# A FUNCTION WHICH BOUNDS TRUTH-TABULAR CALCULATIONS IN S5

M. K. RENNIE

(0). In this paper I define a function for all formulae in a standard formulation of modal logic, and using this function prove that every such formula has a finite characteristic matrix for S5-validity. I then investigate the practical applicability of the decision procedure for S5 which results from the main theorem.

(1). Firstly we define S5-Validity. The semantics involved is a theoretically inessential modification of that in Kripke's [4], which is presupposed for this whole paper. We assume that our modal logic has primitive logical constants $\sim, ., \Box$, propositional variables $p_1, p_2, \ldots$, and standard formation rules. Let $\alpha$ be a wff of this modal logic containing propositional variables $p_1, \ldots, p_k$. A model for $\alpha$ is a matrix (in the algebraical sense of the word) $\Phi_n$ where

$$\Phi_n = \begin{bmatrix} \varphi_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \varphi_n \end{bmatrix} [p_1 \ldots p_k] = \begin{bmatrix} \varphi_1\ p_1 & \ldots & \varphi_1\ p_k \\ \cdot & & \cdot \\ \cdot & & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \varphi_n\ p_1 & \ldots & \varphi_n\ p_k \end{bmatrix}$$

Each $\varphi_i$ is a function from the set $\{p_1, \ldots, p_k\}$ of propositional variables to the set $\{1, 0\}$. So $\Phi_n$ is an $n \times k$ matrix each of whose elements is either 1 or 0. For a given n and k, there are $2^{(n \times k)}$ separate models $\Phi_n$: if needed we will denote these by $\Phi^m{}_n$, where $1 \leqslant m \leqslant 2^{(n \times k)}$.

Given a particular model $\Phi_n$, we define a semantical valuation function for $\alpha$ with respect to the model $\Phi_n$, nd in terms of this function we define S5-Validity. The clauses in the definition are:

(1.1) $\bigwedge_{r=1}^{n} \bigwedge_{i=1}^{k} (\Phi_n \, Val_r(p_i) = \varphi_r p_i)$

(1.2) $\bigwedge_{r=1}^{n} ((\Phi_n \, Val_r (\beta) = 1. \supset . \Phi_n \, Val_r (\sim\beta) = 0) .\&.$
$(\Phi_n \, Val_r (\beta) = 0. \supset . \Phi_n \, Val_r (\sim\beta) = 0))$

(1.3) $\bigwedge_{r=1}^{n} ((\Phi_n \, Val_r (\beta) = 1.\&.\Phi_n \, Val_r (\gamma) = 1 : \supset :$
$\Phi_n \, Val_r (\beta.\gamma) = 1) .\&. (\Phi_n \, Val_r (\beta) = 0 .v.$
$\Phi_n \, Val_r (\gamma) = 0 : \supset : \Phi_n \, Val_r (\beta.\gamma) = 0))$

(1.4) $\bigwedge_{r=1}^{n} ((\bigwedge_{s=1}^{n} (\Phi_n \, Val_s (\beta) = 1) . \supset . \Phi_n \, Val_r (\Box\beta) = 1)$
$.\&. (\bigvee_{s=1}^{n} (\Phi_n \, Val_s (\beta) = 0) . \supset . \Phi_n \, Val_r (\Box\beta) = 0))$

Clauses (1.1) - (1.4) define the n functions $\Phi_n$ $Val_r$, for $1 \leqslant r \leqslant n$. (1.1) is the initial clause, linking the function to the model $\Phi_n$. (1.2) and (1.3) are standard truth-functional clauses, and (1.4) is the peculiarly modal clause, being the recursive clause for $\Box$. From (1.4) it follows that

$$\bigwedge_{r=1}^{n} \bigwedge_{s=1}^{n} (\Phi_n \, Val_r (\Box\beta) = \Phi_n Val_s (\Box\beta)($$

that is, a well-formed part of $\alpha$ of the form $\Box\beta$ will have the same value for any of the functions $\Phi_n$ $Val_r$ for $1 \leqslant r \leqslant n$. This leads, *inter alia*, to the well-known reduction theorem for modalities in S5.

