PROPOSITIONAL IDENTITY

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1. The System FCR.

In [1] we discussed a calculus of functions of propositions [FC] in which the functorial variables f,g,...*etc.* are not truth-functional. We noted that it is possible to augment such a system by adding propositional identity with the following formation rules and axiom schemata (otherwise the basis is as in section 2 of [1]);

1.11 Primitive symbol, $=$.

1.21 Formation rule; If A and B are wffs then $(A=B)$ is a wff.

1.31 Axiom schema I1; $A = A$ (Where A is any wff).

1.32 I2; $(p_1)\ldots(p_n)(A=B) \supset (C\supset D)\ (n\geqslant 0)$,
where C and D are formulae differing only in replacement of A by B and there is no variable p other than $p_1\ldots p_n$ free in A or B such that A or B in C or D occur within the scope of (p).

In such a system it is not possible to prove either $(p\equiv q) \supset (p=q)$ or even:

1.4 (R) $\vdash A \equiv B \rightarrow \vdash A = B$.

The addition of the former reduces FC to protothetic and so its consequences are well known (v.[5] pp. 151-154). It is the purpose of this paper to investigate the consequences of the latter.

Part of the motivation for this is the connection between functorial calculi with a rule such as R and modal logic. In particular between semantics of a kind considered in [1] and the kind of semantics for modal quantificational logic set out in [2]. It is shewn in [3] that the addition of the rule R to a pure calculus of propositional identity gives a system deductively equivalent to S4. To obtain S5 we would add some such formula as:

1.5 (NI) $((p=q)\supset O) \supset ((p=q)=O)$.

The system of functorial logic which we shall investigate is that obtained from FC by adding I2 (I1 can be proved by R from $(A\equiv A)$), R and NI (In schematic form if there are no rules of uniform substitution). We call the system so defined FCR.

We define *validity* for FCR by extending the method of [1]. While [1] assumed a set T of propositions which could be interpreted as the set of all true propositions and so could be said to mirror the 'real world', FCR assumes a set W of sets of propositions each set representing a 'possible world' (1).

2. Validity in FCR

An FCR-model is an ordered triple $\langle V\,W\,P\rangle$ such that P is a set of objects (propositions); $\{p_1,\ldots,p_i,\ldots\}$; W is a set of subsets of P; $\{T_1,\ldots,T_i,\ldots\}$; and V an assignment from wffs to members of P and from functors to n-tuples of P satisfying the following:

2.1 For propositional variable p, V(p) is some member of P.

2.2 V(O) is some member of P not in any $T \varepsilon W$.

2.3 $V(=)$ is a descriptive (v.[1]p. 547) set of ordered triples of P, $\langle p_i p_j p_k\rangle$ such that for any $T\varepsilon W$, $p_i \varepsilon T$ iff $p_j = p_k$.

2.4 $V(\supset)$ is a descriptive set of triples of P, $\langle p_i p_j p_k\rangle$ such that for any $T\varepsilon W$, $p_i \varepsilon T$ iff p_j is not in T or $p_k \varepsilon T$.

2.5 For n-adic functorial variable f, V(f) is a descriptive set of $(n+1)$-tuples of P.

2.6 $V(\forall)$ is a descriptive set of pairs $\langle p_i \alpha\rangle$ such that $p_i \varepsilon P$, and α is a set of pairs of P such that for any $T\varepsilon W$, $p_i \varepsilon T$ iff the first member of each pair in α is in T.

(1) The analogy with modal predicate logic should be by now becoming apparent. For a semantical discussion of this v.[2] and [8].

2.7 Where F is an n-adic functor (variable or constant) then, for wffs $A_1,\ldots,A_n$, $V(F(A_1,\ldots,A_n)) = p_i$, iff $\langle p_i, V(A_1),\ldots,V(A_n)\rangle \varepsilon V(F)$.

2.8 Where V′ differs from V only in assignment to p then, where α is the set of all $\langle V'(A), V'(p)\rangle$ (for every such V′), then $V((p)A) = p_i$ iff $\langle p_i, \alpha\rangle \varepsilon V(\forall)$.

