A UNIVERSALLY VALID SYSTEM OF PREDICATE CALCULUS WITH NO EXISTENTIAL PRESUPPOSITIONS

John T. KEARNS

1. INTRODUCTION. In this paper, I present a system of predicate calculus in which individual variables are not construed in the normal fashion. Before I explain my way of understanding variables, I will state the ordinary view of variables. In dealing with the customary logical systems, individual variables are regarded as "ranging over" a domain of individuals — these individuals are the values of the variables. For a formula to be valid in a given domain, it must be true for all values of its free variables. Quantifiers are used with individual variables in order to make statements about all or some individuals in the domain. This use of quantifiers has given rise to the claim that to be is to be the value of a bound variable.

I am proposing an alternative way to "understand" individual variables (1). These variables are to be regarded as replacements for singular terms, where the singular terms may be empty terms (may be terms without reference). If we consider the distinction that is often made between variables which take entities as values and variables which are schematic letters (2), my proposal is that individual variables (in fact, *all* variables) be construed as schematic letters. If variables are taken to be schematic letters, they cannot be said to take entities as values. For individual variables replace singular terms that may or may not have reference. One might regard these singular terms as the values of the variables, but they would not be values in the same sense that individuals are the values of variables in customary quantificational systems.

(1) I have discussed this "alternative understanding" in [3] and [5].

(2) W. V. Quine frequently refers to the distinction between a variable taking entities as values and a variable which is a schematic letter. See, for example, "Logic and the Reification of Universals", in [9], especially p. 107ff.

When individual variables are regarded as schematic letters, they can still be quantified. Ordinarily, '(x)' is read, "For all individuals x".
But on the view I am proposing,

$$(x)f(x)$$

might be read,
"For every singular term which replaces 'x', $f(x)$ [is true]". An analogous reading can be provided for the particular (existential) quantifier. (These readings are discussed at greater length in section 3).

Customary systems of predicate calculus are not adequate when individual variables are regarded as schematic letters replacing possibly empty singular terms. This is because the customary systems do not possess a means for distinguishing what exists from what does not. In the remainder of this paper, I set up a system of predicate calculus which is suitable for distinguishing empty from non-empty names. But I will present this system by stages. To begin with, I will set up a system that is acceptable on either view of variables. The peculiarity of this system, given the customary treatment of individual variables, is that it is universally valid — it is valid in *all* domains, including the empty domain. After presenting this system, I introduce the additional axioms and rules that are required when individual variables are regarded as replacements for singular terms that may be empty terms.

2. The System UV. The system UV is a simplified version of the system presented in [4] (3). It results from the system of Hilbert and Ackermann (which is found in [1]), when this system is modified to achieve universal validity. The system UV

(3) The system UV is simpler than the system in [4] by virtue of containing fewer axioms and rules. In [4], I was not concerned to "streamline" the system, but only to argue in favor of a universally valid system.

does not contain free individual variables; for when free variables are eliminated, the limitation of validity to non-empty domains becomes less plausible. And formulas containing free individual variables have the same intuitive significance as similar formulas in which the corresponding individual variables are bound by initial universal quantifiers.

In order to present the axiom and rules of procedure, the following definitions are required.

Well-formed formula (4)

(i) A propositional variable standing alone is a wff.

(ii) A predicate variable whose argument places are filled with individual variables is a wff.

(iii) If A[α] is a wff containing a free individual variable α, then (α)A[α] is a wff.

(iv) If A is a wff, then $\sim$A is a wff.

(v) If A, B are wffs such that no individual variable occurs bound in one and free in the other, then [A ⊃ B], [A v B], [A & B], [A ≡ B] are wffs.

It should be noted that an individual variable cannot occur in a wff both bound and free, nor can one quantifier occur within the scope of another containing the same quantified variable.

Initial universal quantifier

(i) If A[α] is a wff containing free individual variable α, then (α) is an initial universal quantifier of (α)A[α].

(ii) If (α_{n-1}) is an initial universal quantifier of a wff $(\alpha_1)(\alpha_2)\ldots(\alpha_{n-1})(\alpha_n)A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n]$, then (α_n) is an initial universal quantifier of this wff.

(4) Propositional variables are chosen from

p, q, r, s, p_1,...

Individual variables are chosen from

w, x, y, z, w_1,...

Predicate variables are chosen from

f, g, h, f_1,...

Step (v) allows different connectives in a wff, because the axioms for propositional calculus that are required in UV will not be specified.

Initial string
If (α_n) is an initial universal quantifier of a wff $(\alpha_1)(\alpha_2)\ldots(\alpha_n)A[\alpha_1, \alpha_2,\ldots,\alpha_n]$, and $A[\alpha_1, \alpha_2,\ldots,\alpha_n]$ does not contain an initial universal quantifier, then $(\alpha_1)(\alpha_2)\ldots(\alpha_n)$ is the initial string of $(\alpha_1)(\alpha_2)\ldots(\alpha_n)A[\alpha_1,\alpha_2,\ldots,\alpha_n]$.