For the wff $\alpha$, there will be n values, one for each of the functions $\Phi_n$ $Val_r$. Of these, we select $\Phi_n$ $Val_1$ as *the* value of $\alpha$: we put

(1.5) $\Phi_n \, Val(\alpha) = {}_{df}\Phi_n \, Val_1(\alpha)$

For each n, there are $2^{(n \times k)}$ separate $\Phi_n$'s: we define n-S5-Validity for each n thus

$$(1.6)\ \text{n-S5-Valid}\ (\alpha) =_{\mathrm{df}} \bigwedge_{m=1}^{2^{(n\times k)}} (\Phi^{m}{}_{n}\ \mathrm{Val}\ (\alpha) = 1).$$

The notion of n-S5-Validity bears an obvious analogy to the notion of k-validity in quantification theory, for which see Hilbert and Bernays [3] vol. I §§4-5. Using the notion, we define S5-Validity by

$$(1.7)\ \text{S5-Valid}\ (\alpha) =_{\mathrm{df}} \bigwedge_{n} (\text{n-S5-Valid}\ (\alpha)).$$

The quantifier here is unbounded: n ranges over all natural numbers $\geqslant 1$. Hence the definition (1.7) does not lead directly to a decision procedure for S5-Validity.

It will be apparent that the given definition of S5-Validity is equivalent to that of Kripke [4], p. 69. In proceeding to the definition, we have diverged at the following points. Firstly, we do not have a "model structure" and a separate "model": the part played by G and K in Kripke's model structures is played by 1 and $\{1, \ldots, n\}$ in our models. Secondly, we have separated Kripke's binary function $\Phi$ into n monadic functions $\varphi_1, \ldots, \varphi_n$: this emphasizes the fact that Kripke's K, so far from being a set of possible worlds, is simply an index-set for a set of functions. Thirdly, we have not extended the model to form a semantical valuation function as well, but instead have defined the valuation function separately, tying it to the model via (1.1). Fourthly, the intermediary definition of n-S5-Validity, (1.6), defines a notion which Kripke nowhere needs.

The divergences from Kripke's treatment allow us to state our main theorem which leads to a decision procedure in terms of familiar truth-tabular methods, and also make the transition from classical formal semantics for propositional calculus to formal semantics for modal logics a little smoother. There is the incidental advantage, already noted, that they make apparent the fact that talk of "possible worlds" is entirely dispensable, and is only ever introduced for its heuristic value.

(2). We define a pair of functions by simultaneous recursion on the length of a wff $\alpha$, with propositional variables $p_1, \ldots, p_k$.

(2.1) $\bigwedge_{i=1}^{k}$ (nopon $(p_i) = 0$)

(2.2) $\bigwedge_{i=1}^{k}$ (nonon $(p_i) = 0$)

(2.3) nopon $(\sim\beta)$ = nonon $(\beta)$

(2.4) nonon $(\sim\beta)$ = nopon $(\beta)$

(2.5) nopon $(\beta.\gamma)$ = nopon $(\beta)$ + nopon $(\gamma)$

(2.6) nonon $(\beta.\gamma)$ = nonon $(\beta)$ + nonon $(\gamma)$

(2.7) nopon $(\Box\beta)$ = 1 + nopon $(\beta)$

(2.8) nonon $(\Box\beta)$ = nonon $(\beta)$

The functions nopon and nonon are both syntactical functions whose arguments are wffs of modal logic and whose values are natural members. They are so-called because nopon (nonon) counts the number of positive (negative) occurrences of necessity in a formula, where a positive (negative) occurrence of necessity is a $\Box$ occurring within the scope of an even (odd) number of $\sim$'s (in primitive notation).

An immediate consequence of the definition is that if $\alpha$ is a wff of classical propositional calculus, then nopon $(\alpha)$ = nonon $(\alpha)$ = 0. We may also note that these functions are independent of the modal depth of a formula: for example we have

nopon $(\Box^n p \supset p) = 0$, whereas the modal depth of $\Box^n p \supset p$ is n, and

nopon $(p \supset \bigvee_{i=1}^{n} \Box p_i) = n$, whereas the modal depth of $p \supset \bigvee_{i=1}^{n} \Box p_i$ is 1.

