ENTAILMENT AND RELEVANT IMPLICATION

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In [2], Anderson and Belnap claim that two kinds of fallacies are involved in the well-known paradoxes of implication — (1) fallacies of *modality,* which "arise when it is claimed that entailments follow from, or are entailed by, *contingent* propositions," and (2) fallacies of *relevance,* which sanction "the inference from A to B even though A and B may be totally disparate in meaning" ([1]). The system E of entailment, as defined for example in [1], is presented by these authors as a sentential logic free from both kinds of fallacies.

It has been observed by J. Michael Dunn, however, that the discussion in [2] of relevance is more or less independent of the accompanying discussion of modality. For Anderson and Belnap use that part of the discussion of [2] which hangs on relevance alone to motivate, in the pure theory of implication, not the implicational part of E but rather the weak theory of implication of Church's [7]. By adding those axioms and rules of E which govern conjunction, disjunction, and negation to Church's system, Anderson and Belnap have constructed a system R, first defined in [4], which avoids fallacies of relevance in their sense but not fallacies of modality. And indeed, one might think of R not as formalizing a theory of entailment in any sense, but merely as the formalization of a kind of conditional which requires relevance of antecedent to consequent as a necessary condition for its truth. And it is hoped that R will be of service in the logical analysis of various kinds of conditional which have hitherto been accounted wayward — counterfactual, subjunctive, lawlike, and so forth.

The question now arises, "Can R be used to support a theory of entailment ?" In a limited sense, the answer is clearly "Yes", for we may proceed along lines well-marked by Quine and others. Indeed, we may distinguish between those systems, like the Lewis

([1]) Cf. [2], esp. pp. 42ff.

calculi, which attempt to *express* a formalized entailment, and those systems, like classical logic, which are content to *indicate* entailment according to the recipe, "If A → B is logically valid, A entails B." R, like the classical system HK, *expresses* not entailment but a variety of the conditional; however, if ever A → B is a *theorem* of R, it can presumably be so only on grounds that are *purely* logical; if we go on to view R as *indicating* entailment in this situation according to the classical recipe, we can then ask what theory of entailment it is that R supports.

The answer is rather satisfying; R supports exactly the theory of entailment which E expresses, in the area where the two systems coincide. To explicate this last remark, let me note that both R and E are built up from a denumerable stock of sentential variables by means of the sentential connectives '&', 'v', '—', and '→'. But though the truth-functional connectives get their usual readings in both systems, the arrow of E has built into it a modal character absent from the arrow of R; a suggested reading for the former is 'that ... entails that ---', while the latter is simply a hardy version of 'if ... then ---' (2).

In virtue of the above remarks, it is clear that although sentences of the form A → B *say* different things in the two systems, such sentences should be *logically* true under the same conditions *provided* that no arrows occur in either A or B. And indeed, in proving a somewhat stronger result in [4], Belnap has showed this to be the case.

On the other hand, when arrows occur nested within arrows the differing interpretations associated with R and E assure that in general it is not sufficient for a sentence to be marked logically true in E that it is logically true according to R. Take, for example, the R-theorem p→.p→q→q (3). This neets An-

(2) In fact, more familiar classical and intuitionistic versions of 'if ... then ---' may be defined on the basis of the arrow of R, as was noted in part in the abstract 10 and will be explored more fully in a subsequent paper. The stilted construal of the arrow of E is a concession to use-mention fans and will henceforth be abandoned.

(3) 'p', 'q', etc., shall henceforth be used as syntactical variables for sentential variables; 'A', 'B', etc., for arbitrary formulas. Among other

derson-Belnap relevance requirements, but since p may be contingent it clearly fails to meet their criteria for avoiding modal fallacies.

One way to pass from a theory of the conditional to a theory which expresses entailment lies in the excision of those theorems of the first theory which commit modal fallacies. This is essentially the course taken by Anderson and Belnap in [2]. Another, and more traditional, approach lies in the addition of modal operators to the underlying theory of the conditional, whatever it may happen to be. The result of such explicit enrichment of the underlying system, as in the Gödel-von Wright-Feys construction of Lewis modal logics, has resulted in systems in which both the conditional and entailment can be *expressed*; the former as $A \rightarrow B$, the latter as $N(A \rightarrow B)$. We ask whether R also can be enriched with modal operators in a natural way so that the resulting theory of entailment meets the conditions set down for such a theory by Anderson and Belnap.

