** + $\Delta p \rightarrow \Delta\Box p$) are each deductively equivalent to **S4**.

Proof: It is immediate from Theorem 2 above that $\Box p \supset \Box\Box p$ is provable in (**S1** + $\Delta p \rightarrow \Delta\Delta p$). Furthermore, since in this system all the axoms are necessitated, a necessitation rule is derivable for the system as in Yonemitsu [6]. Hence $\Box p \rightarrow$

$\Box\Box p$ may be derived in (**S1** + $\Delta p \rightarrow \Delta\Delta p$) and the system deductively includes **S4**. Conversely, since a necessitation rule is derivable in **S4** and $\Delta p \supset \Delta\Delta p$ is provable, $\Delta p \rightarrow \Delta\Delta p$ is provable and so **S4** deductively includes (**S1** + $\Delta p \rightarrow \Delta\Delta p$). A similar argument applies to (**S1** + $\Delta p \rightarrow \Delta\Box p$) using theorem 3 above.

Theorem 15. (**S1** + $\Delta\Delta p = \Delta\Delta\Delta p$) is properly included in **S4** and **S5**.

Proof: It suffices to prove $\Delta\Delta p = \Delta\Delta\Delta p$ in **S4** and to show $\Delta p \rightarrow \Delta\Delta p$ is not provable in (**S1** + $\Delta\Delta p = \Delta\Delta\Delta p$).

ad $\Delta\Delta p = \Delta\Delta\Delta p$.

1.	$\Delta p \rightarrow \Delta\Delta p$	Theorem 14.
2.	$\Delta\Delta p \rightarrow \Delta\Delta\Delta p$	1.
3.	$\sim\Delta\Delta p \rightarrow \sim\Delta p$	F31.34, 2, SD.
4.	$\Box\sim\Delta\Delta p \rightarrow \Box\sim\Delta p$	3, F46.1.
5.	$\Box\sim\Delta\Delta p \rightarrow \Box\Delta p \vee \Box\sim\Delta p$	4, SL, F34.1, F31.021.
6.	$\Box\sim\Delta\Delta p \rightarrow \Delta\Delta p$	5, Df Δ.
7.	$\Box\Delta\Delta p \rightarrow \Delta\Delta p$	F37.12.
8.	$\Box\Delta\Delta p \vee \Box\sim\Delta\Delta p \rightarrow \Delta\Delta p$	6, 7, SL, F42.22, SD.
9.	$\Delta\Delta\Delta p \rightarrow \Delta\Delta p$	8, Df Δ.
10.	$\Delta\Delta p = \Delta\Delta\Delta p$	2, 9, SL.

The matrix F30.5 (Lewis group V) satisfies (**S1** + $\Delta\Delta p = \Delta\Delta\Delta p$) but not $\Delta \quad p \rightarrow \Delta\Delta p$.

Theorem 16. The system (**S1** + $\Delta p = \Delta\Delta p$) is deductively equivalent to McKinsey's **S4.1** (Sobocinski's **K1**) (3).

Proof: By theorems 14 and 7 above, the system (**S1** + $\Delta p = \Delta\Delta p$) is deductively equivalent to the system (**S4** + $\Box\Diamond\Delta p$). But by Df Δ and Sobocinski's axiom **K3** for **S4.1** (see [4, p. 77]), this latter system is deductively equivalent to **S4.1**.

(3) We are indebted to Professor G. E. Hughes for drawing our attention to the essential feature of this result with a proof that (**S4** + $\Delta p \rightarrow \Delta\Delta p$) is deductively equivalent to **S4.1**.

Theorem 17. For each $n \geqslant 1$, the system $(\mathbf{S1}_\Delta + \Delta^n p)$ has at most $2(n+1)$ irreducible modalities.

Proof: The system $(\mathbf{S1}_\Delta + \Delta^n p)$ has SSE and the following strict equivalences:

$p = \sim\sim p$	SL, F34.1.
$\Delta p = \Delta \sim p$	Axiom.
$\Delta^n p = \Delta^{n+1} p$	by theorem 6.

It follows by induction that there are at most the following irreducible modalities: $p, \sim p, \Delta p, \sim \Delta p, \ldots\ \Delta^n p, \sim \Delta^n p$.

Theorem 18. The Trivial system $(\mathbf{S1} + \Delta p)$ has four irreducible modalities.

Proof: By theorem 17 the system has at most four modalities, and by the characteristic matrix for the system namely

.	1	2	$\sim$	Δ
*1	1	2	2	1
2	2	2	1	1

no pair of these is strictly equivalent.