The basic use of these functions is summed up in our main theorem

(2.9) $((1+\text{nopon}(\alpha))$ -S5-Valid $(\alpha))$ $.\supset.$ S5-Valid $(\alpha)$,

that is, if $\alpha$ has the value 1 for all $\Phi_n$'s, where $n=1+\text{nopon}(\alpha)$, then $\alpha$ is S5-Valid. Since for any wff $\alpha$, nopon $(\alpha)$ is finite, (2.9) provides at least a theoretical decision procedure for S5-Validity. Notice that although (2.9) entails that for every wff there is a finite characteristic matrix for S5-Validity, this does not contradict Dugundji's familiar result (in his [2]) that there is no finite characteristic matrix for S5, i.e. for all the wffs of S5. Dugundji's proof shows that for any n, there is a wff which is n-S5-Valid but not (n + 1)-S5-Valid, and hence there can be no finite matrix which is characteristic for all wffs.

The next section is devoted to the proof of (2.9), and may be omitted without loss of continuity.

(3.) For the proof of (2.9), we first prove lemma (3.1). This proof is to be taken to be inserted at 3.05 in Kripke's [4], so that the definitions and terminology for semantic tableaux are established. We state

(3.1) In an S5-construction for $\alpha$ there are at most nopon $(\alpha)$ applications of rule Yr.

The proof proceeds in the first instance by a complete induction on nopon $(\alpha)$. For the basis, we consider nopon $(\alpha) = 0$, and now prove that in an S5-construction for $\alpha$ there are no applications of Yr. This proof proceeds by induction on the length of wffs $\alpha$ such that nopon $(\alpha) = 0$. If $\alpha$ is a propositional variable, then no rules are applicable, and *a fortiori* there are no applications of Yr. If $\alpha$ is of the form $\sim\beta$, then nonon $(\beta) = 0$, and by application of Nr, $\beta$ is placed on the left of the tableau in which $\alpha$ is on the right. So now we need to prove that if nonon $(\beta) = 0$ and $\beta$ is on the left of the main tableau, then there are no applications of Yr, and to do this we carry out a «sub-induction» on the length of $\beta$. If $\beta$ is a propositional variable, then no rules are applicable, so there are no applications of Yr. If $\beta$ is of the form $\sim\gamma$, then Nl is applied: $\gamma$ goes to the right of the tableau and nopon $(\gamma) = 0$. But now this falls under our inductive hypothesis about $\alpha$, since $\gamma$ is of less syntactical length than $\alpha$. If $\beta$ is of the form $(\gamma \,.\, \delta)$, then we apply $\wedge 1$ so that both $\gamma$ and $\delta$ appear on the left. Since nonon $(\gamma \,.\delta) =$ nonon $(\gamma)$ + nonon $(\delta) = 0$, we have nonon $(\gamma) =$ nonon $(\delta) = 0$. Hence ex. ind. hyp. on $\beta$, neither $\gamma$ nor $\delta$ will require any applications of Yr. If $\beta$ is of the form $\Box\gamma$, then we apply Y1. Since there have been no applications of Yr, there is only one tableau (or system of tableaux with alternatives within the system), and so we place $\gamma$ on the left of that tableau (or on the left of each alternative within the system). Now nonon $(\Box\gamma) =$ nonon $(\gamma) = 0$, so ex. ind. hyp. on $\beta$ there will be no applications of Yr caused by $\gamma$. The sub-induction on $\beta$ is complete, and we return to the induction on $\alpha$. If $\alpha$ is of the form $(\beta \,.\, \gamma)$, then we apply $\wedge$r to start two alternative tableaux, one with $\beta$ on the right and one with $\gamma$ on the right. Now nopon $(\alpha) =$ nopon $(\beta)$ + nopon $(\gamma) = 0$, hence nopon $(\beta) = 0$ and nopon $(\gamma) = 0$, and hence ex. ind. hyp. on $\alpha$ there are no applications of Yr. If $\alpha$ is of the form $\Box\beta$, then nopon $(\alpha) = 1 +$ nopon $(\beta)$, so nopon $(\alpha) \neq 0$, so there is nothing to prove. Induction on $\alpha$ is finished, and hence we have established the basis for the complete induction on nopon $(\alpha)$.