The answer again is "Yes"; indeed, such systems were considered independently by Bacon in [3] and by me in my dissertation. Bacon has an interesting hierarchy of such systems, based on the addition of the kind of alternative axioms governing necessity explored by Lewis. Of these systems, however, the one which makes interesting connections with E is the theory which results by imposing an S4-like structure on the underlying non-modal logic R; this is hardly surprising, since the motivating conditions for E give that system the modal character of S4. It is the S4-like version of R, called here NR, which I now define and investigate.

NR is built up by the usual formation rules from $\aleph_0$ sentential variables, the sentential constant f, (4) and the connectives

conventions, I rank the connectives thus in order of increasing scope: '&', 'v', '$\rightarrow$', indicating the first by simple juxtaposition on occasion. Otherwise association shall be to the left, and the conventions of Church shall be employed for the replacement of parentheses by dots.

(4) Use of the sentential constant enables us to define A as $A \rightarrow f$ and to avoid several negation axioms. The extension is conservative with respect to R as defined by Belnap, which is proved in my dissertation (U. of Pittsburgh, 1966) building on Anderson-Belnap results reported in [5] and elsewhere.

'&', 'v', '→', and 'N'. The axiom schemes and rules are the following:

NR1.	A→A	Identity
NR2.	A→B→.B→C→.A→C	Transitivity
NR3.	(A→.A→B)→.A→B	Contraction
NR4.	A→.A→B→B	Assertion
NR5.	AB→A	&E
NR6.	AB→B	&E
NR7.	(A→B) (A→C)→(A→BC)	&I
NR8.	A→AvB	vI
NR9.	B→AvB	vI
NR10.	(A→C)(B→C)→(AvB→C)	vE
NR11.	A→f→f→A	Double negation
NR12.	NA→A	NE
NR13.	N(A→B)→.NA→NB	N Distribution
NR14.	NANB→N(AB)	N&I
NR15.	NA→NNA	NNI
NR16.	A(BvC)→ABvAC	& Distribution
NR17.	A, A→B ⊢ B	*Modus ponens*
NR18.	A, B ⊢ AB	Adjunction
NR19.	A ⊢ NA	Necessitation

Inspection of the non-modal axioms shows them to yield the system R (cf. footnote 4), while the modal axioms and the rule NR19 are the usual ones which produce S4 from the set of classical tautologies. (But NR14 is redundant in that case.) In fact, NR would turn into S4 were we to add in addition the axiom of paradox B→.A→A; though this axiom is consistent with and independent of the other axioms, these facts seem to me no argument on its behalf.

It is readily established that every theorem of E is a theorem of NR when 'A entails B' is defined as 'N(A→B)'. To show this, it suffices to show that all axioms of E are theorems of NR on translation, and that the rules of E are admissible in NR. It would be nice to show also that all non-theorems of E are non-theorems of NR on translation, but of this I have no proof. It is, however, the case that the principal motivating condition for the E theory of modality is met in NR, and indeed in a somewhat strengthened form. The key theorem is the following:

Theorem 1. Suppose A contains no occurrences of the sign N. Then A→NB is a non-theorem of NR.

Proof. Consider the following matrix:

→	+3	+2	+t	−f	−2	−3	&	+3	+2	+t	−f	−2	−3	N
+3	+3	−3	−3	−3	−3	−3		+3	+2	+t	−f	−2	−3	+t
+2	+3	+2	−3	−2	−2	−3		+2	+2	+t	−2	−2	−3	+t
+t	+3	+2	+t	−f	−2	−3		+t	+t	+t	−3	−3	−3	+t
−f	+3	−3	−3	+t	−3	−3		−f	−2	−3	−f	−2	−3	−3
−2	+3	+2	−3	+2	+2	−3		−2	−2	−3	−2	−2	−3	−3
−3	+3	+3	+3	+3	+3	+3		−3	−3	−3	−3	−3	−3	−3

Let +3, +2, and +t be designated, let formulas AvB be evaluated as (A→f) (B→f)→f, and let the sentential constant f be assigned the matrix value −f on all evaluations. Then it is readily seen that axioms of NR are always designated and that this property is preserved under the rules of NR. But for A which satisfies the hypothesis of the theorem, its value on an assignment which gives each sentential variable the value +2 will be ±2; on the other hand, on all assignments NB will have either the value +t or −3; inspection of the table for → shows that if A does not contain N, A→NB will accordingly have the value −3 on an assignment of +2 to all sentential variables; hence no sentence of the suggested form is a theorem of NR. This concludes the proof of theorem 1. (I note that the suggested matrix is an adaptation of matrices due to Ackermann and to Belnap.)

It is also the case in E that no negated entailments entail entailments. We state the corresponding fact for NR as a corollary to the previous theorem.

Corollary 1.1. For all A and B, NA→NB is a non-theorem of NR.