The single axiom of UV (apart from axioms common to propositional calculus) is (5)

A1 $(y).(x)f(x) \supset f(y)$.

The rules of procedure are the following: (6)

α1 A propositional variable μ occurring in a theorem A can be replaced by a wff B, provided that μ is replaced at each of its occurrences in A, and provided that A and B have no individual variable in common. Let the result of this replacement be $\Sigma A'$, where Σ is the initial string; and let $\alpha_1, \alpha_2,\ldots, \alpha_n$ be the individual variables occurring free in $\Sigma A'$. Then $\Sigma(\alpha_1)(\alpha_2)\ldots(\alpha_n)A'$ is a theorem (7).

α2 An individual variable α_1 bound by an initial universal quantifier in a theorem A can be replaced by an individual variable α_2, provided that α_1 is replaced at each of its occurrences in A, and provided that α_2 either does not occur in A or occurs bound by an initial universal quantifier. Let the result of this replacement be $\Sigma_1(\alpha_1)\Sigma_2 A'$

(5) Brackets are abbreviated according to the convention of A. Church, *Introduction to Mathematical Logic,* vol. 1.

(6) These rules are adaptations of the rules presented by David Pager in "An Emendation of the Axiom System of Hilbert and Ackermann for the Restricted Calculus of Predicates", *The Journal of Symbolic Logic,* vol. 27, n⁰. 2. As Pager proved, (adaptations of) the original rules can be derived from these rules.

(7) The rules are stated for theorems, but the axioms are to be included among these theorems. The rules are not stated for wffs (as Hilbert and Ackermann stated their rules), because not all universally valid formulas are theorems.

where $\Sigma_1(\alpha_1)\Sigma_2$ is the initial string of A. If α_2 occurs in A, then $\Sigma_1\Sigma_2A'$ is a theorem. If α_2 does not occur in A, then $\Sigma_1(\alpha_2)\Sigma_2A'$ is a theorem.

α3 Let A be a theorem containing the n-adic predicate variable φ, and let $B[\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots, \alpha_{n+r}]$ be a wff containing (exactly) $n + r$ distinct free individual variables $\alpha_1, \alpha_2, \ldots, \alpha_{n+r}$. Each occurrence of φ in A can be replaced by B, according to the rule that $\varphi(\beta_1, \beta_2, \ldots, \beta_n)$ is to be replaced by $B[\beta_1, \beta_2, \ldots, \beta_n, \alpha_{n+1}, \ldots, \alpha_{n+r}]$, provided that A and $B[\alpha_1, \ldots, \alpha_{n+r}]$ have no individual variables in common. Let the result of this replacement be $\Sigma A'$, where Σ is the initial string. Then $\Sigma(\alpha_{n+1})\ldots(\alpha_{n+r})A'$ is a theorem.

γ1 From a theorem $(\alpha_1)(\alpha_2)\ldots(\alpha_n).A \supset B[\alpha_i]$ in which the consequent contains the free individual variable α_i which does not occur in the antecedent, the theorem $(\alpha_1)\ldots(\alpha_{i-1})(\alpha_{i+1})\ldots(\alpha_n).A \supset (\alpha_i)B[\alpha_i]$ is obtained.

π From a theorem $(\alpha_1)(\alpha_2)\ldots(\alpha_n).A[\alpha_i] \supset B[\alpha_i]$ in which both antecedent and consequent contain the free individual variable α_i, the theorem $(\alpha_1)\ldots(\alpha_{i-1})(\alpha_{i+1})\ldots(\alpha_n).(\alpha_i)A[\alpha_i] \supset (\alpha_i)B[\alpha_i]$ is obtained.

δ All occurrences of an individual variable α_1 in a theorem A which are bound by a single quantifier may be replaced by occurrences of an individual variable α_2, provided that α_2 also replaces α_1 in the quantifier, and provided that the result is a wff.

MP If A, $A \supset B$ are theorems, then B is a theorem.

The system UV requires only a single axiom (other than those common to propositional calculus). A counterpart to the second axiom of Hilbert and Ackermann is

(1) $(y).f(y) \supset (\exists x)f(x)$ ([8]).

But (1) is a consequence of A1 — it can be derived as follows:

(i) $(y).(x)f(x) \supset f(y) \supset . \sim f(y) \supset \sim (x)f(x)$
(ii) $(y)[(x)f(x) \supset f(y)] \supset (y). \sim f(y) \supset \sim (x)f(x)$ (i), π
(iii) $(y). \sim f(y) \supset \sim (x)f(x)$ (ii), A1, MP
(iv) $(y). \sim \sim f(y) \supset (\exists x)f(x)$ (iii), $\alpha 3$
(v) $(y).f(y) \supset (\exists x)f(x)$ (iv)

A rule $\gamma 2$ corresponding to $\gamma 2$ of Hilbert and Ackermann is

> *$\gamma 2$* From a theorem $(\alpha_1)(\alpha_2)\ldots(\alpha_n).A[\alpha_i] \supset B$ in which the antecedent contains the free individual variable α_i which does not occur in the consequent, the theorem $(\alpha_1)\ldots(\alpha_{i-1})(\alpha_{i+1})\ldots(\alpha_n).(\exists\alpha_i)A[\alpha_i] \supset B$ is obtained.