Now make the complete inductive hypothesis that (3.1) holds for all wffs $\alpha$ such that nopon $(\alpha) < n$. We require to prove that (3.1) holds for all wffs $\alpha$ such that nopon $(\alpha) = n$. The structure of this proof parallels the structure of the proof in the basis: we begin by induction on the length of $\alpha$, where nopon $(\alpha) = n$ and $n \geqslant 1$. Under these conditions, $\alpha$ cannot be a propositional variable, so we begin with the case where $\alpha$ is of the form $\sim\beta$. We apply Nr to place $\beta$ on the left of the tableau, and we now require to prove that if nonon $(\beta) = n$ there are at most n applications of Yr in a construction begun with $\beta$ on the left of a tableau. As in the basis, we proceed by a sub-induction on the length of $\beta$. Since nonon $(\beta) \geqslant 1$, $\beta$ is not a propositional variable. If $\beta$ is of the form $\sim\gamma$, we apply N1, $\gamma$ goes to the right, nopon $(\gamma) = n$, and now the inductive hypothesis about $\alpha$ establishes what we want, since $\gamma$ is of less syntactical length than $\alpha$. If $\beta$ is of the form $(\gamma . \delta)$, we apply $\wedge 1$ and place both $\gamma$ and $\delta$ on the left. Now nonon $(\gamma . \delta)$ = nonon $(\gamma)$ + nonon $(\delta)$ = n: hence either (a) nonon $(\gamma)$ = 0 and nonon $(\delta)$ = n, or nonon $(\delta)$ = 0 and nonon $(\gamma)$ = n, or (b) nonon $(\gamma)<$n and nonon $(\delta)<$n. The two subcases of (a) are quite symmetrical, so we treat just the first: if nonon $(\gamma)$ = 0 then, by the proof of the major basis, $\gamma$ on the left causes no applications of Yr, and if nonon $(\delta)$ = n then ex. ind. hyp. on $\beta$, $\delta$ on the left causes at most n applications of Yr, so overall we have at most n applications of Yr. For case (b), it follows that nopon $(\sim\gamma)<$n and nopon $(\sim\delta)<$n. So, by the complete inductive hypothesis, there will be at most n applications of Yr overall in S5-constructions for $\sim\gamma$ and $\sim\delta$. But there are no less applications of Yr in constructions for $\sim\gamma$ and $\sim\delta$ than there are applications of Yr in a construction beginning with a tableau with both $\gamma$ and $\delta$ on the right. Hence case (b) is established. If $\beta$ is of the form $\Box\gamma$, we apply Y1 so that $\gamma$ appears on the left of all tableaux in the set (since we are considering an S5-construction). Now nonon $(\Box\gamma)$ = nonon $(\gamma)$ = n. Ex. ind. hyp. on $\beta$, a single occurrence of $\gamma$ on the left will cause at most n applications of Yr: given however that the, at most n, applications of Yr were carried out for one occurrence of $\gamma$ on the left, or for more than one occurrence of $\gamma$ on the left causing at most n separate applications of Yr, then there will be no further applications of Yr because further applications would be superfluous. They will be superfluous since the previous applications of Yr will have created tableaux which are related to all others, in particular those with $\gamma$ on the left, (because we are considering S5-constructions), and which contain the appropriate formulae on the right to make Yr superfluous. Now if a rule is superfluous then we do not apply it, so for the several occurrences of $\gamma$ on the left there are at most n applications of Yr overall. The sub-induction on $\beta$ is now finished, and we return to the induction on $\alpha$.