2.9 If for some p_i, p_j, $p_i \varepsilon T$ iff $p_j \varepsilon T$, for every $T \varepsilon W$, then $p_i = p_j$.

For consistency we observe that a two-valued model ($P=\{p_1, p_2\}$) defined by the usual truth-functional conditions will verify all the axioms and rules.

A formula A is FCR valid iff for every FCR model $\langle V\ W\ P\rangle$ $V(A)\ \varepsilon T$ for every $T \varepsilon W$. (By 2.9 there will only be one such proposition in any model so we could call it p_1, and say that A is valid iff $V(A) = p_1$).

THEOREM 1 Every FCR theorem is FCR-valid.

This follows from the validity of the axioms and the validity-preservingness of the transformation rules.

3. Completeness in FCR

THEOREM 2 Every FCR-valid formula is a theorem.

We establish some preliminary results:

3.1 $\vdash (A \neq B) \supset (C \equiv D) \rightarrow \vdash (A \neq B) \supset (C = D)$.

Proof:

ex hypothesi, PC	(1)	$((A\neq B)\supset C) \equiv ((A\neq B)\supset D)$
(1) R	(2)	$((A\neq B)\supset C) = ((A\neq B)\supset D)$
NI	(3)	$(A\neq B) \supset ((A=B)=O)$
(3) I2, $(1 =_{df} (O\supset O))$	(4)	$(A\neq B) \supset ((A\neq B)=1)$
(4) I2	(5)	$(A\neq B) \supset (((A\neq B)\supset C)=(1\supset C))$

(2)(4) I2	(6)	$(A \neq B) \supset (((A \neq B) \supset C) = (1 \supset D))$
PC, R	(7)	$(1 \supset C) = C$
PC, R	(8)	$(1 \supset D) = D$
(5)(6)(7)(8)I2	(9)	$(A \neq B) \supset (C = D)$ QED

Clearly 3.1 can be extended to the case where the antecedent is a conjunction of statements of identity or their negations. For in FCR any true identity statement (whether $A=B$ or $A \neq B$) can be replaced by 1. Thus we have:

3.2 $\vdash (A_1. \ldots .A_n) \supset (B \equiv C) \rightarrow \vdash (A_1. \ldots .A_n) \supset (B=C)$, provided each A_i $(1 \leqslant i \leqslant n)$ is either of the form $D=D'$ or $D \neq D'$.

3.3 $(p)(A=B) \supset :(p)A = (p)B$ [2] (from I2)

3.4 $((\exists p)(A.B) \neq O) \supset (\exists p)((A.B) \neq O)$

Proof:

R, I2	(1)	$(\sim p = q) = (p = \sim q)$
(1) Def 1	(2)	$\sim(\sim(p) \sim (A.B) = O) \supset \sim((p) \sim (A.B) = 1)$
3.3	(3)	$(p)(\sim(A.B) = 1) \supset ((p) \sim (A.B) = 1)$
(3) PC (2) syll	(4)	$\sim(\sim(p) \sim (A.B) = O) \supset \sim(p)(\sim(A.B) = 1)$
PC, R	(5)	$(1 \supset O) = O$
(5)(1)(4)	(6)	$\sim(\sim(p) \sim (A.B) = O) \supset \sim(p)((A.B) = O)$
(6) Quantification	(7)	$((\exists p)(A.B) \neq O) \supset (\exists p)((A.B) \neq O)$ QED

3.5 $\vdash A \rightarrow \vdash (B \neq O) \supset ((A.B) \neq O)$.

(2) This formula looks very like the *Barcan* formula of quantified modal logic. I have in fact been able to deduce it from the weaker identity schema, I2′ $(A=B) \supset (C \supset D)$ with the same conditions as in I2, except that *no* variable free in A or B may be bound as a result of replacement). The proof is a simple adaptation of Prior's [10] of the Barcan formula in S5 and relies on NI. I suspect that without NI, 3.3 could not be deduced from I2′, but I have no proof that this is so.

