THE DECIDABILITY AND SEMANTICAL INCOMPLETENESS OF LEMMON'S SYSTEM SO.5.

Richard ROUTLEY

Two decision procedures are given for the modal system SO.5, a Gentzen style decision procedure and a von Wright-Anderson type decision procedure. The second procedure leads easily, as Cresswell has indicated in [2] using Kripke semantics, to a completeness result for SO.5. However under the intended interpretation for SO.5 proposed by Lemmon in [6] and [7] and adopted by Cresswell, namely that the necessity connective '$\Box$' is interpreted as 'it is tautologous (by truth-table) that', the system is incomplete. To complete the system under the intended interpretation conventionalistic theses like $\nabla\Box p$ (contingently necessarily p) must be added to SO.5. But this augmented SO.5 is under its intended interpretation a formalised metalogic of classical sentential logic (see [7]). Hence *the* (usual) *metalogic of classical sentential logic is conventionalistic about modality.* In fact consistent extensions of the familiar explanation of the contingency of a wff, in terms of its having truth-value T for some assignments to its variables and F for other assignments, to modal wff lead directly to conventionalist theses. The related doctrine of necessity as truth (under all assignments) in virtue of logical form also entails conventionalism, reflected in theses like $\Diamond\Diamond p$ and $\nabla p \supset \Diamond\Box p$ which are inconsistent with normal modal logics. Consider, to illustrate, the verification of $\nabla p \supset \nabla\Box p$ given classical doctrine. If ∇p, then p has value T for some assignments and value F for some assignments. For the assignments for which p has value T only, $\Box p$ has value T, since p is true for all these values; and for the assignments for which p has value F (obtainable by restricting the values of p, for instance by substitution), $\Box p$ has the assignment F. Since $\Box p$ has value T for some assignments and F for some, $\nabla\Box$**p.**

The modal system SO.5, formulated with primitive connectives $\sim, \supset, \Box$, has as postulates:

I. *Some* axiomatisation of (classical) sentential logic, to be specific the schemes:

A1. $A \supset . B \supset A$
A2. $A \supset (B \supset C) \supset . A \supset B \supset . A \supset C$
A3. $\sim A \supset \sim B \supset . B \supset A$, and the rule
R1. $A, A \supset B \rightarrow B$.

II. These modal postulates:

A4. $\Box A \supset A$
A5. $\Box(A \supset B) \supset . \Box A \supset \Box B$
R2. $A \rightarrow \Box A$, provided A is a theorem of the sentential logic I.

An equivalent basis is provided by A4, A5, R1 and R2′: $A \rightarrow \Box A$, provided A is a classical tautology.

The sequential system *SO.5 has as postulates all the sentential schemes of Gentzen's system LK (for there see [4]), or of Kleene's system G1 (in [5]), and the following two modal schemes:

$$\frac{A, \Gamma \rightarrow \Theta}{\Box A, \Gamma \rightarrow \Theta} \quad (\Box \rightarrow) \qquad\qquad \frac{\Gamma \rightarrow A}{\Box \Gamma \rightarrow \Box A} \quad (\rightarrow \Box),$$

where Γ, Θ, etc. are empty or non-empty sequences of wff, and $\Box\Gamma$ is the sequence obtained by prefixing each wff in Γ with $\Box$. The modal schemes are subject to the following restriction:
Restriction on $(\rightarrow \Box)$: in an SO.5 proof figure an application of $(\rightarrow \Box)$ never appears under an application of $(\rightarrow \Box)$ or of $(\Box \rightarrow)$.

*The cut-elimination theorem for *SO.5.* Any *SO.5 proof-figure can be transferred into an *SO.5 proof-figure with the same endsequent and without any cut.
Proof: The proof is the same as Ohnishi's "proof" in [9] of the cut-elimination theorem for S2*, except that it should be

observed that the restriction on $(\rightarrow\Box)$ is met in each case. The only case where the restriction is relevant is Ohnishi's case 3. Since the upper parts of the sequents $\Gamma\rightarrow A$ and $\Sigma\rightarrow B$ occurring in case 3, do not contain applications of $(\rightarrow\Box)$, by the restriction on $(\rightarrow\Box)$, an application of the $(\rightarrow\Box)$ rule can be made in the new transformation which decreases the grade of the mix.

*The equivalence theorem for SO.5 and *SO.5.* A wff B is provable in SO.5 if the sequent $\rightarrow B$ is provable in *SO.5.