If $\alpha$ is of the form $(\beta . \gamma)$, we apply $\wedge$r and start two alternative tableaux, one with $\beta$ on the right and one with $\gamma$ on the right. nopon $(\alpha)$ = nopon $(\beta)$ + nopon $(\gamma)$ = n, so either (a) nopon $(\beta)$ = 0 and nopon $(\gamma)$ = n, or nopon $(\gamma)$ = 0 and nopon $(\beta)$= n, or (b) nopon $(\beta)<$n and nopon $(\gamma)<$n. The two subcases of case (a) are symmetrical: say nopon ($\beta$) = 0 and nopon ($\gamma$) = n. Then by the proof of the major basis, the occurrence of $\beta$ on the right causes no applications of Yr and ex. ind. hyp. on $\alpha$, that of $\gamma$ on the right causes at most n applications of Yr. Hence case (a) is covered. In case (b) we apply the complete inductive hypothesis, which immediately entails that there are at most n applications of Yr caused by $\beta$ and $\gamma$ between them. If $\alpha$ is of the form $\Box\beta$, then nopon ($\alpha$) = 1 + nopon ($\beta$) = n, so nopon ($\beta$) = n—1. We apply Yr once to $\alpha$, starting out a new tableau which starts a construction for $\beta$ since $\beta$ is on the right. From the complete inductive hypothesis there are at most n—1 applications of Yr from this point onwards, hence there are at most n applications of Yr in the construction for $\alpha$. The induction on $\alpha$ is now finished, and with it the complete induction on nopon ($\alpha$), and hence (3.1) is proved.

In the proof just given, every one of the rules N1—Yr for semantic tableaux have been appealed to, and every one of the definitional clauses (2.1)—(2.8) has been used. Moreover, we have used the condition that the construction for $\alpha$ is an S5-construction. The "at most" proviso is necessary, since it is possible for a set of tableaux to close before all n applications of Yr have been carried out. To illustrate this, consider $\Box p \;.\!\supset\!.\; \Box p \vee \Box p$; its nopon is 2, yet since it is a substitution-instance of a tautology it will be validated by just one tableau thus:

| | |
|---|---|
| $\Box p$ | $\Box p \vee \Box p$ |
| | $\Box p$ |
| | $\Box p$ |

These considerations show that there are no essentially weaker forms of (3.1).

(3.2) There is one more tableau in an S5-construction for a wff $\alpha$ than there are applications of Yr in the construction.

This follows since the only rule which generates new tableaux in the set of tableaux, as distinct from alternative tableaux within a system of tableaux, is Yr, and because in a construction for $\alpha$ one sets out one tableau to begin with (with $\alpha$ on the right) regardless of the form of $\alpha$.

Hence there are $1+$nopon $(\alpha)$ tableaux in a set of tableaux forming an S5-construction for the wff $\alpha$. Now if $\alpha$ is not valid, by Kripke's § 3.2 Lemma 1, the construction for $\alpha$ is not closed: by Kripke's proof of Lemma 1, $\alpha$ will in this case be invalidated by a model defined on a model structure $\langle G, K, R \rangle$, where the cardinality of K is equal to the number of tableaux in the construction. Hence if $\alpha$ is to be invalidated, it will be invalidated by a model defined on an m.s. $\langle G, K, R \rangle$ where $\overline{\overline{K}} = 1 +$ nopon $(\alpha)$.

Now to prove (2.9), all that is required is a formal proof of the remark in §1 that Kripke's K plays the same role in his semantics as $\{1, \ldots, n\}$ does in ours, and hence that $\overline{\overline{K}}$ plays the same role as n does in ours. Such a proof is totally tedious, and will not be given: we now take (2.9) to have been proved.

(4). We now investigate the practical implications of the semantics laid down in § 1, and the main theorem (2.9) of § 2, for truth-tabular calculations in S5. If $\alpha$ has the k propositional variables $p_1, \ldots, p_k$, and if nopon $(\alpha) = n$, then we need to demonstrate $(n+1)$ -S5-Validity of $\alpha$ in order to demonstrate its S5-Validity, and in order to do this we need to calculate the value of $\alpha$ with respect to the $2^{(n+1)\times k}$ different $\varphi^{m}{}_{n+1}$'s, where $1 \leqslant m \leqslant 2^{(n+1)\times k}$.
Each $\varphi^{m}{}_{n+1}$ will have k columns and $n+1$ rows.