Proof:

PC,R	(1)	$(1.B) = B$
(1) I2	(2)	$(B \neq O) \supset ((1.B) \neq O)$
ex hypothesi	(3)	A
(3) PC, R	(4)	$A = 1$
(2)(4) I2	(5)	$(B \neq O) \supset ((A.B) \neq O)$ QED

For completeness we shew how to construct a set of maximal consistent (²) sets of wffs of FCR $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_i, \ldots\}$ such that Γ_1 contains a given consistent formula H. This time however we have to be sure that the set of identity formulae, (i.e. formulae of the form $A = B$) is the same in each. This causes a slight complication in obtaining the requirement that, where $(\exists p)A \ \varepsilon \ \Gamma_i$, then there is some appropriate A′ also in Γ_i.
We introduce the notion of an *E-formula* (⁴) as follows:

3.61 $(\exists p)A \supset A$ is an E-form.

3.62 Where B is a formula not containing free p, then $(B \neq O) \supset (((\exists p)A \supset A).B) \neq O)$ is an E-form.

For both 3.61 and 3.62 p is said to be the *replacement variable* of the E-form. Where some formula for which p is free replaces every occurrence of free p in the E-form, we obtain an E-formula of that form. A set of formulae is said to have the *E-property* if it contains a formula of every E-form.
We prove the following lemmata:

(³) The method of maximal consistent sets (taken from [4]) is applied to functorial calculi in [1]. (We have relativized the domain P to each model and have proceeded without individual constants, relying on the assignment to give a value to the variables). The rule R complicates slightly the definition of a proof from hypotheses as used in [1] (p. 548) and taken from [4]. For present purposes it is simplest to define $A_1, \ldots, A_n \vdash B$ as $\vdash (A_1. \ldots .A_n) \supset B$. This enables the proofs to go through as before.

(⁴) This method has, on the semantical level, analogies with the subordinate maximal consistent sets of [6] and [7]. Since FCR has a semantics akin to that of S5, we are in act using the version of this method found in [6] rather than the more complicated one of [7].

3.63 If A is an E-form with p as its replacement variable, then $\vdash (\exists p)A$.

Proof by induction on the construction of E-forms:
Clearly $\vdash (\exists p)((\exists p)A \supset A)$.
Given $\vdash (\exists p)A$ we have, by 3.5 $\vdash (B \neq O) \supset (((\exists p)A.B) \neq O)$, hence (p not free in B) $\vdash (B \neq O) \supset ((\exists p)(A.B) \neq O)$, hence (by 3.4) $\vdash (B \neq O) \supset (\exists p)((A.B) \neq O)$, hence (p not free in B) $\vdash (\exists p)((B \neq O) \supset ((A.B) \neq O)$.

3.64 If $\wedge$ is a consistent set of wffs none of whose members contains free p, then, if A is an E-formula whose replacement variable is p, then A can be consistently added to $\wedge$. Suppose it could not, then $\wedge \vdash \sim A$, hence (p not free in any member of $\wedge$) $\wedge \vdash (p) \sim A$, hence $\wedge \vdash \sim (\exists p)A$. But by 3.63 $\vdash (\exists p)A$ and so $\wedge$ is inconsistent, contrary to hypothesis.

It is thus possible, beginning with H, to construct a maximal consistent set of wffs Γ_1 with the E-property and containing H. We shew how to construct a maximal consistent set Γ_i such that:

3.71 Every $A = B$ and $A \neq B$ in Γ_1 is in Γ_i;

3.72 For every wff A there is some $(\exists p)A \supset A'$ in Γ_i (for some appropriate A');

3.73 For some $C \neq D$ in Γ_1, $C \not\equiv D \; \varepsilon \; \Gamma_i$.

3.8 Construct Γ_i as follows:

3.81 Let Γ_{i1} be $C \not\equiv D$.

Clearly Γ_{i1} is consistent for, if not, $\vdash C \equiv D$ hence, (by R) $\vdash C = D$, but $C \neq D \; \varepsilon \; \Gamma_1$ and Γ_1 is consistent.