1. If B is provable in SO.5, $\rightarrow B$ is provable in *SO.5.
Proof: The theorems of sentential logic follow using LK (or G1) and the elimination theorem. It remains to prove the correlates of A4, A5 and R2 in *SO.5.

*A4.

$$\cfrac{\cfrac{A\rightarrow A}{\Box A\rightarrow A}\;(\Box\rightarrow)}{\rightarrow\Box A\supset A}\;(\rightarrow\supset)$$

*A5.

$$\cfrac{\cfrac{A\supset B, A\rightarrow B}{\Box(A\supset B), \Box A\rightarrow\Box B}\;\text{(by LK rules }(\rightarrow\Box))}{\rightarrow\Box(A\supset B)\supset.\Box A\supset\Box B}\;\text{(applying LK rules).}$$

*R2. If A is a tautology of sentential logic then $\rightarrow A$ using purely LK rules and no applications of the modal rules. Hence

$$\cfrac{\rightarrow A}{\rightarrow\Box A}\;(\rightarrow\Box)$$

$(\rightarrow\Box)$ can be applied since the restriction is satisfied.

2. If $\rightarrow B$ is provable in *SO.5, B is provable in SO.5.
Proof: The sequents: $\Gamma\rightarrow\Theta$, $\rightarrow\Theta$, $\Gamma\rightarrow$ are represented, respectively, in SO.5 by the wff $\Gamma\&\supset\Theta v$, Θv, $*(\Gamma\&)$, where $\Gamma\&$ is the conjunction of the wff in sequence Γ, and Θv is the disjunction of the wff in sequence Θ. It is proved that if a sequent is provable in *SO.5 its representation is provable in SO.5; hence

in particular if $\rightarrow B$ is provable in *SO.5, B is provable in SO.5. It is sufficient to show that the representations of the schemes of *SO.5 are derived rules of SO.5.

(a) The LK rules are all representable in SO.5 since SO.5 contains sentential logic.

(b) $(\Box\rightarrow)$ is represented by: $\dfrac{A\&\Gamma\&\supset.\Theta\vee}{\Box A\&\Gamma\&\supset.\Theta\vee}$

That this is a derived rule of SO.5 follows from the theorem — where $A\&\Gamma\&\supset\Theta\vee\supset.\Box A\&\Gamma\&\supset\Theta\vee$ and the rule R1.

(c) $(\rightarrow\Box)$. *Case 1* Γ is empty. Then $(\rightarrow\Box)$ is represented by: $\dfrac{A}{\Box A}$,where by the restriction A is derivable only by LK rules. This is rule R2 of SO.5.

Case 2. Γ is not empty. Then $(\rightarrow\Box)$ is represented by $\dfrac{\Gamma\&\supset A}{(\Box\Gamma)\&\supset\Box A}$, where above this rule no modal rule appears.

As then $\Gamma\&\supset A$ is provable by sentential logic rules,

$\Box(\Gamma\&\supset A)$ is provable by rule R2.

Hence $\Box(\Gamma\&)\supset A$ is provable by A5 and R1.

It remains to show $(\Box\Gamma)\&\supset\Box(\Gamma\&)$; & for this it will suffice to show

(d)$\Box A\&\Box B\supset).\Box(A\&B)$, as then the previous scheme follows by inductive iteration. (d) is proved by this sequence:

$A\supset.B\supset A\&B$ by sentential logic
$\Box(A\supset.B\supset.A\&B)$ by R2.
$\Box A\supset.\Box B\supset.\Box(A\&B)$ applying A5 twice.
$\Box A\ \&\ \Box B\supset.\Box(A\&B)$ by sentential logic.

The decidability theorem for SO.5. The system SO.5 is a decidable system.

The system *SO.5 provides a direct Gentzen decision procedure for SO.5.

The decision procedure can be used to prove directly the following:

Theorem. There are infinitely many modalities in SO.5.

For all attempted proofs of $\Box^m p \dashv \Box^n p$, where $m<n$, are stopped

by the restriction on ($\rightarrow\Box$). Hence $\vdash \Box^m p$ ⥼—⥽ $\Box^n p$ iff $m = n$. This theorem does, however, follow immediately from the fact that SO.5 is a sublogic of S2 and S2 has infinitely many modalities.

A decision procedure for SO.5 cannot be provided by a finite characteristic matrix.

Theorem. There is no finite characteristic matrix for SO.5. Since Henle's matrix satisfies SO.5 (as SO.5 is a subsystem of S1), and since $\vdash_{SO.5}$ (p⥼—⥽p) $\vee$ q, Dugundji's proof in [3] will work.