To illustrate the method of calculation, let $\alpha$ be $\Box(p \supset q) . \supset . \Box p \supset \Box q$. We calculate nopon $(\alpha)$ thus:

$$\begin{aligned}
\text{nopon } (\alpha) &= \text{nopon } (\Box(p \supset q) . \supset . \Box p \supset \Box q) \\
&= \text{nopon } (\sim(\Box(\sim p \vee q) . \sim \sim(\Box p . \sim \Box q))) \\
&= \text{nonon } (\Box(\sim p \vee q) . (\Box p . \sim \Box q)) \\
&= \text{nonon } (\Box(\sim p \vee q)) + \text{nonon } (\Box p . \sim \Box q) \\
&= \text{nonon } (\sim p \vee q) + \text{nonon } (\Box p) + \text{nonon} \\
&\quad\ (\sim \Box q) \\
&= 0 + \text{nonon } (p) + \text{nopon } (\Box q) \\
&\quad\ 0 + 0 + 1 + \text{nopon } (q) \\
&= 1
\end{aligned}$$

Hence by (2.9) we need to demonstrate 2-S5-Validity. The calculation may be set out thus

| p | q | □ | (p⊃q) | .⊃. | □p⊃□q |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 1 1 | 1 | 1 1 1 1 1 |
| 1 | 1 | | 1 1 1 | | 1   1 |
| 1 | 1 | 0 | 1 1 1 | 1 | 1 1 0 0 1 |
| 1 | 0 | | 1 0 0 | | 1   0 |
| 1 | 1 | 1 | 1 1 1 | 1 | 0 1 1 1 1 |
| 0 | 1 | | 0 1 1 | | 0   1 |
| 1 | 1 | 1 | 1 1 1 | 1 | 0 1 1 0 1 |
| 0 | 0 | | 0 1 0 | | 0   0 |
| 1 | 0 | 0 | 1 0 0 | 1 | 1 1 0 0 0 |
| 1 | 1 | | 1 1 1 | | 1   1 |
| 1 | 0 | 0 | 1 0 0 | 1 | 1 1 0 0 0 |
| 1 | 0 | | 1 0 0 | | 1   0 |
| 1 | 0 | 0 | 1 0 0 | 1 | 0 1 1 0 0 |
| 0 | 1 | | 0 1 1 | | 0   1 |
| 1 | 0 | 0 | 1 0 0 | 1 | 0 1 1 0 0 |
| 0 | 0 | | 0 1 0 | | 0   0 |
| 0 | 1 | 1 | 0 1 1 | 1 | 0 0 1 1 1 |
| 1 | 1 | | 1 1 1 | | 1   1 |
| 0 | 1 | 0 | 0 1 1 | 1 | 0 0 1 0 1 |
| 1 | 0 | | 1 0 0 | | 1   0 |
| 0 | 1 | 1 | 0 1 1 | 1 | 0 0 1 1 1 |
| 0 | 1 | | 0 1 1 | | 0   1 |
| 0 | 1 | 1 | 0 1 1 | 1 | 0 0 1 0 1 |
| 0 | 0 | | 0 1 0 | | 0   0 |
| 0 | 0 | 1 | 0 1 0 | 1 | 0 0 1 0 0 |
| 1 | 1 | | 1 1 1 | | 1   1 |
| 0 | 0 | 0 | 0 1 0 | 1 | 0 0 1 0 0 |
| 1 | 0 | | 1 0 0 | | 1   0 |
| 0 | 0 | 1 | 0 1 0 | 1 | 0 0 1 0 0 |
| 0 | 1 | | 0 1 1 | | 0   1 |
| 0 | 0 | 1 | 0 1 0 | 1 | 0 0 1 0 0 |
| 0 | 0 | | 0 1 0 | | 0   0 |