3.82 Given Γ_{in} take the n'th formula of the form $(\exists p)A \supset A$. Suppose that the members of Γ_{in} are $(C \not\equiv D), A_1, \ldots, A_{n-1}$. Now $((((C \not\equiv D)A_1. \ldots .A_{n-1}) \neq O) \supset \vdash ((C \not\equiv D).A_1 \ldots .A_{n-1}).(\exists p)A \supset A) \neq O)$ is an E-form. Hence there will be some E-formula of that form in Γ_1 (since Γ_1 has the E-property). Since the replacement variable will only occur in A, let A' be A with the formula which replaces the replacement in the E-formula in Γ_1 variable replacing the replacement variable in A. Let $\Gamma_{i(n+1)}$ be $\Gamma_{in}U\{(\exists p) A \supset A'\}$.

Γ_{in} (for each n) is consistent; for suppose not, then
$\vdash ((C \not\equiv D).A_1. \ldots .A_{n-1}.((\exists p)A \supset A')) \equiv O$, hence by R
$\vdash ((C \not\equiv D).A_1. \ldots .A_{n-1}.((\exists p)A \supset A')) = O$, contrary to the consistency of Γ_1.

3.84 Let Γ_i' be the union of all the Γ_{in}'s.

Clearly Γ_i' satisfies 3.72 (for the n'th $(\exists p)A \supset A$ has an appropriate instance as the n'th member of Γ_i'). Further since $((C \not\equiv D) \neq O)\ \varepsilon\ \Gamma_1$, then for every conjunction $(B_1. \ldots .B_n)$ of members of Γ_i, $((B_1. \ldots .B_n) \neq O)\ \varepsilon\ \Gamma_1$.

3.85 Form Γ_i'' by adding every formula $A=B, A \neq B\ \varepsilon\ \Gamma_1$.

Γ_i is consistent for, suppose not, then for some finite subset where $X_1 = Y_1, \ldots, X_m = Y_m, X_{m+1} \neq Y_{m+1}, \ldots, X_n \neq Y_n$ are all in Γ_1 and $A_1, \ldots, A_k$ are all in Γ_i',

$\vdash \sim(X_1 = Y_1. \ldots .X_m = Y_m.X_{m+1} \neq Y_{m+1}. \ldots .X_n \neq Y_n.A_1. \ldots .A_k.C \not\equiv D)$,

hence $\vdash (X_1 = Y_1. \ldots .x_m = Y_m.X_{m+1} \neq Y_{m+1}. \ldots X_n \neq Y_n)$
$\supset ((A_1. \ldots .A_k.C \not\equiv D) \equiv O)$
$\supset ((A_1. \ldots .A_k.C \not\equiv D) = O)$ (by 3.2).

But $(A_1. \ldots .A_k.C \not\equiv D) \neq O\ \varepsilon\ \Gamma_1$, hence Γ_1 is inconsistent, contrary to hypothesis.

3.86 Increase Γ_i'' to a maximal consistent set of Wffs Γ_i. Γ_i is subordinate to Γ_1. Construct such a subordinate for each $C \not\equiv D$ such that not $(C \neq D)\ \varepsilon\ \Gamma_1$.

3.9 We define an FCR model $\langle V\ W\ P \rangle$ which makes $V(H)\ \varepsilon\ T$ for some $T\ \varepsilon\ W$.

Make the assignment as follows:

3.91 Assume some suitable domain P.

3.92 For wff A let V(A) be some member of P as follows:

If there is some earlier B such that $A=B\ \varepsilon\ \Gamma_1$, then let $V(A) = V(B)$. Otherwise let V(A) be the first member of P not already assigned to a formula.

3.93 For every wff A let $V(A)\ \varepsilon\ T_i$ iff $A\ \varepsilon\ \Gamma_i$.

Let W be the set of all T_i's determined in this way.
We shew that $\langle V\ W\ P\rangle$ is an FCR model (for clearly $V(H)\ \varepsilon\ T_1$).

