A VINDICATION OF SYSTEM E

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In their paper *Tautological Entailments* [2] Anderson and Belnap develop the system E of entailment which is sound and complete relative to tautological entailments. In our view, this is a logical advance of considerable significance. Recently the philosophical motives which led to the development of E have called into question by Professors Woods and Pollock [6, 8, 9]. It is our purpose to show that their criticisms are ineffectual, and our hope that some measure of agreement may be reached.

I

Relevance postulations. One of the principle merits of E, as well as one of the motives for developing it, is that it avoids fallacies of relevance. In a previous paper [4] we defined Belnap's principle of relevance as an adequacy condition for any systematic treatment of entailment. Woods has argued that our defence is in error [9]. If he is correct then it would appear that the philosophical grounds for the construction of E are unacceptable — a most serious matter indeed.

One of the arguments he advances concerns the postulation of

CIII. If A and B have no variables in common, 'A entails B' is rejected as a theorem of the system.

We have contended that it is too strong to require that a sufficient condition for the irrelevance of A to B entail 'A does not entail B'. By this we do not mean to *deny* that this sufficient condition, namely 'A and B have no variables in common', entails 'A does not entail B'. In fact we hold that CIII may be re-expressed as such an entailment. What we intended to call attention to by our contention was that there is no *formal* means by which CIII is established. Given the condition sufficient for the irrelevance of A to B, there is good reason to postulate CIII, since CIII is the formal counterpart of our informal thought about relevance and entailment (see [4] p. 215). (In order to avoid further misunderstanding we hasten to point out that CIII is meant to apply to a propositional system and therefore is not self-refuting.)

Now Woods attempts to show that, due to the notion of consistency, we are in difficulties, if we hold that 'A is irrelevant to B' does not entail 'A does not entail B' ([9] p. 364-366). His argument is misguided precisely because we do not hold that 'A does not entail B' is not entailed by either 'A is irrelevant to B' or 'A and B have no variables in common'. Moreover the relevant claim for Woods to have attended to (if we had made it) would be that 'A and B have no variables in common' does not entail 'A does not entail B'. If he had done so his argument against us would be the same, but it would fail because we hold that CIII may be expressed as the entailment that Woods takes us to deny. In short, Woods has confused the remarks we made concerning the postulation of CIII with the status of CIII itself.

As we pointed out above, one of the merits of E is that it avoids fallacies of relevance. Pollock has recently urged that the view that the paradoxes of strict implication commit a fallacy of relevance and are therefore unacceptable, is mistaken. His argument may be construed as an attempt to show that it is a mistake to postulate CIII as an adequacy condition for a system of entailment.

Pollock considers the following arguments advanced by Belnap: Geometry teachers could shorten their work by explaining that since the sides of an equilateral triangle are all equal, and necessarily so, hence the proposition follows from Euclid's axioms. A single axiom, 'Justice is, and is not a virtue', would suffice for the Hegelian deduction of the world; and the literal truth (or falsity) of the *Iliad* would suffice to prove the binomial theorem. Peano need not have bothered to show that '7 + 5 = 12' follows from his postulates, for P. 3 [the statement of paradox] guarantees this antecedently ([3] p. 5) (1).

(1) Also see [1] p. 33 for a similar argument.

If Belnap's argument is sound then there is good reason to postulate CIII, for a system constructed in accordance with CIII will be devoid of the paradoxes. However Pollock claims that the argument rests on a confusion between implication and valid inference. According to him, Belnap's argument is premised on the mistaken idea that if P implies Q then "the inference from P to Q (in a single step) is a valid argument." This is said to be a mistake because: to say that P implies Q means merely that *there* is some valid argument by which we can infer Q from P — not that the inference from P to Q (in a single step) *is* a valid argument ([6], p. 185).

It seems clear to us that Belnap is not guilty of the confusion attributed to him. What he has contended is that by making *use* of the paradox that a necessary proposition is implied by any proposition, one can prove that because '7 + 5 = 12' is a necessary proposition, there is some valid argument by which it can be inferred from Peano's axioms or if you like the axioms of the propositional calculus. Obviously there is no confusion here between valid inferences and implications. If one wishes to hold, as did Lewis, that the paradoxes state truths about deducibility, then of course it is a truth about deducibility that a necessary statement can be deduced from any statement. Belnap's argument is designed to show that this leads to absurd consequences. Indeed, there are no valid rules of inference by which we may deduce '7 + 5 = 12' from the axioms of the propositional calculus, but if Lewis is correct there should be.

