THE LOGIC OF LOGICAL NECESSITY

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A number of different theories of propositional modal logic can be found in the literature. Most of these theories have at one time or another been claimed to adequately formalize the logic of logical necessity. The problem that arises is to determine which of these theories is really a correct formalization of that concept. There are two ways one might go about assessing such a theory. On the one hand, one might try to assess it intuitively, trying to find axioms that are intuitively evident, and then prove completeness in the same way by arguing that the addition of any other axioms yields theorems that are intuitively invalid. This is the way that most philosophers have approached this problem in the past, but unfortunately it has been largely unsuccessful. It seems that philosophers' intuitions are not sufficiently precise here to decide this issue directly.

On the other hand, one might try to formulate a semantics for modal logic which could then be used to prove the completeness and soundness of a particular axiomatic theory of modal logic. There have been several noteworthy recent attempts along this line. But unfortunately, it is not at all easy to see how the purely mathematical concepts of validity employed in these attempts relate to the intuitive concept of a truth of the logic of logical necessity which the concept of validity is supposed to explicate. Many of the details of these definitions of validity seem to be philosophically unmotivated. And this is mirrored by the rash of different theories of modal logic that have been proven complete using different concepts of validity. Only one of these concepts of validity can correctly formalize our intuitive concept of a truth of the logic of logical necessity, and the problem of deciding which one does seems to be no less difficult than the problem of deciding directly which axiomatic theory is intuitively complete (1).

(1) For a detailed discussion of this question see [1].

I think that the best way to attack the problem of deciding between different theories of modal logic may be to look directly at the intuitive concept of a truth of the logic of logical necessity, try to clarify it, and then work from that to a mathematical concept of logical validity. I take it that what it means to say a sentence (of propositional modal logic) is a truth of logic is that every assignment of meanings to the atomic parts of the sentence yields a necessarily true statement (²). This much is fairly simple. But to go further we must become clearer on just what we mean by "logical necessity" and thus how the modal operator is to be interpreted.

Logical necessity has traditionally been defined, or characterized, in a number of different ways. Let me list a few of the more common characterizations:

(i) A statement is necessarily true iff its denial entails an explicit contradiction.

(ii) A statement is necessarily true iff it is entailed by its own denial.

(iii) A statement, P, is necessarily true iff it is logically equivalent to (P v ~P).

(iv) A statement is necessarily true iff it is entailed by every statement.

(v) A statement is necessarily true iff its denial entails every statement.

(vi) A statement is necessarily true iff any state of affairs whatsoever would make it true.

(vii) A statement is necessarily true iff it is true in all possible worlds.

These are seven common characterizations of logical necessity, but they are not as clear as we might desire. And it is not all obvious that they are all characterizations of the same concept. Philosophers commonly use these characterizations interchangeably, assuming that they are all equivalent, but such an assump-

(²) For a defense of this see [1].

tion would seem to be entirely unwarranted unless some proof can be given to that effect.

The first thing I shall do will be to try to find an interpretation of these seven characterizations of logical necessity which will allow us to develop their mathematical properties, and then I will go on to investigate whether these characterizations are all equivalent under that interpretation. Then we can see how these results bear on the problem of deciding between alternative theories of modal logic.

2. *States of Affairs.*

In order to clarify the concept of logical necessity, I will give an analysis which is in the general tradition of Wittgenstein [2], Carnap [3], and C. I. Lewis [4]. The attempt will be made to interpret the above seven characterizations of logical necessity in terms of states of affairs. By a "state of affairs" I mean things like John's being a bachelor, Bill and Bob's being brothers, the third house on the block's being green, Mary's making pies, our not being able to travel faster than the velocity of light, there being ten million unicorns on Mars, etc. A state of affairs is just something's being the case. The concept of a state of affairs brings together under one heading a number of different kinds of things between which we ordinarily distinguish, such as events (the Giant's winning the pennant), situations (the division's being trapped at Dien Bien Phu), conditions (X's being a locally compact Hausdorff topological group), causes (the five ball's striking the eight ball), occurrences, accidents, happenings, incidents, circumstances, consequences, etc. Given an English indicative sentence P, there is a gerund clause P_g which stands to P as "John's being a bachelor" stands to "John is a bachelor". A state of affairs is that *kind* of thing to which we can refer by using these gerund clauses. This is of course not to say that we can refer to every state of affairs by an actual English gerund clause of this form. Let Φ be the class of all states of affairs.