Proof. Consulting the previous matrix, we note that NA must take the value $-f$ or $+3$ on all assignments, while NB must take the value $+t$ or -3. In any case, the value of NA→NB on any assignment will be -3.

The semantical situation of E is unclear, as Anderson reported in [1]. There are, however, some important partial results, and we now ask whether these results apply to NR also.

First, Belnap and Wallace investigated in [6] the fragment E_I of E determined by its axioms which contain only occurrences of '→' and '—' and the rule of *modus ponens*; by providing a Gentzen consecution calculus for E_I^-, they were able to show this system decidable. The corresponding fragment NR_I^- is that determined by NR1-NR4, NR11-NR13, NR15, NR17, and NR19. If we consider only such formulas involving f as are definitional abbreviations for formulas involving negation, we obtain a Gentzen system from axioms $A \vdash A$ and the rules *C*, *W*, *P*, *N*, and *Y* of formulation II of LKY in Curry's [8]. By examining a proof of a translation of a formula of E_I^- in this system, it is easy to show that the same formula can be proved in the Belnap-Wallace consecution calculus for E_I^-. Since it continues to be the case that all theorems of E_I^- are theorems of NR_I^-, it follows that the negation-*entailment* fragments of E and NR are identical, in the sense in which these fragments have been specified above.

A second area in which the semantical character of E is clear lies in its set of first-degree formulas — formulas in which no subformula of the form $C \rightarrow D$ contains as its proper subformula an item of the form $A \rightarrow B$, or, briefly, formulas in which no entailments are nested within entailments. Belnap provided in [4] a complete semantics and a decision procedure for the first-degree fragment of E and of R; in fact, the set of first-degree theorems of these two systems is the same. Our second theorem records the fact that this fragment of E is exactly translatable into NR.

Theorem 2. Let A be a first-degree formula of E. Then A is a theorem of E iff A is on translation a theorem of NR.

Proof. Suppose that A is on translation a theorem of NR. Let $A_1, \ldots, A_n$ be a proof of the translated A, and let A_i^* result from A_i by deleting all occurrences of N. Noting that all axioms of NR remain axioms of R when all N's are erased, and that the rules continue to hold after erasure, it is clear that A_n^* is a theorem of R. But A_n^* with all the N's erased is just A. Hence A is provable in R; if A is first-degree, it is by Belnap's result also provable in E. Hence if A is on translation a theorem of NR, A is a theorem of E if it is a first degree formula of E. On the other hand, all theorems of E are theorems of NR on translation, as noted above, thus concluding the proof of theorem 2.

So far as can be presently ascertained, accordingly, E and NR express the same theory of entailment. A choice between the two systems is accordingly to some degree a matter of taste, but it seems to me that the following considerations favor NR. First, NR separates those issues in the critique of classical logic which are generated by the classical analysis of the conditional from the issues which lead to the explicit introduction of modal operators. Since as Quine has rightly pointed out, it is the classical analysis of the conditional — not the absence of squares — which leads to what is felt to be strange in the classical doctrine of entailment, one might expect a full-fledged theory of entailment to include a theory of the kind of conditional which, in the vocabulary introduced here, *indicates* entailment. Second, since the question, "Which axioms shall govern the modal operators ?" is notoriously in doubt, formulating one's theory of entailment via NR leaves open the possibility of changing one's assumptions about the modalities while leaving one's underlying theory of the conditional alone (5). Third, however, those attempts to motivate modal logic which rest on the analysis of deduction rather than on classical semantic analysis seem very frequently to motivate a system in the neighborhood of S4; one thinks not only of [2] but of Curry's analysis in [8], and in a certain sense of McKinsey's [9]. NR shows that it is possible to

(5) It is important in this connection again to note that the kind of conditional which indicates entailment in the Anderson-Belnap sense is not the only conditional expressible in R. Cf. footnote 2 and the remarks to which it is appended.

preserve these insights in a semantical frame otherwise highly non-classical.

I conclude, as I began, with an observation of Dunn. In his dissertation, Dunn provides an algebraic semantics for R, and he notes that axioms like my NR12-NR15 turn the algebra of R into a closure algebra. For those who, unlike Dunn and me, consider non-classical logics philosophically meaningless but worth investigating for the mathematical structures to which they give rise, I submit NR also (6).

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(6) My thanks are due to the National Science Foundation of the U.S.A., whose partial support of Anderson and Belnap's studies of entailment at one time partially supported me as well. Since I have depended throughout on Belnap's work and occasionnally on his advice, any errors are probably his fault; errors not so traceable may perhaps be blamed on my conversations with Dunn; cf. his dissertation (U. of Pittsburgh, 1966), before crediting me with *original* folly.

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