II

Counter-examples to adequacy conditions. Woods has apparently failed to understand our criticism of his alleged counter-examples. Basically, our point was simply this: Woods' examples are irrelevant (in the ordinary sense of the term). This is so because the logistic system under discussion purports to treat only logical connectives between sentences (or propositions) independently of the internal structure of sentences. In short, the system is not a propositional and functional logic, but only a propositional logic. Woods' examples are formulated in terms of (the names of) functional expressions, and so their truth may be expected to involve, though, strictly, it *need* not involve, the internal structure of sentences, and not merely the logical structure of logically unanalyzed (atomic) sentences. Since Woods' examples involve names of functional expressions, i.e., *mention* and do not *use,* sentence forms, one need not expect them to be examples of the sort of sentences with which this system is intended to deal. In short, the calculus is developed in terms of a *connector* between sentences (and compounds of sentences) and not as a predicator between nouns or names. A functional logic developed along similar lines *might* be expected to cover examples similar to those Woods' presents. Why it should *not* be expected to will be discussed subsequently. Woods should have produced valid statements which are not certifiable as such by the micro-structure dealt with by means of this propositional logic. But Wood's alleged counter-examples are not of this sort. To advance them as counter-examples is to complain that a propositional logic is not adequate for all of logic, which is silly, since no one supposed otherwise.

Our discussion [4] concerning analysing the predicates in examples similar to those of Woods' (via the object language counterparts of Woods' examples) was intended as a *suggestion* as to how an entailment theory for predicate logic might handle such examples such as:

1a) If something is blue, then something is colored.

1b) If something, is blue, then it is colored.

These examples might be thought to present difficulties for a predicate calculus of entailment.

Our suggestion is that in a predicate calculus for entailment based an a relevance condition of sharing predicate letters, that analysis of the predicates involved in the above examples would be necessary in order to certify these statements as logically true entailments (2). Woods shows, with some plausibility, that this fails to resolve the difficulty, since the analysandum must entail

(2) Notice two things: 1) As they stand, these examples are not logical truths, a point we shall return to later; 2) 'logically true' is used in a broad sense here.

the analysans for the analysis to be correct; but this entailment cannot hold, since there is no commonality of predicates ([3]).

Even accepting this framework, we believe that there is a way out of the problem posed by Woods, In this sort of case, a special proviso might be made, similar to a definition rule. We might require that before CIII be used in an evaluation, every predicate be replaced by its analysans. We are still faced with the problem of certifying that a given analysandum entails the proposed analysans, even in violation of CIII. At this juncture we are prepared to go part of the way with Woods: showing that an analysis is satisfactory, however this might be done, is sufficient for certifying that the analysandum entails the analysans.

This seems to us to admit only that there may be primitive entailments (for which CIII fails) in the axiomatization of a given (*descriptive*) discipline. But is this so damaging an admission ? Before giving the reasons why we think that it is not damaging, remember that 'entails' is to be taken as the converse of 'is deducible from', and that 'if... then...' is to be taken as the object language counterpart of entails'. So if 'x is blue, then x is colored' will be true if 'x is colored' can be deduced from 'x is blue'. If 'x is blue' is taken as a premise, then by what means are we to deduce 'x is colored' ? This is the problem we are worried about; what justifies this move, or inference ? ([4])

Our reasons for thinking that admitting primitive entailments is harmless follow. (1) Once the required analysis are carried out, we can dispense with each analysandum, in the rational reconstruction of the discipline with which we are concerned, in favor of its analysans. CIII will then hold in this rational reconstruction. (2) The force of the admission can be seen if we look at the situation "in the opposite direction". Suppose we have a family of predicates, such as 'x is blue', 'x is red', etc., and we have noticed that each of the objects in a particular

([3]) Note that this is a very strong criterion for evaluating an analysis, and many philosophers would insist only on extensional equivalence or even weaker conditions.