It is advantageous to introduce the following locution: Given a gerund clause P_g that refers to some state of affairs, let us define

'p_g obtains' to mean simply P. Thus, "John's being a bachelor, obtains", means, "John is a bachelor". The use of this locution will allow us to say things that would otherwise be very difficult to express.

Now of course, some philosophers will object that there are no such things as states of affairs, or that the gerund clauses which have been said to refer to states of affairs are really non-referential. On the face of it, to say that there are no such things as states of affairs seems just false. It would involve us in saying that there are no such things as events, causes, happenings, incidents, occurrences, etc. and those are certainly not philosophers' inventions. But perhaps this is too summary a dismissal of a serious philosophical position. Let us consider for a moment just what is involved in saying that expressions of a certain type are referential. As far as I can see, this involves no more than saying that we can use "thing-talk" with respect to these expressions. For example, if A is an expression of that type, then from 'A is a φ', we can conclude, 'Something is a φ'. Applying this to states of affairs, from "The eight ball's striking the five ball caused the five ball to go in the pocket", we can conclude, "Something caused the five ball to go in the pocket". And we can say either, "Halley's comet's appearing on the wrong day was widely discussed", or, "Halley's comet's appearing on the wrong day was something that was widely discussed", or, "One thing that was widely discussed was Halley's comet's appearing on the wrong day", and from any of these we can conclude, "Something was widely discussed". And from, "His being color blind was responsible for his going through the red light", we can conclude, "Something was responsible for his going through the red light". The use of thing-talk with respect to states of affairs is perfectly normal, and I see no other basis on which to judge whether states of affairs are things.

Those who would deny that there are such things as states of affairs take themselves to be saying something very deep and profound. They make such proclamations as "States of affairs are not among the ultimate furniture of the world". Frankly, I don't understand such talk. But, if there really is anything at issue here (which I rather doubt), I suspect that what I mean by saying that states of affairs exist is not incompatible with what they mean to be saying. When I say that there is a state of affairs having a certain property, I am using "there is" in the same sense as the mathematician who says "There is a prime number between 15 and 20". The people who deny the existence of states of affairs frequently deny, in the same sense, the existence of numbers. But in doing so they do not mean to deny that there is a prime number between 15 and 20. Thus I am inclined to think that my talking about the existence of states of affairs is really philosophically neutral. I am only talking about interrelations between states of affairs, and not any deeper questions about the "ontological status" of states of affairs.

I think that much the same thing can be said to forestall possible objections to my saying that there are conjunctive states of affairs, negative states of affairs, general states of affairs, etc. This is really no more than a convention on my part concerning how I will use the term "state of affairs". The disputes, for example, of the logical atomists, are irrelevant here. They purported to be concerned with a deep ontological question that I prefer to leave untouched.

3. *The Algebra of States of Affairs*

Now I want to say a little about the algebraic structure of states of affairs.

We sometimes have occasion to say that two states of affairs are really one and the same state of affairs. For example, we may say that John's being a bachelor is the same thing as John's being an unmarried man, or that Smith's being a widower is the same thing as Smith's being a man whose wife died, or that X's being a compact topological space is the same thing as X's being a topological space such that each net in X has a cluster point. It is not entirely clear just what our grounds for such claims are, but the fact remains that we do make them. At least sometimes they are based on an argument to the effect that two statements imply one another. For example, the claim that John's being a bachelor is the same thing as John's being an unmarried man might be supported by saying that the statement that John is a bachelor and the statement that John is an unmarried man imply one another.