On the left hand side of the calculation, the columns are headed by designations for each propositional variable occurring in $\alpha$. These columns are the columns of each matrix $\Phi'''_2$. Each such matrix occupies two rows: $2^{(1+1)\times 2} = 2^4 = 16$, so there are 16 matrices and hence 32 rows in the whole calculation. The initial clause (1.1) justifies the transference for each row and for each propositional variable, of the value of that propositional variable in the matrix to the value of that propositional variable as occurring in the formula $\alpha$ on the right-hand-side of the calculation. Clauses (1.2) and (1.3) justify the carrying out of ordinary truth-tabular calculations on each row, and clause (1.4) provides the means of calculating the value to appear under a $\Box$ sign in any row. By (1.5), the main column is only calculated in the top row of each matrix and then by (1.6) the formula is 2-S5-Valid since it has all l's in its main column. Then by (2.9) the formula is S5-Valid.

In general, for a wff $\alpha$ such that nopon $(\alpha)$ = n and containing k propositional variables, there will be $(2^{(n+1)\times k}) \times (n+1)$ rows in the truth-table for $\alpha$ if the calculation is set out in the same way as the given example. This number will become large very rapidly: e.g. in the truth-table for $\Box(p \vee q) \, . \supset . \, \Box p \vee \Box q$ there will be 192 rows, and in the truth-table for $\sim\nabla\nabla p$, where $\nabla$ is the sign for contingency, there will be 160 rows (since nopon $(\sim \nabla \nabla p) = 4$). Even though shortened truth-table techniques can be used and nopon can sometimes be reduced by making strict-equivalence transformations on $\alpha$, the class of formulae for which (2.9) leads to a humanly feasible decision procedure is fairly small. The decision procedure is of interest on two counts, however. Firstly, it may be exhibited as a rather smooth extension of the familiar truth-tabular procedure for classical propositional calculus: in algebraic terms the extension parallels that from the notion of row-vector to the notion of matrix. Secondly, it lends itself particularly well to a computer technique. The nopon function may be easily calculated by a computer, particularly if the program is written in a language like ALGOL, and since all the calculations are purely Boolean (even that for $\Box$, which is a logical 'and' on a column rather than a row of digits) there will usually be hardware functions available for the calculations.

(5). We may reduce the labour of calculation somewhat if we set out the calculation using many-valued logical matrices. If we take the columns of a model $\Phi_n$, and transpose them into rows, we will have a row of n 1's and 0's. Now if we treat this as a binary number, and convert to the decimal scale, we will have $2^n$ decimal numbers ranging from 0 to $2^n—1$. These numbers may be taken as the values of a $2^n$-valued logic, the connectives of which are defined thus:

$\sim p$ is defined as $(2^n—1)—p$,

p . q is defined as the decimal value of the logical 'and' of the binary representations of p and q,

$\Box p$ is defined as $2^n—1$ if $p = 2^n—1$ and 0 otherwise.

The logical matrices which result, with $2^n—1$ as the only designated value, may be seen to be summaries of the truth-tabular calculation process set out in 1's and 0's for all the $\Phi'''_n$'s, as in § 4. With inessential reletterings, they are the powers of the matrices

| | . | 1 | 0 | ~ | □ |
|---|---|---|---|---|---|
| * | 1 | 1 | 0 | 0 | 1 |
| | 0 | 0 | 0 | 1 | 0 |

where the powers are formed according to methods of Łukasiewicz, as in his [6], p. 158ff. They are also, with different primitive symbols, the Henle matrices, as referred to in Lewis and Langford [5] fn. 1, p. 492, and as characterized by Scroggs in his [7], p. 115.