([4]) Woods fails to address himself to this problem, and merely assumes that the corresponding metalinguistic entailment statement is true. He leaves us in the dark as to how the deduction would proceed.

domain of discourse satisfies exactly one of these predicates. For certain practical purposes we might find it convenient to introduce a term, 'x is colored', to abbreviate the disjunction of these predicates. Naturally, we would want to hold that 'if x is blue, then x is colored' is a true entailment in our language in light of this abbreviation. Consequently, we would want a rule of definitional elimination to cover this case, so as to make our resulting theory a definitional extension of our previous theory; it seems plausible to introduce such a rule, waiving CIII in such cases, for when the defined term is eliminated. CIII can be applied. (3) Our basic response, however, is to reject the framework of the question. The entire issue seems beside the point. It is useful to distinguish between an axiomatization of a pure calculus of entailment, which would contain no non-logical (i.e., descriptive) terms, and an axiomatized (descriptive) subject whose logical theory is that of the entailment calculus. Anderson and Belnap have not attempted to axiomatize all the various subjects from which Woods draws his examples. (1a) and (1b) would appear to us to be more like genuine counter-examples only if Anderson and Belnap were committed to the following thesis:

(L). All necessary (i.e., *a priori*) truths, whether containing descriptive expressions or not, are truths of logic.

We might call this the logicist thesis of necessity. Furthermore, we must be persuaded that any logistic system which fails to certify all necessary truths as such (whether they are purely logical or not) is inadequate or unsatisfactory, a thesis for which no arguments have been given by Woods.

Presumably, '$A \rightarrow B$' will be, if true, necessarily true, and if false, necessarily false, where '$\rightarrow$' is read 'if-then (in its entailment sense). Suppose in a particular case, 'A' and 'B' abbreviate expressions having non-vacuous occurrences of descriptive terms. Then to present such a statement as a counter-example to Anderson's and Belnap's *formal system* surely presupposes that '$A \rightarrow B$' is true, and, if true, then necessarily so. But it also presupposes that Anderson and Belnap are committed, or should be committed, to the thesis that if '$A \rightarrow B$' is true, then their logical apparatus, if it is to be regarded as satisfactory, must certify 'A → B' as true. For otherwise if thre are necessary statements of this form, having descriptive expressions which appear to occur non-vacuously and which are not certifiable as such by a given formal system, then it might be held that there are necessary truths which are neither truths of logic nor statements exemplifying the form of any truth of logic. This latter is simply to hold that there are necessary statements in which descriptive terms occur essentially. We would then have prima facie genuine counter-examples to the theory which incorporates the formal system and the metatheoretical claim that there are no necessary statements other than the logical truths of the logical system(and interpretations of these) in ordinary language, perhaps. But are Anderson and Belnap committed to (L) ? Do they believe that no true entailment statement contains descriptive terms essentially ? Surely they need not be committed to this thesis. As Quine ([7], p. 139) and others have pointed out, "not all analytic statements are instances of logical forms all of whose instances are analytic." And it is acknowledged by many logicians (e.g., Quine) that such statements as (1a) and (1b) are not the business of logicians, "are not supposed to be provided for by the rules of logic" ([7], p. 139). The reason for this is that such statements strictly are not logical statements in the sense of a statement's being logical if and only it it contains no constants beyond a specified logical vocabulary.

The logicist thesis is, however, an interesting one, for if it is true, and if it be granted that all logically true statements are analytic, then the problem of synthetic *a priori* truth could certainly be laid to rest. Since many philosophers accept the view that necessarily every logical truth is analytic, the logicist thesis seems equivalent to the denial of synthetic *a priori* truths. From certain points of view, the logicist thesis is a very attractive one. Logicians might try to extend the class of logical truths so as to be coextensive with the class of necessary truths, as Carnap has attempted to do, his explication also making use of the notion of 'meaning postulates.' In axiomatizing an extra-logical subject, introducing "ludicrously many special postulates, one for every particular case of such entailments" (Woods, p. 368) would not seem at all ludicrous when introduced as meaning postulates or implicit definitions for the descriptive terms of that particular subject. But why should it appear damaging to Anderson and Belnap that additional postulates asserting primitive entailments to obtain, are needed in the axiomatization of a non-logical subject ? For it is certainly not the logician's task, *qua* logician, to provide a logical system that would capture all the necessary truths of any subject matter, for this could involve axiomatizing all subjects. Besides, it would be an impossible task, as Gödel's incompleteness result shows, assuming that every purely mathematical truth is a necessary truth. At any rate, we believe that we have shown, perhaps more clearly than before, that the examples which Woods advances are not damaging to the enterprise which Anderson and Belnap and other logicians have set for themselves.