There is another way of saying that two states of affairs are really one and the same state of affairs. For example, rather than saying, "John's being a bachelor is the same thing as John's being an unmarried man", we can sometime more easily say, "For John to be a bachelor is the same thing as for John to be being an unmarried man", we can sometime more easily say, place of the gerund constructions is often more natural. Note also that the way I have introduced the verb "obtain", the statement "For John to be a bachelor is the same thing as for John to be an unmarried man", is equivalent to, "For John's being a bachelor to obtain, is the same thing as for John's being an unmarried man to obtain". Thus, in general, given two states of affairs, X and Y, we can read "$X = Y$" as either, "X is the same thing as Y", or, "For X to obtain is the same thing as for Y to obtain".

We sometimes have occasion to say that one state of affairs is *part of* another state of affairs. For example, we may say that John's being unmarried is part of John's being a bachelor, or Smith's having had a wife is part of Smith's being a widower, or X's being a compact topological space is part of X's being a Lindelöf space such that every sequence in X has a cluster point. As for our grounds for claiming that one state of affairs is part of another, they are sometimes based on one statement's implying another. For example, to support the assertion that John's being unmarried is part of John's being a bachelor, we might appeal to the fact that the statement that John is a bachelor implies the statement that John is unmarried.

Just as in the case of identity, there is an alternative way of expressing this relation. For example, rather than saying, "John's being unmarried is part of John's being a bachelor", we can if we wish say, "For John to be unmarried is part of what it is for John to be a bachelor". Given two states of affairs, X and Y, let us symbolize the statement that X is part of Y by writing "$X < Y$".

We can define certain interesting operators on states of affairs:

DF 3.1 Given any two states of affairs X and Y, and any set A of states of affairs:

(i) X+Y is its being the case that either X obtains or Y obtains (which is a state of affairs, given the way I introduced states of affairs — in terms of gerund clauses);

(ii) X·Y is its being the case that both X and Y obtain;

(iii) —X is its being the case that X does not obtain;

(iv) ∩ A is its being the case that all of the states of affairs in A obtain.

Given these operators, we need not take "<" as primitive. It can be defined in terms of our other concepts: Part of what it is for Y to obtain is for X to obtain, just in case for Y to obtain is the same thing as for *both* X and Y to obtain. That is,

AX 3.2: $X < Y$ iff $Y = X \cdot Y$.

We can list a number of other things which are also intuitively true of states of affairs:

AX 3.3: $X \cdot Y = Y \cdot X$ (for both X and Y to obtain is the same thing as for both Y and X to obtain);

AX 3.4: $X \cdot X = X$ (for both X and X to obtain is the same thing as for X to obtain);

AX 3.5: $X \cdot (Y \cdot Z) = X \cdot Y) \cdot Z$ (for both X and both Y and Z to obtain, is the same thing as for both, both X and Y, and Z to obtain);

AX 3.6: $-(-X) = X$ (for it not to be the case that X doesn't obtain is the same thing as for X to obtain);

AX 3.7: $-(X+Y) = (-X) \cdot (-Y)$ (for it not to be the case that either X or Y obtain is the same thing as for neither X nor Y to obtain);

AX 3.8: $X \cdot (Y+Z) = [(X \cdot Y)+(X \cdot Z)]$ (for both X and either Y or Z to obtain, is the same thing as for either both X and Y to obtain, or both X and Z to obtain);

AX 3.9: $X < Y$ iff $—Y < —X$ (part of what it is for Y to obtain is for X to obtain, iff part of what it is for X not to obtain is for Y not to obtain);

AX 3.10: $X < [(X+Y) \cdot (—Y)]$ (part of what it is for both, either X or Y to obtain, and Y not to obtain, is for X to obtain).