Theorem 19. The non-contingency based **S5** system $(\mathbf{S1}_\Delta + \Delta\Delta p)$ has six irreducible modalities.

Proof. By theorem 17 the system has at most six modalities and by the matrix F56.3 (Lewis group III) no pair of these is strictly equivalent.

Theorem 20. The system $(\mathbf{S1}_\Delta + \Delta\Delta\Delta p)$ has eight irreducible modalities

Proof: By theorem 17 the system has at most eight modalities and by the matrix of theorem 4 no pair of these is strictly equivalent.

Theorem 21. The non-contingency based $\mathbf{S4}_\Delta$ system $(\mathbf{S1}_\Delta + \Delta p \rightarrow \Delta\Delta p)$ has six irreducible modalities.

Proof: By theorem 15 and following the proof of theorem 17, the system has at most six modalities, and since it is included in $(\mathbf{S1}_\Delta + \Delta\Delta p)$, by theorem 19 there are exactly six.

Theorem 22. The non-contingency based **S4.1** system $(\mathbf{S1}_\Delta + \Delta p = \Delta\Delta p)$ has four irreducible modalities.

Proof: By the same proof as theorem 17 the system has at most four modalities, and by the matrix F56.2 (Lewis group II) it follows that these are distinct.

From theorems 17, 18, 19, 20 we conjecture the system $(\mathbf{S1} + \Delta^n p)$ has $2(n+1)$ irreducible modalities. If this conjecture is correct, then by theorem 5 the system $\mathbf{T}_\Delta$ has an infinity of irreducible modalities.

Finally we note that the system $(\mathbf{S1} + \Delta\Box p)$ is also deductively equivalent to **S5**, but this basis can be further weakened. The system $(\mathbf{S1}^{\circ} + \Box p \supset p)$, which is properly included in **S1**, is labelled "$\mathbf{S1}^{e}$". Then we have:

Theorem 23. The system $(\mathbf{S1}^{e} + \Delta\Box p)$ is deductively equivalent to **S5.**

Proof: We first prove some theorems of $(\mathbf{S1}^{e} + \Delta\Box p)$:

1.	$\Box\Box p \supset \Box p$	Axiom $\Box p \supset p$
2.	$\Diamond\Box p \supset \Box\Box p$	Axiom $\Delta\Box p$, Df Δ, Df $\Box$, SL.
3.	$\Diamond\Box p \supset \Box p$	1, 2, SL.
4.	$\Box\sim p \supset \Box(\sim p \vee \Diamond p)$	SL, F34.1, F33.311, SD.
5.	$\Box\Diamond p \supset \Box(\sim p \vee \Diamond p)$	SL,F34.1,F33.311,SD.
6.	$\Box\Diamond p \vee \Box\sim p \supset \Box(\sim p \vee \Diamond p)$	4, 5, SL.
7.	$\Diamond\Box\sim p \supset \Box\sim p \supset . p \rightarrow \Diamond p$	6, F33.2, SL, F34.1, SSE, Df $\rightarrow$.
8.	$p \rightarrow \Diamond p$	7, 3, MD.
9.	$\Box p \supset \Diamond\Box p$	8, F37.12, SD.
10.	$\Box p \supset \Box\Box p$	9, 2, SL.
11.	$\Box\Box\Box p \supset \Box(\Box\Box p \vee \Box\sim\Box p)$	SL, F34.1, F33.311, SD.
12.	$\Box\Box\sim\Box p \supset \Box(\Box\Box p \vee \Box\sim\Box p)$	SL, F34.1, F33.311, SD.
13.	$\Box\Box\Box p \vee \Box\Box\sim\Box p \supset \Box\Delta\Box p$	11, 12, SL, Df Δ.
14.	$\Box\Box p \vee \Box\sim\Box p$	Axiom, Df Δ.

15. $\Box\Box\Box p \vee \Box\Box \sim \Box p$ 14, 10, SL.
16. $\Box\Delta\Box p$ 13, 15, MD.
17. $\Box(p \rightarrow p)$ 10, SL, F34.1, MD.

By lines 8 and 16, ($\mathbf{S1}^e + \Delta\Box p$) is deductively equivalent to ($\mathbf{S1} + \Box\Delta\Box p$). By line 17 and the results of Yonemitsu referred to earlier a necessitation rule is derivable in this system, and hence the **S5** axiom $\Diamond\Box p \rightarrow \Box p$ follows from line 3. Hence ($\mathbf{S1}^e + \Delta\Box p$) deductively includes **S5** and the converse follows from theorem 4 of [3].

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