The rule $\gamma 2$ is a derived rule of UV; its proof is omitted. The derived rule of transitivity (hypothetical syllogism) is somewhat different from the corresponding derived rule in the system of Hilbert and Ackermann. The rule, which will not be proved, is

> *HS* If $(\alpha_1)(\alpha_2)\ldots(\alpha_n).A \supset B$, $(\alpha_1)(\alpha_2)\ldots(\alpha_n).B \supset C$ are theorems, then if $(\alpha_1)(\alpha_2)\ldots(\alpha_n).A \supset C$ is a wff, it is a theorem.

When individual variables are construed in the normal fashion, the system UV is both consistent and complete. The proof of consistency is trivial, since UV is part of the original system of Hilbert and Ackermann, which is consistent. A proof of completeness is constructed in the same fashion as the proof by Mostowski in [8]. To understand what is meant by the completeness of UV (on the customary treatment of individual variables), it is necessary to understand what it is for a formula to be universally valid. Validity in non-empty domains is the same as for ordinary systems of predicate calculus. But validity

([8]) The universal quantifier is basic; the particular quantifier is an abbreviation.

must be specially defined for the empty domain. This is done in such a way that a universally quantified formula is always assigned the value truth in the empty domain.

The system UV contains as theorems the universal closures of most of the important theorems of the system of Hilbert and Ackermann. Some theorems of the system of Hilbert and Ackermann which are not theorems of the system UV are

(2) $(x)f(x) \supset (\exists x)f(x)$
(3) $(x)[p \,\&\, f(x)] \supset p \,\&\, (x)f(x)$ (9)
(4) $p \vee (\exists x)f(x) \supset (\exists x).p \vee f(x)$.

3. Individual variables as schematic letters. If individual variables are regarded as replacements for possibly empty singular terms, the theorems of UV will have a different significance than they do on the customary view. But this change in significance does not constitute an argument against my treatment of variables, for this treatment is a perfectly legitimate one.

To understand the significance of formulas whose variables are regarded as schematic letters, it will be helpful to consider quantification. As indicated in section 1, the formula

$(x)f(x)$

will be read, "For every singular term which replaces 'x', $f(x)$". This reading must not be understood as presupposing a linguistic ontology that is replete with abstract entities. For a statement whose form is

(9) The reason why (3) is not universally valid is not immediately evident. But if (3) were universally valid, so would be

(i) $(x)\ [p \,\&\, f\ (x)] \supset p.$

But from (i) we can obtain

(ii) $\sim p \supset \sim (x)\ p. \,\&\, f\ (x)$
(iii) $\sim\sim p \supset \sim (x).\ \sim p \,\&\, f\ (x)$
(iv) $p \supset (\exists x).\ p \vee \sim f\ (x)$

If (iv) were a theorem of UV, then any true statement would imply that there is at least one individual.

$(x)\mathrm{f}(x)$

is not making a claim about some eternal domain of singular terms. The statement simply asserts that any singular term which has been, or can be, formed can be written in place of 'x' to form a true sentence (or a sentence expressing a true proposition).

The formula

$(\exists x)\mathrm{f}(x)$

will be read, "There is a singular term which can replace 'x' so that $\mathrm{f}(x)$". This should not be understood as claiming that such a singular term "exists" at the present time. A statement of form

$(\exists x)\mathrm{f}(x)$

simply claims that a singular term can be formed which, when written in place of 'x', will make a true sentence from

$\mathrm{f}(x)$.

When individual variables are construed in the present fashion, the system requires some device for introducing new singular terms. For if there are true statements of the form

$(\exists x)\mathrm{f}(x)$,

there ought to be some way to add a singular term "answering" to the bound variable. A description operator will be added to the system (in section 7) to satisfy this requirement ([10]).

([10]) In a system which permitted quantification of variables belonging to all categories, it would be appropriate to introduce rules for defining expressions of different categories. Lesniewski, whose treatment of variables is similar to that which I am proposing (an account of Lesniewski's views can be found in [2]), devoted considerable attention to the formulation of rules for introducing definitions into his formal systems.

Someone might object to the above readings of quantifiers on the grounds that they are meta-linguistic — they confuse use with mention. Such an objection is without force. The readings presented above seem to me to be the (almost) (11) ordinary-language expressions that best bring out the role of individual variables. But alternative readings could be introduced which are not "meta-linguistic" (12) — though these might be expressions that do not belong to ordinary English. A more serious objection claims that the proposed treatment of variables limits the application of the logical system to domains containing at most an enumerably infinite number of individuals. For it is clear that no system (or language) can contain more than a finite number of singular terms. Even if rules for introducing an infinite number of singular terms are allowed (the system in this case will contain a *potentially* infinite number of names), the singular terms will be enumerable. Thus it seems that it will not be possible to say anything about all individuals if there are non-denumerably many of them. This objection applies only to languages containing a fixed and final number of singular terms, where the significance of these terms is specified "in advance". For a universally quantified statement is an inference warrant which entitles us to replace the quantified variable by a singular term. No matter how large the domain, there is no individual which cannot be named — though it is clear that we cannot have a name for every individual. Because it is possible to name any individual, a universally quantified statement will have a truly universal force.