Our main theorem entails that for a wff α where nopon (α) = n, the $2^{n+1}$-valued Henle matrices are characteristic for S5-Validity. In practical terms, this means that we may calculate the Henle matrices up to, say, the 8-valued tables, both for the primitive connectives $\sim$, $.$, $\Box$ and for the useful defined connectives such as $\vee$, $\supset$, $\equiv$, $\Diamond$, ⥽, once and for all, and use these tables to validate formulae $\alpha$ where nopon $(\alpha) \leqslant 2$. As compared to the method of § 4, the number of rows in the many-valued calculation will be divided by a factor of nopon $(\alpha)+1$. As an example, we re-work the calculation for $\Box(p\supset q) .\supset. \Box p \supset \Box q$. The 4-valued Henle matrices are

| | | p . q | | | | p ⊃ q | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | 2 | 1 | 0 | 3 | 2 | 1 | 0 | ~ | □ |
| * | 3 | 3 | 2 | 1 | 0 | 3 | 2 | 1 | 0 | 0 | 3 |
| | 2 | 2 | 2 | 0 | 0 | 3 | 3 | 1 | 1 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 3 | 2 | 3 | 3 | 2 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 3 | 3 | 3 | 3 | 3 | 0 |

and the calculation is

| p | q | □ | (p⊃q) | .⊃. | □ | p ⊃ | □q |
|---|---|---|---|---|---|---|---|
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3 | 2 | 0 | 2 | 3 | 3 | 0 | 0 |
| 3 | 1 | 0 | 1 | 3 | 3 | 0 | 0 |
| 3 | 0 | 0 | 0 | 3 | 3 | 0 | 0 |
| 2 | 3 | 3 | 3 | 3 | 0 | 3 | 3 |
| 2 | 2 | 3 | 3 | 3 | 0 | 3 | 0 |
| 2 | 1 | 0 | 1 | 3 | 0 | 3 | 0 |
| 2 | 0 | 0 | 1 | 3 | 0 | 3 | 0 |
| 1 | 3 | 3 | 3 | 3 | 0 | 3 | 3 |
| 1 | 2 | 0 | 2 | 3 | 0 | 3 | 0 |
| 1 | 1 | 3 | 3 | 3 | 0 | 3 | 0 |
| 1 | 0 | 3 | 3 | 3 | 0 | 3 | 0 |
| 0 | 3 | 3 | 3 | 3 | 0 | 3 | 3 |
| 0 | 2 | 3 | 3 | 3 | 0 | 3 | 0 |
| 0 | 1 | 3 | 3 | 3 | 0 | 3 | 0 |
| 0 | 0 | 3 | 3 | 3 | 0 | 3 | 0 |

Thus showing again that $\Box(p\supset q) .\supset. \Box p \supset \Box q$ is S5-Valid.

(6). We may gain a more practical decision procedure by combining (2.9) with Castañeda's theorem (in his [1]). We put

(6.1) $W\,C\,B\,(\alpha) =_{df} 2^k$,

where W C B ($\alpha$) is the "Wajsberg-Castañeda bound" on truth-tabular calculation for $\alpha$, and k is the number of propositional variables in $\alpha$. Now since (n+m)-S5-Valid ($\alpha$) $.\supset.$ n-S5-Valid ($\alpha$), we may amalgamate (2.9) with Castañeda's theorem thus:

(6.2) (min(nopon($\alpha$)+1, W C B ($\alpha$))) -S5-Valid ($\alpha$) $.\supset.$ S5-Valid ($\alpha$).

In terms of the many-valued approach of § 5, this means that we may validate $\alpha$ by truth-tabular calculation using the Henle matrices of the smallest size out of $2^{n+1}$, where nopon ($\alpha$) = n, and $2^{(2^k)}$ For some formulae such as $\sim \nabla \nabla p$, $2^k < n+1$, and for others such as $\Box(p \vee q) .\supset. \Box p \vee \Box q$, $n+1 < 2^k$. The W C B function is especially simple when formulae involving modal iteration are to be tested, and the nopon function will take advantage of cases where most of the modal functions either appear in the antecedent of a formula or have a wide scope in the consequent of a formula.

To conclude: there is a small but by no means trivial class of modal formulae which may be directly validated by a humanly feasible truth-tabular calculation, either wholly in terms of 1's and 0's or by using many-valued matrices. The methods of calculation are, in the most direct sense possible, extensions of the familiar methods employed in classical (assertoric) propositional calculus.

*University of Auckland, N.Z.* M. K. RENNIE