The proof proceeds by taking each condition in turn. The conditions for quantification and the truth functors may be proved for each T by an argument parallel to the one used on p. 551 of [1]. For identity we note first that in Γ_1, $V(A) = V(B)$ iff $A=B\ \varepsilon\ \Gamma_1$ (by the method of assignment). But by construction the set of identity formulae is the same in every Γ_i. Hence, for every $T\ \varepsilon\ W$, $V(A=B)\ \varepsilon\ T$ iff $V(A) = V(B)$ thus satisfying 2.3. From I2 we may be sure that in any Γ_i any formulae C and D differing only in replacement of A by B, $C=D\ \varepsilon\ \Gamma_i$, and hence $V(C) = V(D)$. For 2.9 suppose $p_i \neq p_j$ where $p_i = V(A)$ and $p_j = V(B)$, then $A=B \notin \Gamma_1$, hence $A \neq B\ \varepsilon\ \Gamma_1$, hence for some Γ_h $A \not\equiv B\ \varepsilon\ \Gamma_h$, hence one but not both of $V(A)$, $V(B)\ \varepsilon\ T_h$ thus satisfying 2.9.

Hence FCR is complete.

4. FCR and Modal Logic

We mentioned the connection between modal logic and functorial calculi. In FCR we have the rule R ($\vdash A \equiv B \rightarrow\ \vdash A=B$) and thus identity amounts to provable equivalence. This being so we can introduce the necessity operator L as $LA =_{df} A = (O \supset O)$ and obtain systems of modal functorial logic mirroring S5. If we do not have the rule R, we cannot define L analogously (we could not without R prove, e.g. $\vdash A \rightarrow\ \vdash LA$). But one might wish to add modal operators to functorial systems which do not have the rule R.

FCS5 is the system obtained by adding to FC (with or without identity) the constant monadic functor L and the following schemata:

LA1	$LA \supset A$,
LA2	$L(A \supset B) \supset (LA \supset LB)$,
LA3	$\sim LA \supset L \sim LA$,
LR1	$\vdash A \rightarrow\ \vdash LA$.

Note that we can no longer prove R, though we can prove $\vdash A \equiv B \rightarrow \vdash L(A \equiv B)$ and, if we define strict equivalence as necessary material equivalence, we must not confuse identity of propositions with strict equivalence. We cannot prove in FCS5 $L(A \equiv B) \supset (C \supset D)$ where C and D are as in the identity schema. This fails where A and B occur within the scope of a functorial variable.

For a semantics for FCS5 we take the semantics for FCR, but without 2.9, and make the following addition:

4.1 V(L) is a descriptive set of pairs of P, $\langle p_i p_j \rangle$ such that for any $T \varepsilon W$, $p_i \varepsilon T$ iff for every $T' \varepsilon W, p_j \varepsilon T'$.

The completeness of FCS5 can be proved in a manner similar to that of FCR. The difference is that Γ_i is constructed to have, not the same set of identity formulae that Γ_1 has, but the same set of fully modalized formulae and the initial member of each Γ_i is some $\sim A$ for which $\sim LA \ \varepsilon \ \Gamma_1$. For this reason, and since completeness proofs for modal systems have been developed elswhere, we shall not go further into the completeness of FCS5.

The connection between this kind of semantics and the semantics of [2] can very easily be exhibited. For corresponding to each of our T-sets there will be a possible world, and a proposition will be true in that world iff it is in the T-set. It would in fact be easy to shew how, given an FCR model, one could construct a more orthodox kind of S5 model to verify the same formula, and *vice versa*.

From the results of [3] it would seem that the omission of NI from the basis would give a system corresponding to S4 (5) in the way FCR corresponds to S5, but since it is difficult to see what kind of a semantics for identity would verify R and I2 but fail to verify NI, we shall not investigate such a system.

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(5) The independence of NI may be proved by appropriately adapting the Group II matrix of [9] p. 493.

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