(11) The qualification "almost" is added because the proposed readings should not be understood in the most straightforward way. These readings require a commentary to eliminate the suggestion of a linguistic ontology „replete with abstract entities."

(12) Lejewski, in [6] and [7], proposes to read '(x)' as "For all x," and '$(\exists x)$' as "For some x." His readings are not ordinary-language expressions in common use. They can be regarded as expressions introduced into ordinary language to serve certain logical purposes.

There is no requirement that a logical symbol have an exact translation in a natural language, though there ought to be some way of explaining in the natural language how the logical symbol is to be understood, or used.

Given my treatment of individual variables, it is necessary to define a concept of validity for formulas containing such variables. Conventional definitions of validity make use of the concept of an assignment of values to variables. When individual variables are regarded as schematic letters, an interpretation cannot be made in a domain of individuals. Some concept other than that of assignment is required to define validity.

On the proposed view of variables, the logical system is regarded as the basis (or foundation) of a language. The system is capable of being extended "in different directions" (to extend the logical system is to add predicates and singular terms to it). Validity is defined with respect to an extension — though an extension might be open ended: it need not (though it may) consist of a completely specified list of predicates and constants, whose significance is fully determined. To define validity, it is necessary to employ the concept of replacement. In a replacement, propositional variables are replaced by sentences, predicate variables by predicates, and free individual variables by singular terms. For a given replacement of its free variables, a formula $(\alpha)A[\alpha]$ is replaced by a true sentence if $A[\alpha]$ is replaced by a true sentence for every replacement of α by a singular term; otherwise $(\alpha)A[\alpha]$ is replaced by a false sentence. A formula is valid for a given extension if it becomes a true sentence for all replacements of its components. A formula is valid if it is valid for all extensions ([13]).

Let us consider what happens to the consistency and completeness of UV when individual variables are viewed as replacements for possibly empty singular terms. Consistency is not affected, since this is established in a purely formal manner. But the system UV is also complete, although this completeness is rather strange. For the theorems of UV are valid for every extension of UV — this includes extensions that (a) contain only

([13]) Validity can also be defined in such a way that the formal system is not regarded as a language being used. Instead of dealing with extensions of the formal system, the system would be interpreted in various languages which are distinct from the formal system (but these must be languages which are meaningful). The concept of replacement will still be necessary to the definition of validity.

empty singular terms, (b) contain only non-empty singular terms, (c) contain both empty and non-empty singular terms, and (d) contain no singular terms. The system UV contains as a theorem (the universal closure of) every formula valid in all such extensions. The universal validity of the axiom and theorems of UV is easily established. (In an extension of UV without singular terms, a universally quantified statement is vacuously true). The completeness of UV is established by the same proof that establishes the completeness of UV given the ordinary treatment of individual variables. For a universally valid formula must be valid in extensions that do not contain empty singular terms. But validity in such cases reduces to the normal concept ([14]). Hence, if the system is complete when individual variables are regarded as taking individuals as values, it is also complete when these variables are regarded as replacements for singular terms.

4. A SPECULATIVE ASIDE. In this section I will make a claim about the customary treatment of individual variables, and the conventional sort of interpretation. This claim represents an anti-platonist bias on my part, and I wish to separate it from the rest of this paper. For I think that even if my claim can be refuted, the major aim of this paper will not be affected.

The treatment of variables that I have proposed, and the definition of validity that I have sketched appear very different from the conventional ones. And it may be claimed that my approach is of limited value when compared to the customary approach. For when variables are interpreted in domains of individuals, it seems possible to escape the limitation to a language or linguistic system that characterizes my view. Now my claim is that what seems to be the case is just not so. An interpretation must be made by someone, and he must employ some language — there is no direct way to "assign" things to symbols.

This claim about the nature of (ordinary) interpretation is the

([14]) Perhaps I should say instead that validity in such cases comes to the same thing as the normal concept.

application to formal systems of a point made by Professor Wilfrid Sellars in other contexts. For Sellars is continually pointing out that a statement of the form

(i) 'A' means B.

must not be understood to be assigning an entity to an expression ([15]).
Instead, (i) should be understood along the lines of

(ii) 'A' [in whatever language] means the same as 'B' in our language.

But (i) differs from (ii) in presupposing that the speaker and his audience understand 'B'. Even conventional interpretations of formal systems take one language into another. The chief difference between conventional interpretations and the interpretations I am proposing is this: in conventional interpretations, the interpreting language does not contain empty singular terms.