These principles are intuitively evident in the same sense in which the principle of mathematical induction is intuitively evident. They may not seem obvious at first, but once you think about them for a while, and try them out for specific cases, you will come to see just what they say and see that they are true. To show how this is the case, let me go through one instance of 3.6:

—(—Mary's making pies)
= its not being the case that (—Mary's making pies) obtains
= its not being the case that (its not being the case that Mary's making pies obtains) obtains
= its not being the case that (its not being the case that Mary is making pies) obtains
= its not being the case that it is not the case that Mary is making pies
= its being the case that Mary is making pies
= Mary's making pies.

This same schema of transformations works in general to show that 3.6 is true.

Huntington's axioms for Boolean algebra (in [5]) follow easily from 3.2-3.10, and so:

TH 3.11: $\langle \Phi, +, \cdot, — \rangle$ is a Boolean algebra. "$<$" is the Boolean "greater than or equal to".

Let 0 and 1 be the zero and unit elements of this algebra respectively.

We can make several interesting observations about our fourth operator "$\cap$". Clearly, for any set A of states of affairs, if $X \varepsilon A$, then part of what it is for all of the states of affairs in A to obtain is for X to obtain:

AX 3.12: If $X \varepsilon A$, then $X < \cap A$.

Furthermore, if there is some state of affairs Y such that for every $X \varepsilon A$, part of what it is for Y to obtain is for X to obtain, then part of what it is for Y to obtain is for all of the states of affairs in A to obtain:

AX 3.13: If $(\forall X)$(if $X \varepsilon A$, then $X < Y$), then $\cap A < Y$.

3.12 and 3.13 together, mean that $\cap A$ is the *infimum* of A, and that $\langle \Phi, +, \cdot, - \rangle$ is a *complete* Boolean algebra.

4. *Entailment and Logical Necessity*

Now we are in a position to characterize entailment, logical equivalence, and logical necessity, in terms of states of affairs. First let us define a function T which maps Σ, the class of statements, onto Φ, the class of states of affairs. Given a statement P, T(P) is just P's being true (which is a state of affairs). T(P) might be called the *truth condition* of P. T maps Σ *onto* Φ because corresponding to each state of affairs is the statement that it obtains.

Clearly the following will be true if we let "&" symbolize conjunction, "v" symbolize disjunction, and "$\sim$" symbolize negation:

AX 4.1: $T(P \,\&\, Q) = T(P) \cdot T(Q)$ (for P & Q) to be true is the same thing as for both P and Q to be true);

AX 4.2: $T(P \vee Q) = T(P) + T(Q)$ (for $(P \vee Q)$ to be true is the same thing as for either P to be true or Q to be true);

AX 4.3: $T(\sim P) = -T(P)$ (for $\sim P$ to be true is the same thing as for P not to be true).

We can now give formal definitions of entailment and logical equivalence. We can define entailment by saying that P entails Q just in case any state of affairs which would make P true would also make Q true. And analogously, we can say that P is logically equivalent to Q just in case any state of affairs which would make one true would also make the other true. A state of affairs, X, makes a statement, P, true, just in case part of what it is for X to obtain is for P to be true, that is, just in case $T(P) < X$. So we can state our definitions of entailment and logical equivalence in the following manner:

DF 4.4: $P \rightarrow Q$ (P entails Q) iff for every state of affairs X, if $T(P) < X$, then $T(Q) < X$;

DF 4.5: $P \leftrightarrow Q$ (P is equivalent to Q) iff for every state of affairs X, $T(P) < X$ iff $T(Q) < X$.

Because $\langle \Phi, +, \cdot, - \rangle$ is a Boolean algebra, we are immediately led to the following simpler characterizations of entailment and logical equivalence:

TH 4.6: $P \rightarrow Q$ iff $T(Q) < T(P)$;

TH 4.7: $P \leftrightarrow Q$ iff $T(P) = T(Q)$.

This means that P entails Q just in case part of what it is for P to be true is for Q to be true, and P is equivalent to Q just in case for P to be true is the same thing as for Q to be true.