However SO.5 has the finite modal property, and is decidable by extended truth-table techniques. The method presented is a variation of that of Anderson in [1], and some of the terminology of [1] and [10] is presupposed. As Ohnishi has observed in [10], Gentzen formulations of modal systems both suggest requirements on eliminated F-rows in von Wright-Anderson truth-tables, and facilitate proofs of the adequacy of the decision procedures. Cut-free formulations of modal systems also lead through these extended truth tables to completeness theorems for the modal systems.

Definition: A is an *SO.5-tautology* iff every F-row r of the truth table $\mathfrak{T}(A)$ for A satisfies at least one of the following requirements:

I. Some constituent of the form $\Box B$ has the value T in r where B has value F in row r.

II. Some constituents of the form $\Box C_1, \ldots, \Box C_n$ $(n \geqslant o)$ all have the value T in r and some constituent of the form $\Box B$ has the value F in row r, where $C_1 \& C_2 \& \ldots \& C_n \supset B$ is a (substitution instance of a) tautology.

Theorem: If $\rightarrow A$ is provable in *SO.5, then A is an SO.5-tautology.

Proof: The result follows from

(*) If $\Gamma \rightarrow \Theta$ is derivable in *SO.5 then its representation is an SO.5-tautology. For the representation of $\rightarrow A$ is A. Now

(1) The representation of prime statements $A \rightarrow A$ are SO.5-tautologies. To prove (*) it suffices to show for every rule of inference of *SO.5 that SO.5-tautologousness of the upper se-

quent implies SO.5-tautologousness of the lower sequent. That the LK rules preserve SO.5-tautologousness follows from
(2) Every (classical) tautology is an SO.5-tautology. Now
(3) ($\Box\!\rightarrow$) preserves SO.5-tautologousness. This is guaranteed by requirement I, since the only way the lower sequent in an ($\Box\!\rightarrow$) rule application differs from the upper sequent is in the introduction of the symbol $\Box$.
(4) ($\rightarrow\!\Box$) preserves SO.5-tautologousness. This is similarly guaranteed by requirement II and the restriction on ($\rightarrow\!\Box$).

Theorem. If A is an SO.5-tautology, then $\vdash_{SO.5}A$.
Proof: If there are no F-rows, then A is a classical tautology. Therefore A is a substitution-instance of a theorem of sentential logic, and so of SO.5. If there are F-rows, define a wff Dr for every F-row r of $\mathfrak{T}(A)$, where r = 1,2,...,k, as follows: If F-row r satisfies requirement I for some wff $\Box B$, let Dr be $\Box B \supset B$; otherwise F-row r satisfies II for some wff $\Box C_1$, $\Box C_2$, ..., $\Box C_n$ ($n > o$), $\Box B$: let Dr be $\Box C_1, \& \Box C_2 \& \ldots \& \Box C_n \supset \Box B$.
(*) $\vdash_{SO.5}$Dr (r = 1,2 ... k).
When Dr is $\Box C_1 \& \ldots \& \Box C_n \supset \Box B$, $C_1 \& C_2 \& \ldots \& C_n \supset B$ is a theorem since it is a substitution-instance of a tautology. Hence Dr is a theorem.
(**) $D_1, D_2, \ldots D_k \vdash A$
(substituting in classical logic), since apart from F-rows which are excepted by the hypotheses, A is a substitution-instance of a classical tautology. By (*), (**) and the deduction theorem, $\vdash_{SO.5}A$.

Theorem. $\vdash_{SO.5}A$ iff A is an SO.5 tautology.

This theorem furnishes a completeness result for SO.5. To make the result appear in a more semantic light, first define a *classical interpretation* of a wff A over the values T and F inductively as follows for any subformulae B and C of A:
1. If B is atomic the value of B is T or F;
2. If the value of B is T(F) then the value of $\sim B$ is F(T);
3. If the value of B is T and the value of C is F then the value of ($B \supset C$) is F; if the value of B is F or the value of C is T then the value of ($B \supset C$) is T.
4. The value of $\Box B$ is T or F (whatever the value of B).