When an interpretation is regarded as assigning things to expressions, it appears that we can regard a formal system from an absolute point of view. I am denying that there is such a thing as an absolute point of view. No one can ever get outside all linguistic (conceptual) frameworks.

5. Additional axioms. Although the system UV is complete in the sense explained in section 3, it is inadequate. For when individual variables are regarded as replacements for possibly empty singular terms, there is no reason to be interested in formulas valid in extensions that do not contain singular terms. Hence, it is desirable to limit the system (which is really to expand the system) to those formulas valid in extensions that (a) contain only empty singular terms, (b) contain only non-empty singular terms, and (c) contain both empty and non-empty

([15]) In nearly every essay of [10], Sellars uses this point to resolve some philosophical perplexity.

singular terms. Another shortcoming of UV is that it possesses no means for distinguishing what exists from what does not (for distinguishing non-empty from empty names).

When the system UV is considered with respect to extensions containing singular terms, there is no objection to admitting the formulas

(2) $(x)\mathrm{f}(x) \supset (\exists x)\mathrm{f}(x)$
(3) $(x)[\mathrm{p} \,\&\, \mathrm{f}(x)] \supset \mathrm{p} \,\&\, (x)\mathrm{f}(x)$
(4) $\mathrm{p} \vee (\exists x)\mathrm{f}(x) \supset (\exists x).\mathrm{p} \vee \mathrm{f}(x)$

Formula (2), for example, claims that if

$\mathrm{f}(x)$

is true no matter what singular term replaces 'x', then (since there is at least one singular term) there is a singular term which can replace 'x' so that

$\mathrm{f}(x)$

is true (becomes true).

To correct the shortcomings of the system UV that were mentioned above, additional axioms will be formulated. The system UV with the additional axioms and rules introduced in this paper constitutes the system UV$^+$. The next four axioms of UV$^+$ are

A2 $(x)(y).x \overset{e}{=} y \supset .\mathrm{f}(x) \supset \mathrm{f}(y)$
A3 $(x)(y).x \overset{e}{=} y \supset x \overset{e}{=} x$
A4 $(x)(y).\sim[x \overset{e}{=} x] \,\&\, \sim[y \overset{e}{=} y] \supset .\mathrm{f}(x) \supset \mathrm{f}(y)$
A5 $\sim[\wedge \overset{e}{=} \wedge]$ (16).

The symbol '$\overset{e}{=}$' is an existential identity symbol. If

$x \overset{e}{=} y,$

(16) The system UV is quite similar to the system L4 presented by Lejewski in [7].

then x is identical with y, and x exists (i.e., 'x' is a non-empty name). '$\wedge$' is the standard empty name ([17]). The addition of these axioms makes it necessary to change the definition of a wff to allow formulas containing $\overset{\text{‘e’}}{=}$ and '$\wedge$'. It is also necessary to change rule $\alpha 2$ to allow the substitution of '$\wedge$' for an individual variable bound by an initial universal quantifier.

The addition of '$\wedge$' to UV limits this system to extensions which contain empty singular terms (when '$\wedge$' is understood in the intended fashion). This addition also makes possible the proof of all those theorems (not containing free individual variables) of the system of Hilbert and Ackermann that were eliminated from UV. For from

(5) $(x).\mathrm{f}(x) \vee \sim \mathrm{f}(x)$

we can obtain

(6) $\mathrm{f}(\wedge) \vee \sim \mathrm{f}(\wedge)$
(7) $\mathrm{f}(\wedge) \vee \sim \mathrm{f}(\wedge) \supset (\exists x).\mathrm{f}(x) \vee \sim \mathrm{f}(x)$ From (1)
(8) $(\exists x).\mathrm{f}(x) \vee \sim \mathrm{f}(x)$

In [8], Mostowski has shown that the addition of (8) to UV yields the system of Hilbert and Ackermann ([18]).

Axioms A2—A5 enable us to proceed in UV+ exactly as one proceeds in the system of Hilbert and Ackermann. This is justified by the following derived rules of procedure.

([17]) Since '$\wedge$' is an empty name, it is not the name of any entity. In particular, it is not the name of a "null individual," as has been proposed by R. M. Martin ("Of Time and the Null Individual," *Journal of philosophy*, 62 (1965)). The proposal to admit such an individual represents an attempt to retain the customary treatment of individual variables while gaining the advantages of UV+. However, this attempt makes the concept of value of a variable unintelligible.

([18]) This proves that UV+ contains all formulas (without free individual variables) that are universally valid with respect to extensions of UV+ that contain at least one singular term; this includes formulas valid for extensions that do not contain empty terms. This characteristic (of UV+) is a consequence of the completeness of customary systems of predicate calculus.

$\gamma 3$ From a theorem $(\alpha_1)(\alpha_2)\ldots(\alpha_n).A[\alpha_i]\supset B$ in which the antecedent contains the free individual variable α_i which does not occur in the consequent, the theorem $(\alpha_1)\ldots(\alpha_{i-1})(\alpha_{i+1})\ldots(\alpha_n).(\alpha_i)A[\alpha_i]\supset B$ is obtained.