Now we can go back and examine our first few concepts of logical necessity and show that they are all equivalent. On our interpretation, (i) - (v) will be, respectively:

(i) $(\exists Q)[\sim P \rightarrow (Q \ \& \sim Q)]$;
(ii) $\sim P \rightarrow P$;
(iii) $P \leftrightarrow (P \vee \sim P)$;
(iv) $(\forall Q)(Q \rightarrow P)$;
(v) $(\forall Q)(\sim P \rightarrow Q)$.

The interpretations of (i) - (v) in terms of our Boolean algebra of states of affairs are all equivalent to the condition that $T(P) = 1$. Therefore,

TH 4.8: (i) - (v) are all equivalent descriptions of the same concept of logical necessity.

Now let us consider (vi). According to (vi), a statement is necessarily true iff any state of affairs whatsoever is sufficient to make it true:

(vi) For any state of affairs X, $T(P) < X$.

Remembering that "$<$" is the Boolean "greater than or equal to", (vi) is also equivalent to the condition that $T(P) = 1$, and so

TH 4.9: (vi) is equivalent to characterizations (i) - (v) of logical necessity.

Finally, let us consider (vii), which is probably the historically most important concept of logical necessity. According to (vii), a statement is necessarily true iff it is true in all possible worlds. Without some explanation it is not at all clear just what a possible world is, although we do seem to have some intuitive concept of a possible world. It seems advisible therefore to substitute an apparently more precise notion for the notion of a possible world. We can simply identify the possible world with the set of states of affairs that obtain in it. For us then a possible world is a set of states of affairs. However, not just any set of states of affairs will qualify as a possible world. First of all, a possible world W, must be *maximal* in the sense that for any state of affairs X, either $X \varepsilon W$ or $—X \varepsilon W$. This is just the law of the excluded middle. Furthermore, W must be *consistent* in the sense that for no state of affairs X, part of what it is for all of the states of affairs in W to obtain is for both X and —X to obtain. This is just the requirement that the law of non-contradiction hold. Given that W is maximal, this is the same thing as requiring that $\cap W \neq O$. If W is a maximal consistent set of states of affairs, then it is a possible world.

Let us take Ψ to be the set of all of these maximal consistent sets of states of affairs. Then Ψ is the set of possible worlds. Mathematically, Ψ is the set of maximal complete proper filters of $\langle \Phi, +, \cdot, - \rangle$.

But now, how do we know that Ψ has any members ? In other words, how do we know that there are any possible worlds ? There are Boolean algebras in which there are no maximal complete proper filters. Consider an arbitrary state of affairs X. Suppose there is no element of Ψ containing X. Then it must be logically impossible for X to obtain, because in order for X to obtain, the set of states of affairs obtaining along with X must constitute a possible world — an element of Ψ. Let P_X be the statement that X obtains. It is logically impossible for X to obtain just in case it is logically impossible for P_X to be true, that is, just in case $\sim P_X$ is necessarily true. So if there is no element of Ψ containing X, then $\sim P_X$ is necessarily true. Utilizing our characterization of logical necessity in terms of (i) - (vi), $\sim P_X$ is necessarily true iff $T(\sim P_X) = 1$, or iff $T(P_X) = 0$. But now consider, what is $T(P_X)$? For P_X to be true is the same thing as for X to obtain, so $T(P_X)$ is just X. Thus $\sim P_X$ is necessarily true just in case $X = 0$. Therefore, if there is no element of Ψ which contains X, then $X = 0$. Thus, in general,

TH 4.10: For any state of affairs X, if $X \neq 0$ then there is a $W \varepsilon \Psi$ such that $X \varepsilon W$.

This has the consequence that $\langle \Phi, +, \cdot, - \rangle$ is an *atomic* Boolean algebra, the set of atoms being $\{ \cap W; W \varepsilon \Psi \}$. These atoms are states of affairs that are "complete" in the sense that they by themselves uniquely determine a possible world. An atom has the form $\cap W$ where W is a possible world, so for that atom to obtain is the same thing as for all of the states of affairs in W to obtain, that is, for W to be the real world. This suggests another way of interpreting the concept of a possible world. Rather than take a possible world to be a set of states of affairs, we might take it to be an atomic state of affairs. The consequences for logical necessity would be the same.