In connection with the above discussion, it is to be noted that Woods has not, to our knowledge, offered *any* criteria for determining whether one statement entails another, nor has he even offered criteria for establishing the truth or necessity of any 'if-then' statement. Since some statements of the 'if-then' form are contingent, and so presumably not true entailments, a constructive account would surely be expected to do at least this. Would Woods, for example, hold that there are "ludicrously many" intuitive and for some mysterious reason true, entailment statements which are such that one must simply "see" that the "deduction" could be performed ?

III

The independent proofs. In our comments of Lewis' so-called independent proof of the paradox that any proposition is implied by an impossible proposition, we rejected the disjunctive syllogism (DS) as a rule of inference. Both Pollock and Woods have expressed the view that such a rejection is unwarranted.

Pollock's complaint is that Belnap's proof that the DS commits a fallacy of relevance amounts to nothing more than showing that if we accept it together with all of our other rules of inference, we can prove the paradoxes of strict implication. This merely shows that if something is wrong with the paradoxes, then something is wrong with one of our rules, but not necessarily the rule of detachment (5). Thus he has not succeeded in pinpointing an error in our proofs of the paradoxes. (Pollock [6] pp. 188-189).

This is an unsound argument. Belnap and Anderson give an independent proof in addition to other evidence to show that the inference from $\sim$A and (A $\vee$ B) to B is "a simple inferential mistake" and commits a fallacy of relevance ([2], pp. 18-22). It is simply false that the arguments advanced by Belnap and Anderson for rejecting the rule of DS amount of showing that "if something is wrong with the paradoxes then something is wrong with one of our rules." They first show that something is wrong with the paradoxes ([1], pp. 333-340) and then in their later paper show just precisely what is wrong with the rule of DS which is employed by Lewis in his independent proof of the paradox that any proposition is implied by an impossible proposition. Pollock's argument is therefore without force. However, Woods has attended to the details of Anderson's and Belnap's argument for the claim that rule DS is not a valid principle of inference. It therefore behooves us to examine Woods' argument, for he believes that it is decisive against us as well (See [9], p. 369).

The essence of Woods' argument is that Anderson and Belnap confuse invalidity with unsoundness.

Before we consider Woods' point, let us be clear about how Anderson and Belnap argue in the passage to which Woods addresses himself (6). Their argument may be put as follows:

(1) A necessary condition for a disjunction supporting the DS is that it support a corresponding subjunctive conditional.

(2) The truth of A-or-B, with truth-functional 'or' is not sufficient for the truth of 'If it were not the case that A it would be the case that B'.

Therefore

(5) This remark also applies to the rule of DS since Pollock mentions it in the same context.

(6) It should be noted that Woods fails to consider the "independent proof" for the claim that DS is inferential mistake. (See [2], p. 21).

(3) 'A-or-B', with truth-functional 'or' does not support the DS.

Obviously this is a valid argument. Since Woods holds that premise (1) is true ([8], p. 316), he must therefore show that premise (2) is false. Presumably he believes himself to have done this by his argument to show that (2) is unacceptable because it is premised on a confusion of invalidity with unsoundness. He argues as follows:

(4) Anderson and Belnap hold that the truth functional 'or' "cannot generate the corresponding conditional because in those cases where p∨q is affirmed solely on the strength of p, one could not say: "p∨q is true and p is false, so q is true", just because one could no longer say that p∨q". ([8], p. 316).

(5) At most this shows that the deduction of q from p∨q and p is unsound. ([8], p. 317).