PROOF Since

(i) $(\alpha_1)(\alpha_2)\ldots(\alpha_n).A[\alpha_i]\supset B$

is a theorem, so is

(ii) $(\alpha_1)(\alpha_2)\ldots(\alpha_n).A[\alpha_i]\supset .f(\alpha_i) \vee \sim f(\alpha_i)\supset B.$

But then we can obtain

(iii) $(\alpha_1)\ldots(\alpha_{i-1})(\alpha_{i+1})\ldots(\alpha_n).(\alpha_i).A[\alpha_i]\supset(\alpha_i).$
$f(\alpha_i) \vee \sim f(\alpha_i)\supset B$ (ii), π

(iv) $(\alpha_1)\ldots(\alpha_{i-1})(\alpha_{i+1})\ldots(\alpha_n).(\alpha_i)[f(\alpha_i) \vee \sim f(\alpha_i)\supset B]\supset .$
$(\exists\alpha_i)[f(\alpha_i) \vee \sim f(\alpha_i)]\supset B$ From a theorem of UV

(v) $(\alpha_1)\ldots(\alpha_{i-1})(\alpha_{i+1})\ldots(\alpha_n).(\alpha_i)A[\alpha_i]\supset .$
$(\exists\alpha_i)[f(\alpha_i) \vee \sim f(\alpha_i)]\supset B$ (iii), (iv),HS

(vi) $(\alpha_1)\ldots(\alpha_{i-1})(\alpha_{i+1})\ldots(\alpha_n)(\exists\alpha_i)[f\alpha_i) \vee \sim f(\alpha_i)]\supset .$
$(\alpha_i)A[\alpha_i]\supset B$ (v)

(vii) $(\exists\alpha_i)[f(\alpha_i) \vee \sim f(\alpha_i)]\supset (\alpha_i)\ldots(\alpha_{i-1})(\alpha_{i+1})\ldots(\alpha_n).$
$(\alpha_i)A[\alpha_i]\supset B$ (vi), $\gamma 1$

(viii) $(\alpha_1)\ldots(\alpha_{i-1})(\alpha_{i+1})\ldots(\alpha_n).(\alpha_i)A[\alpha_i]\supset B$
(vi), (8), MP [19]

The following two rules will not be proved — their proofs are straightforward.

MP′ If the universal closures of A, $A\supset B$ are theorems, then so is the universal closure of B.

HS′ If the universal closures of $A\supset B$, $B\supset C$ are theorems, then so is the universal closure of $A\supset C$.

Theorems can now be abbreviated by dropping initial universal quantifiers. For MP′ and HS′ justify the use of these abbreviated theorems in proofs. However, to make sense of theorems con-

[19] This proof follows the outlines of Mostowski's proof (in [8]) that adding (8) to UV yields the conventional system of Hilbert and Ackermann (minus free individual variables).

taining "free" individual variables, it must not be forgotten that these variables are bound.

6. DEFINITIONS. To develop UV+ in a natural manner, it is necessary to introduce defined predicates. These predicates cannot be regarded simply as abbreviations, because of the treatment of the description operator that is adopted in section 7. Two rules for introducing defined terms are required — these are:

1 An n-adic predicate Γ can be introduced by a definition having the form

$$(\alpha_1)(\alpha_2)\ldots(\alpha_n).\Gamma\alpha_1, \alpha_2,\ldots, \alpha_n)\equiv A[\alpha_1,\ldots,\alpha_n]$$

provided that the definition is well-formed and provided that $\alpha_1, \alpha_2,\ldots, \alpha_n$ are n distinct individual variables, each of which occurs in the definiens. The definiens must contain no other free individual variables, and must contain no predicate variables. Γ must not have been previously introduced.

δ2 An $(r_{\varphi_1}\ r_{\varphi_2},\ldots, r_{\varphi_n}, m)$-adic predicate-forming functor Λ can be introduced by a definition having the form

$$(\alpha_1)\ldots(\alpha_m).\Lambda\{\varphi_1, \varphi_2,\ldots, \varphi_n\}(\alpha_1,\ldots,\alpha_m)\equiv A[\varphi_1, \varphi_2,\ldots, \varphi_n, \alpha_1,\ldots,\alpha_m]$$

provided that the definition is well-formed, that φ_1 is an r_{φ_1}-adic predicate variable, φ_2 is an r_{φ_2}-adic predicate variable,..., φ_n is an r_{φ_n}-adic predicate variable, that $\varphi_1, \varphi_2,\ldots, \varphi_n$ are n distinct predicate variables each of which occurs in the definiens, and provided that $\alpha_1, \ldots, \alpha_m$ are m distinct individual variables each of which occurs in the definiens. The definiens must contain no other predicate variables and no other free individual variables. Λ must not have been previously introduced.

The elimination of predicate variables and free individual variables from the definiens (when they do not occur in the definiendum) is required to keep UV+ consistent. Without this restriction, the following definition would be acceptable:

(i) Paradoxical $(x) \equiv \sim \mathrm{f}(x)$.

