AN INDUCTIVE DECISION-PROCEDURE FOR THE MONADIC PREDICATE CALCULUS

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In [1], pp. 167-8, Leblanc suggests using an inductive procedure to ascertain whether quantificational formulas are finitely valid. The purpose of this paper is to develop this suggestion into a decision procedure for the monadic case.

If a closed monadic quantificational formula is 1-valid, and is n-valid only if it is n′-valid ($n'=n+1$), it is valid in all finite domains, and hence, by the well-known theorem according to which validity in the monadic predicate calculus is 2^k-validity, where k is the number of predicates (see [2], pp. 34-37), the formula is valid. If it is valid, it is 1-valid and n′valid for any n; and if it is n′-valid, then the statement that it is n-valid only if it is n′-valid is true. All that remains to be shown is that there is a decision procedure to determine not only whether it is 1-valid but also whether the statement that it is n-valid only if n′-valid is true.

Behmann (in [3]) gave a method by which any monadic schema can be transformed into a standard schema; i.e., a truth-function of components each of which is a uniform quantification or sentence letter. By a "uniform quantification" is meant a quantification that consists of '(∀x)' or '(∃x)' followed by a truthfunction exclusively of F(x), G(x), H(x), etc.

Let S be a standard schema in which every occurrence of '(∃x)' has been replaced by '∼(∀x)∼'. S is thus of the form

$$(1)\quad f[\forall x)\varphi_1(x), \dots, (\forall x)\varphi_i(x), p_1, \dots, p_j],$$

in which each φ_r is a distinct truthfunction of F(x), G(x), H(x), etc. Now for each r, let Φ_r be the conjunction $\varphi_r(1)\&\varphi_r(2)\&\dots\&\varphi_r(n)$. Then (1) is n-valid if and only if

$$2)\quad f(\Phi_1, \dots, \Phi_i, p_1 \dots, p_j),$$

is a tautology. Hence (1) is n′-valid if and only if

(3) $f\{[\Phi_1\&\varphi_i(n')], \ldots, [\Phi_i\&\varphi_i(n')], p_i, \ldots, p_j\}$.

is a tautology. Thus (1) is n-valid only if it is n′valid, if and only if (2) ⊃ (3) is a tautology.

The first step of the procedure to be developed here is to ascertain whether (1) is 1-valid. If not, (1) is invalid. If it is 1-valid, we proceed to the second step. This step involves assigning truth-values to the sentential constituents of the conditional (2) ⊃ (3). It presupposes that for each *r*, Φ_r, which is actually a conjunction of n sentences, can be treated as a single sentence. Let us prove that this is so in the special case of the conditional

(4) $f[\varphi(1)\&\ldots\&\varphi(n)] \supset f[\varphi(1)\&\ldots\&\varphi(n)\&\varphi(n')]$.

What we must show is that if (4) is valid, so is

(5) $f(\Phi) \supset f[\Phi\&\varphi(n')]$.

(of course, if (5) is valid, (4) follows immediately as a substitution-instance.) Our proof is as follows. Assume that (5) is invalid. Then it must be possible for $f(\Phi)$ to be true when $f[\Phi\&\varphi(n')]$ is false. This is possible only when Φ is true and $\varphi(n')$ is false; for otherwise, the truthvalue of the conjunction is the same as that of Φ. But whenever Φ can be made true and $\varphi(n')$ false, $\varphi(1)\&\ldots$ & $\varphi(n)$ can be made true and $\varphi(n')$ false. For since $\varphi(1), \ldots, \varphi(n)$ are all distinct, no two of them can contradict each other. Of course, for some i, φ (i) might be a contradiction. But in this case, $\varphi(i)$ would be a contradiction for every i; in particular, $\varphi(1)$ is a contradiction. In this case, (5) cannot be invalid. For since we have already ascertained that (1) is 1-valid, we know that $f[\varphi(1)]$ is true when $\varphi(1)$ is false. This means that $f(\Phi)$ is true when Φ is false. Assume now that (5) is invalid. We have already seen that in this case $f(\Phi)$ is true when Φ is true. Hence the truthvalue of $f(\Phi)$ does not depend on the truthvalue of Φ, and we can accordingly substitue $\Phi\&\varphi(n')$ for Φ in $f(\Phi)$. This substitution yields (5) as a valid conditional, contradicting the assumption that (5) is invalid.

The special case can now readily be extended to cover (2) and (3), since (2) can be true and (3) false only if at least one of the Φ's is true and its corresponding $\varphi(n')$ is false. In case for some j $\varphi_j(i)$ is a contradiction for every i, but (2) is 1-valid, the conditional $(2) \supset (3)$ can be invalid only if the truthvalue of (2) is independent of the truthvalue of Φ_j; whence $\Phi_j \& \varphi_j(n')$ can be substituted for Φ_j.

In view of the fact that for each r, Φ_r is a conjunction of n sentences which may be treated as a single sentence, and $\varphi_r(n')$ is a sentence, the truth of the inductive conditional having (2) as its antecedent and (3) as its consequent can be decided by assigning truthvalues to the Φ's and φ's. So can the truth of the formula expressing the claim that (1) is 1-valid. Hence the validity of (1) can be decided.

As an example, let us apply the procedure to the following result of replacing '$(\exists x)$' by '$\sim(\forall x)\sim$' throughout a standard schema:

(5) $(\forall x)[F(x) \vee G(x) \vee H(x)] \supset [(\forall x)F\ (x) \vee \sim(\forall x)\sim G(x) \vee \sim(\forall x) \sim H(x)]$

Now (5) is clearly 1-valid. Accordingly, we test the conditional

(6) $[\Phi_1 \supset (\Phi_2 \vee \sim\Phi_3 \vee \sim\Phi_4)] \supset [\{\Phi_1\&[F(n') \vee G(n') \vee H(n')]\} \supset \{[\Phi_2\&F(n')] \vee \sim[\Phi_3\&\sim G(n')] \vee \sim[\Phi_4\&\sim H(n')]\}]$

by assigning truthvalues to the sentential constituents Φ_1, Φ_2, Φ_3, Φ_4, $F(n')$, $G(n')$, and $H(n')$. Since (6) is a tautology, we conclude that (5) is valid.

This example shows that the procedure under discussion sometimes requires the evaluation of truthfunctions with a smaller number of sentential variables than the procedure consequent upon [2], pp. 34-37, would require. For the example involves a truthfunction of only seven variables, while the latter procedure, consisting of a validity test in a domain of 2^k individuals (k in this case being 3), gives rise to a truthfunction of the twenty-four variables F(1),...,F(8), G(1),...,G(8), H(1),...,H(8). However, the present procedure is not always the simpler. It calls for three variables in the validation of $(\forall x)F(x) \supset \sim(\forall x) \sim F(x)$, while the older procedure requires only two.

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[2] W. ACKERMAN, *Solvable Cases of the Decision Problem,* North-Holland, Amsterdam, 1954.

[3] H. BEHMANN, "Beiträge zur Algebra der Logik, insbesondere zum Entscheidungsproblem", *Mathematische Annalen,* Vol. 86 (1922), pp. 163-229. Quine also uses this method in "On the logic of quantification". *Journal of Symbolic Logic,* Vol. 10 (1945), pp. 1-12.