Therefore

(6) When Anderson and Belnap claim that such a deduction is invalid they confuse unsoundness with unvalidity. ([8], p. 317).

To begin with, it should be pointed out that DS is only rejected in the general case. It is not rejected in the case where there is an inference from $A_1 \vee A_2 \vee \ldots \vee A_n$ and $\sim A_1$ to $A_2 \vee \ldots \vee A_n$. When $A_1 \vee A_2 \vee \ldots \vee A_n$ ($n = 2^k$, where $k =$ the number of distinct sentence variables) is a Boolean expansion. We point this out to make it clear that the rejection of DS for the truth-functional 'or' is not categorical, but is only confined to those cases where there is no relevance between the disjuncts. With this in mind let us now examine premise (4).

It is clear from premise (4) that Woods interprets Anderson and Belnap as basing their premise (2) on epistemological considerations. (Note that Woods speaks of affirming p∨q solely on the strength of p ([8], p. 316).) This is a mistake. The issue regarding (2) is, and is presented by Anderson and Belnap as, a purely logical one, not as an epistemological one. In order to establish premise 2) it is sufficient for them to point out that "It is true that either Napoleon was born in Corsica or else the number of the beast is perfect (with truth-functional "or"); but it does not follow that had Napoleon *not* been born in Corsica, 666 would equal the sum of its factors." The reason why it does not follow is the elementary one that the premise can be true and the conclusion false. This has nothing to do with unsoundness — it has only to do with a sufficient condition for invalidity. Woods for changing that Anderson and Belnap have confused invalidity with unsoundness is to be found in his mistaken interpretation (in premise (4)) of why Anderson and Belnap think that the truth-functional 'or' does not generate the corresponding subjunctive conditional. In premise (4) he says that one could no longer say that pvq, for the reason that it is affirmed that p is false and pvq is affirmed solely on the strength of p. (Note that in a latter passage he speaks of the "categorical assertion" of p). Thus Anderson and Belnap are accused of thinking that in the subjunctive conditional the antecedent is affirmed ! It is to this that the alleged confusion is traced. But there is absolutely no evidence that would suggest that Anderson and Belnap are guilty of such an elementary error. The reason why the corresponding subjunctive conditional does not follow is simply that there is no relevance between the disjuncts (See [2] pp. 20 and 22). Woods' premise (4) is therefore false. Hence the argument advanced by Anderson and Belnap remains undefeated.

In our criticism of Lewis' independent proof to show that a necessary proposition is entailed by any proposition we argued that the second step of the proof is inadmissible because

$$A \rightarrow (A.(A \vee B \,.\, (A \vee \sim B) \,.\, (B \vee \sim B)\,)$$

is not a tautological entailment. Woods charges us with circular reasoning here ([9], p. 370) because he thinks that our rejection of the second step is, in the end, premised on rejecting what the second paradox purports to prove, namely that A entails $(B \vee \sim B)$. This is simple not the case. The rejection of the second step is, in the end, premised on considerations having to do with relevance. The reason why $A \rightarrow (B \vee \sim B)$ is rejected is because of CIII. And since it is possible to determine whether A is relevant to $(B \vee \sim B)$ without having to first determine whether A entails $(B \vee \sim B)$, (Woods' second condition for defining irrelevance), Woods is thus mistaken in charging us with circularity.

Uniform substitution. Woods asks, "Is uniform substitution a valid mode of inference ?» ([9], p. 370). The answer is that in one sense of 'valid', certainly not ! The sense of valid involved is a common one, a necessary condition for which is: A rule is valid only if whenever it is applied to a true statement, the result is a true statement. It is very easy to see that a rule of uniform substitution is not valid in this sense of 'valid'. If a logically false statement is substituted for each component of a *contingently* true statement, a logically false statement results; thus, uniform substitution does preserve "logical truth" and does preserve inconsistency. As a rule of inference in logistic systems whose axioms are valid, uniform substitution does not result in invalid inferences (in the above sense of 'valid'), as can be proved for many such systems. This, however, does not mean that substitution preserves validity in any stronger sense of that term. But the above remarks, hopefully, provide some justification for our observation that substitution as a rule of inference must be relativized to particular axiom systems.

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