A wff A is *classically valid* iff it has the value T for all classical interpretations.
Next define an *SO.5-interpretation* of a wff A over values T and F as follows for subformulae of A:
1. — 3. as in the classical interpretation
4. If the value of B is F then the value of $\Box B$ is F; if the value of B is T then, if for any $C_1, C_2, \ldots, C_n$ such that $C_1 \& C_2 \& \ldots \& C_n \supset B$ is classically valid, $\Box C_1, \Box C_2, \ldots, \Box C_n$ are subformulae of A which have value T, $\Box B$ has value T; otherwise if the value of B is T then the value of $\Box B$ is either T or F.
Although 4. is not fully inductive it is effective since A is a finite wff.
A wff is *SO.5-valid* iff it has value T for all SO.5-interpretations.

Theorem. $\vdash_{SO.5} A$ iff A is SO.5-valid.

In spite of this result, and in spite of Cresswell's completeness result in [2], SO.5 is not complete under the canvassed interpretation, where '$\Box$' is interpreted as 'it is tautologous (by truth table) that'. Indeed without a rather liberal reading of 'it is tautologous that (1), the interpretation is not even correct; for as '$\sim \Box p$' taken literally reads 'it is not the case that p is tautologous' the interpretation would, inconsistently with the logic, bring out $\sim \Box p$ as true.

The intended interpretation, when modified so as to be correct, turns out to be very interesting. For an interpretation select as a *domain* d of interpretation a non-empty set of wff of classical sentential logic. An *assignment* i of elements of the domain to variables of a wff A is a uniform simultaneous substitution of selected wff of the domain for variables of A, resulting in a wff A^1. Under an assignment for each variable, whether in A or in some other wff B, a definite wff of d is selected. Once an assignment of elements of the domain to variables of A is made, the value of A over the domain for that assignment is

(1) Of the sort suggested by M. RENNIE in [11]. Both M. Rennie and H. Montgomery have independently noted the incompleteness of SO.5 under the Lemmon-Cresswell interpretation.

defined. The value of A is the value of A^1 which is as follows, where B and C are subformulae of A:
Clauses 1 — 3 as before
4. If B is classically valid, $\Box B$ has value T; and if B is not classically valid, $\Box B$ has value F.
A wff A is *valid in* a domain d iff A has value T for all assignments of elements of the domain to its variables. A wff is *semantically valid* (*s-valid*) iff it is valid in all non-empty domains.

Theorem. If $\vdash_{SO.5} A$ then A is semantically valid.
Proof: As usual by induction over the length of proofs of theorems of SO.5.
A1 — A3. If A is a theorem of sentential logic then all its instances are classically valid, so it is semantically valid.
R2. If A is a substitution-instance of a tautology then, for any d and i, A^1 is also such a substitution-instance. As then A^1 is classically valid, by 4. $\Box A^1$ has value T.
R1. If A is s-valid and $A \supset B$ is s-valid, then for any assignment i in domain d A^1 has the value T and $(A \supset B)^1$ has value T. Thus $(A^1 \supset B^1)$ has value T; hence B^1 has value T by rule 3. As this is for any i in any d B is s-valid.
A4. If $(\Box A)^1$, i.e. $\Box A^1$, has value T in classical d under i then A^1 is classically valid, so A^1 has value T for i in d. Otherwise if $\Box A^1$ has value F, $\Box A^1 \supset A^1$ has value T for that i and d by rule 3. Hence A4 is s-valid.
A5. If for an i over a d $(\Box(A \supset B))^1$ and $(\Box A)^1$ have value T then $(A \supset B)^1$ is classically valid and A^1 is classically valid. Then $A^1 \supset B^1$ and A^1 have value T for all classical interpretations. Hence by the rule for '$\supset$' in classical interpretations B^1 has value T for all classical interpretations, and so is classically valid. Hence $\Box B^1$ has value T. Otherwise if $(\Box(A \supset B))^1$ or $(\Box B)^1$ has value F some d and i $(\Box(A \supset B) \supset . \Box A \supset \Box B)^1$ has value T by rule 3. Since A5 has this value for all i in all d, A5 is s-valid.

Theorem. SO.5 is semantically incomplete, i.e. for some wff A of SO.5, A is s-valid but $\sim \vdash_{SO.5} A$.

Proof. 1. $\sim\Box\Box A$ is s-valid. For any i in any d $\Box\Box A'$ has value F since $\Box A'$ is not classically valid. Hence $\sim\Box\Box A$ has value T for all i over all non-empty d.

2. $\sim\vdash_{SO.5}\sim\Box\Box A$. By either decision procedure for SO.5.

Completing SO.5 leads to contingency oriented modal systems with conventionalist theses $\nabla\Box p$ and $\nabla\nabla p$, as well as the S6 thesis $\sim\Box\Box p$. These systems are studied in [8].

Monash University, Australia R. ROUTLEY

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