But then,

(ii) Paradoxical $(x) \equiv \sim$ Paradoxical (x).

The rule δ2 prescribes a form for predicate-forming functors that will not be adhered to in practice. But such a form *could* always be employed (departures from it have the nature of abbreviations).

The definition of a wff must be modified to allow predicates as well as predicate variables, and to allow formulas containing predicate-forming functors [20]. An expression of the form

$$\Delta\{\varphi_1,\ldots,\varphi_n\}$$

will be called a predicate schema. An additional clause can be added to the definition of a wff so that

> If Δ is an $(r_{\varphi_1}, r_{\varphi_2},\ldots, r_{\varphi_n}, m)$-adic predicate-forming functor, φ_1 is an r_{φ_1}-adic predicate, predicate variable or predicate schema,..., φ_n is an r_{φ_n}-adic predicate, predicate variable or predicate schema, and $\alpha_1,\ldots,\alpha_m$ are individual variables or singular terms, then $\Delta\{\varphi_1, \varphi_2,\ldots, \varphi_n\}(\alpha_1, \alpha_2,\ldots, \alpha_m)$ is a wff.

Rule α3 must be modified to eliminate substitution for predicate variables having an occurrence as argument of a predicate-forming functor. And an additional rule of substitution must be added:

> *α4* Let A be a theorem containing the n-adic predicate variable ψ. Let Γ be one of

[20] No variables belonging to the category of predicate-forming functors will be employed in UV+.

(a) an n-adic predicate
(b) an n-adic predicate variable
(c) an n-adic predicate schema having the form

$$\Delta\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$$

where Δ is an $(r_{\varphi_1}, \ldots, r_{\varphi_m}, n)$-adic predicate-forming functor, φ_1 is an r_{φ_1}-adic predicate, predicate variable or predicate schema,..., φ_m is an r_{φ_m}-adic predicate, predicate variable or predicate schema.

The result of replacing each occurrence of ψ in A by Γ is a theorem.

Rule ծ1 makes it possible to define ordinary identity:

D1 $x = y \equiv (z).x \overset{e}{=} z \equiv y \overset{e}{=} z$

This identity has the customary properties — i.e.,

(9) $x = y \supset .f(x) \supset f(y)$
(10) $x = x$.

It is even the case that

(11) $\wedge = \wedge$.

The relation between '$=$' and '$\overset{e}{=}$' is brought out by

(12) $x \overset{e}{=} y \supset x = y$
(13) $x = y \;\&\; x \overset{e}{=} x \supset x \overset{e}{=} y$

A predicate 'Exists' can also be defined.

D2 Exists $(x) \equiv x \overset{e}{=} x$.

From this definition, it is possible to derive

(14) $\sim$Exists $(\wedge)$
(15) $(\exists x)\sim$Exists (x)
(16) Exists $(x) \equiv \sim[x = \wedge]$.

The predicate 'Exists' is fundamental with respect to other predicates of UV+. For the distinction between empty and non-empty names is basic to this system. To bring out the fundamental character of 'Exists', the following definitions are helpful.

D3 $f^{+}(x) \equiv$ Exists (x) & $f(x)$
D4 $f^{-}(x) \equiv$ .Exists $(x) \supset f(x)$ [21]

The functor '+' is used to form "strong" predicates, while '—' is used to form "weak" predicates. Some theorems illustrating the roles of strong and weak predicates are:

(17) $f^{+}(x) \supset f(x)$
(18) $f(x) \supset f^{-}(x)$
(19) $\sim$Exists $(x) \supset f^{-}(x)$
(20) $\sim$Exists $(x) \supset \sim f^{+}(x)$
(21) $f^{-}(\wedge)$
(22) $\sim f^{+}(\wedge)$
(23) Exists$^{+}(x) \equiv$ Exists (x)
(24) Exists$^{-}(x)$.

The difference between strong and weak predicates is illustrated by

(25) $\sim f^{+}(x) \equiv \sim$Exists $(x) \vee \sim f(x)$
(26) $\sim f^{-}(x) \equiv$ Exists (x) & $\sim f(x)$.

A strong predicate can go wrong in two ways, while a weak predicate can go wrong in only one way [22]. The fundamental

[21] The expressions '+' and '—' are predicate-forming functors.
[22] It is interesting to compare this treatment of predicates to discussions of definite descriptions and other singular terms. For it is com-

character of the predicate 'Exists' is shown by the fact that every predicate of the system UV+ is either a strong or a weak predicate. For

(27) $(x)[f(x) \equiv f^{+}(x)] \vee (x)[f(x) \equiv f^{-}(x)]$

is a theorem.

7. DESCRIPTIONS. It is necessary to have a convenient means for adding singular terms to UV^{+}. For a statement

$(\exists x)A[x]$

means that a name can be formed to make

$A[x]$

true. But this need not be a name that is already part of the language. For if it were a requirement that the name already be part of the language, it would make no sense to wonder whether

($\exists x$) Abominable Snowman (x)

is true (if this amounted to wondering whether there are any abominable snowmen).