Now, for a statement P, let us define M(P) to be the set of possible worlds in which P is true:

DF 4.11: $M(P) = \{W; W \varepsilon \Psi \text{ and } T(P) \varepsilon W\}$.

To say that P is true in all possible worlds is just to say that $M(P) = \Psi$. This then is our interpretation of (vii):

(vii) $M(P) = \Psi$.

From the fact that $\langle \Phi, +, \cdot, — \rangle$ is a complete, atomic, Boolean algebra, we can conclude the following:

TH 4.12: $M(\sim P) = \Psi — M(P) = \overline{M(P)}$;

TH 4.13: $M(P \vee Q) = M(P) \cup M(Q)$;

TH 4.14: $M(P \& Q) = M(P) \cap M(Q)$;

TH 4.15: $P \rightarrow Q$ iff $M(P) \subseteq M(Q)$;

TH 4.16: $P \leftrightarrow Q$ iff $M(P) = M(Q)$.

Also, because T maps Σ *onto* Φ, it follows that M maps Σ *onto* the class of subclasses of Ψ. And by TH. 4.16, we can determine a statement P to within logical equivalence by specifying M(P).

Thus we can characterize entailment and logical equivalence in terms of sets of possible worlds. And from theorems 4.12-4.16, it follows that $M(P) = \Psi$ iff $T(P) = 1$. Thus,

TH 4.17: Characterization (vii) is equivalent to characterizations (i) - (vi) of logical necessity.

On the interpretation I have given, all seven of the traditional characterizations of logical necessity turn out to be equivalent.

We can think of statements as being of the general form "the statement that P" (e.g., the statement that $2 + 2 = 4$). Now let us introduce a new statement operator "*L*" such that, if P is the statement that p, then *LP* is the statement that it is necessarily true that p. By (vii), *LP* is true in a possible world W iff $M(P) = \Psi$. That is,

AX 4.18: $$M(LP) = \begin{cases} \Psi \text{ if } M(P) = \overline{\Psi} \\ \varnothing \text{ otherwise} \end{cases}$$

Now we have a fairly well developed algebraic theory of logical necessity. In the next section I will attempt to show this theory can be used to decide between the different theories of propositional modal logic.

It is perhaps of some interest to contrast this treatment of logical necessity with that of Carnap [3] and his many followers. The two treatments seem superficially very similar. But I don't think that the similarity really extends any further than the fact that both *result in* a complete atomic Boolean algebra of states of affairs. Their philosophical underpinnings are quite different. Carnap bases his theory on the assumption that there is a class $\mathfrak{B}$ of propositions which has the following two properties:

(1) given any partition $\langle \mathfrak{B}_1, \mathfrak{B}_2 \rangle$ of $\mathfrak{B}$, there is a consistent proposition (a "state description") which is true when, and only when, all of the propositions in $\mathfrak{B}_1$ are true and all of the propositions in $\mathfrak{B}_2$ are false;

(2) every consistent proposition is logically equivalent to a (perhaps infinite) disjunction of propositions formed as in (1) from partitions of $\mathfrak{B}$.

I can see no reason at all for thinking that there is such a class $\mathfrak{B}$ of propositions. Its existence is not entailed by anything in my treatment of logical necessity, and I am convinced on general philosophical grounds that it cannot exist. Thus it seems that the similarity between the two approaches is only superficial. They have quite different philosophical foundations.

5. *Propositional Modal Logic*

Now we can turn to propositional modal logic. Let us introduce a language L within which to formulate propositional modal logic. In constructing L, we begin with a countably infinite set AT of *atomic sentences*. Then we define ST, the set of *sentences* of L, recursively as follows:

DF 5.1: (i) If $p \varepsilon AT$, then $p \varepsilon ST$;
(ii) If $p \varepsilon ST$, then $[\sim p] \varepsilon ST$;

(iii) If $p \varepsilon ST$, then $[\Box p] \varepsilon ST$;
(iv) If $p,q \varepsilon ST$, then $[p \mathbin{\&} q] \varepsilon ST$.

To formalize our concept of a truth of the logic of L, we next define the notion of an N-interpretation:

DF 5.2: An *N-interpretation* of L is a function δ mapping ST into Σ in such a way that:
(i) $\delta(\sim p) = \sim\delta(p)$;
(ii) $\delta(\Box p) = L\delta(p)$;
(iii) $\delta(p \mathbin{\&} q) = [\delta(p) \mathbin{\&} \delta(q)]$.

Let us say that a sentence p is *N-valid* iff for every N-interpretation δ, $\delta(p)$ is necessarily true. This formalizes our concept of a truth of the logic of L (3). The mathematical development of the algebra of states of affairs in sections three and four will now allow us to develop the mathematical properties of this concept of validity, and ultimately to decide between the different theories of propositional modal logic.

Let us define:

DF 5.3: A *representation* of L is a function μ assigning to each sentence a subclass of Ψ in such a way that:
(i) $\mu(\sim p) = \overline{\mu(p)}\ (=\Psi - \mu(p))$;
(ii) $\mu(p \mathbin{\&} q) = [\mu(p) \cap \mu(q)]$;
(iii) $\mu(\Box p) = \begin{cases} \Psi \text{ if } \mu(p) = \Psi; \\ \varnothing \text{ otherwise.} \end{cases}$

Clearly, if δ is an N-interpretation, then $M \circ \delta$ is a representation. Furthermore, for every representation μ, there is some N-interpretation δ such that $\mu = M \circ \delta$, because M maps Σ *onto* the class of subclasses of Ψ. Thus a sentence p is N-valid iff for every representation μ, $\mu(p) = \Psi$.

Now what will be shown is that a sentence is N-valid iff it is a theorem of S5 (4). The theorems of S5 can be generated by the following set of axiom schemata and rules (5):

(3) In the terminology of [1], this is semantical validity$_2$.
(4) This theory of propositional modal logic is described in [6], p. 501.
(5) See [7].

A1. All tautologies;

A2. $\Box p \supset p$;

A3. $\sim \Box p \supset \Box \sim \Box p$;

A4. $\Box[p \supset q] \supset [\Box p \supset \Box q]$;

R1. if p and $[p \supset q]$ are theorems, so is q;

R2. if p is a theorem, so is $\Box p$.

It is easily verified that all of these axioms are N-valid, and that the rules preserve N-validity. For example, let us look at A3. Suppose $\mu(p) \neq \Psi$. Then $\mu(\Box p) = \varnothing$, and $\mu(\sim \Box p) = \Psi$. Then $\mu(\Box \sim \Box p) = \Psi$, and so $\mu(\sim \Box p \supset \Box \sim \Box p) = \Psi$. Conversely, suppose $\mu(p) = \Psi$. Then $\mu(\Box p) = \Psi$. Thus once again, $\mu(\sim \Box p \supset \Box \sim \Box p) = \Psi$.

Thus,

TH 5.4: If p is a theorem of S5, then p is N-valid.

So S5 is sound.

The proof that S5 is complete follows very easily from theorem 6 of [8]. We have seen that S5 is sound. It is also obvious that the set of N-valid sentences constitutes a regular theory in the sense of [8]. Therefore, S5 is complete:

TH 5.5: If $p \varepsilon ST$ is N-valid, then p is a theorem of S5.

Thus

TH 5.6: If $p \varepsilon ST$, then p is N-valid iff p is a theorem of S5.

Thus our partial analysis of logical necessity allows us to determine which of the various theories of propositional modal logic correctly formalizes the logic of logical necessity.

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