In conventional systems, a description operator is used with individual variables — e.g.,

$(\iota x)f(x)$.

But this is not appropriate when variables are understood as replacements for singular terms. Instead of "the x such that $f(x)$", we shall deal with "the f" — i.e.,

mon to regard existence as either presupposed or asserted by the use of a singular term. But in UV+, it is predicates that carry existential force.

ιf.

(This is closer to ordinary English than the normal description.) The axioms required are:

A6 $\sim(\exists x)f^{+}(x) \supset \iota f = \wedge$
A7 $f^{+}(x) \& f^{+}(y) \& \sim[x=y] \supset \iota f = \wedge$
A8 $f^{+}(x) \& (y)[f^{+}(y) \supset x=y] \supset \iota f = x$ ([23]).

Descriptions require another modification in the definition of a wff. Rule α2 must be changed to allow the substitution of a description for an individual variable bound by an initial universal quantifier. And α3 must be further restricted to prevent the substitution for a predicate variable which occurs as part of a description ([24]).

([23]) These axions are such that if

$f\ (\wedge)$

$(\exists x) . f+(x) \& (y) . f+(y) \supset x = y,$

then

$f+(x) \supset x = \iota f.$

Alternatively, the axioms might have been such that

$\wedge = \iota f$

in this case — the following

$f+(x) \& (y)[f+(y) \supset x = y] \supset x = \iota f$

would hold only if

$\sim f\ (\wedge).$

([24]) As it now stands, the system UV+ does not contain the resources for forming a description from a relational predicate — e.g., the father of Tom Jones. This shortcoming can be remedied by adding two more rules for definitions. Rule δ3 would allow predicate-forming functors that take singular terms as arguments — such a definition could have the form

$(\alpha_1) \ldots (\alpha_n)\ (\beta).\ \Delta\langle\alpha_1,\ldots,\alpha_n\rangle\ (\beta) \equiv A\ [\alpha_1,\ldots,\alpha_n,\ \beta].$

And δ4 would allow predicate-forming functors which take both predicates and singular terms as arguments -- these could have the form

$(\alpha_1) \ldots (\alpha_n)\ (\beta).\Delta\{\Phi_1,\ldots,\Phi_m\}\ \langle\alpha_1,\ldots,\alpha_n\rangle\ (\beta) \equiv$
$A[\Phi_1,\ldots,\Phi_m,\ \alpha_1,\ldots,\alpha_n,\ \beta]$

These additional rules have been omitted for the sake of brevity of exposition.

8. CONSISTENCY. The system UV+ is demonstrably consistent. This can be established by stages. For the system UV is consistent. And the addition of the axioms A2-A5 preserves consistency. For the system UV with these axioms can be interpreted in a language which does not contain empty singular terms — '$\wedge$' will be the name of some selected individual. But then the system has been given the normal sort of interpretation. It is easy to find a model which establishes the consistency of the system.

The addition of rules $\delta 1$, $\delta 2$, and $\alpha 4$ does not render the system inconsistent. For all defined terms are eliminable. And any theorem (not containing defined terms) which is proved by means of theorems containing defined terms can also be proved without the use of defined terms (the proof of this will be omitted).

Once the axioms for the description operator have been added, defined terms are not always eliminable. But any formula containing a description can be replaced by a formula without a description, according to the rule:

> A formula $A[\iota\varphi]$ shall be replaced by
>
> $$(\exists\alpha_1)[\varphi^+(\alpha_1) \;\&\; (\alpha_2)[\varphi^+(\alpha_2) \supset \alpha_1 = \alpha_2 \;\&\; A[\alpha_1]] \vee.$$
> $$\sim(\exists\alpha_1)[\varphi^+(\alpha_1) \;\&\; (\alpha_2).\varphi^+(\alpha_2) \supset \alpha_1 = \alpha_2] \;\&\; A[\wedge]$$
>
> where α_1, α_2 are individual variables not occurring in $A[\iota\varphi]$.

The resulting formula is equivalent to the original. Any theorem containing a description yields a theorem when this replacement is performed, and the resulting theorem can be proved using only axioms A1-A5.

9. CONCLUSION. The system UV+ is a system of predicate calculus in the spirit of the systems of Lesniewski (but he did not permit any free variables, not even free propositional variables). This system is a useful vehicle for getting clear about the logical concept of existence. Customary quantificational systems are unsuitable for this purpose, because they are based on an

existential presupposition. Existence is usually "built-in" to the category of individual variables. It has even been claimed that this treatment of individual variables is necessary, or that "to be is to be the value of a bound variable". But this claim is incorrect. The customary treatment of individual variables is just one possible treatment. If individual variables are regarded as schematic letters replacing singular terms that may be empty or non-empty, there is no reason to make existential presuppositions. For existence need not be smuggled in with a category of variables taking entities as values, it can be dealt with explicitly.

State University of New York at Buffalo John T. KEARNS

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