A CORRECTION TO MACKIE'S NATURAL DEDUCTION

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In two papers (1), Professor Mackie has set out a justification and explanation of the rules of natural deduction. He has proposed a new symbolism, which uses three different styles for variables, in which natural deduction can be carried out in a somewhat intuitive fashion. There is, however, a defect in the system, which may be illustrated by the following derivation:

1.	$(\exists x)Fax$	Spec.
2.	Fab	1, EI
3.	$(\exists x)Fkx . \supset . Fkb$	1 — 2, CP (arb.)
4.	$(y)((\exists x)Fyx . \supset . Fyb)$	3, UG.
5.	$(\exists z)(y)((\exists x)Fyx . \supset . Fyz)$	4, EG.

Now line 5, although freed from all assumptions, is not a logical truth. We can apply the usual principles of quantifier distribution to it, and obtain as an implication of it

$$(\exists z)((y)(\exists x)Fyx . \supset . (y)Fyz)$$

and then, since 'z' is not free in '$(y)(\exists x)Fyx$',

$$(y)(\exists x)Fyx . \supset . (\exists z)(y)Fyz,$$

which, on rewriting 'x' for 'z' in the consequent, becomes

$$(y)(\exists x)Fyx . \supset . (\exists x)(y)Fyx.$$

Thus we have effectively reversed the quantifiers in the fashion which Mackie's subscripting device in his rule EI (arb) was designed to prevent.

There are three ways in which we may correct the system so as to make the given derivation invalid. We will call these the first, second

(1) 'The Rules of Natural Deduction', *Analysis* 19 (1958) and 'The Symbolizing of Natural Deduction', *Analysis* 20 (1959).

and third 'solutions'. The first solution is to invalidate line 2, by adding to the rule EI (arb.) an additional clause referring to instantiation in the presence of an individual variable introduced in a specimen assumption. We add clauses such as

$$(\exists x)\varphi(a',x),\ \{a^*,b^*,c^*,d,e,f\};\ \ldots\ \varphi(a',d_{a'})$$

where a' is any variable free in a specimen assumption.

Then since 'a' is free in line 1, line 2 must be changed to

2′. Fab_a.

The second solution, which has the same ultimate effect as the first, is to invalidate line 3, by modifying the rule CP (arb.), so that it reads

$$\{a^*,b^*,c^*,d,e,f\}\ \Gamma\ \ \varphi d \text{ spec} \rightarrow \psi(d,a');$$
$$\therefore \varphi k\,.\supset.\,\psi(k,a'_k),$$

where a' is any variable introduced by EI within the scope of the specimen assumption (2).

Then, since 'b' is introduced by EI within the scope of the specimen assumption, line 3 must be changed to

3′. $(\exists x)Fkx\,.\supset.\,Fkb_k$.

Line 3′ would also be the consequence of 1 — 2′ under the first solution. From 3′, line 4 must be

4′. $(y)((\exists x)Fyx\,.\supset.\,Fyb_y)$,

which is harmless and cannot have EG applied to it.

The disadvantage of the first and second solutions lies in the underlined provisos. For no longer is a derivation "valid from line to line", in Mackie's phrase; we must now look back in a derivation to check the mode of introduction of a-type variables. The first solution is the better of the two, in that it requires only that specimen assumptions be checked; conversely the second solution is better than the first in that the checking is only required when cashing specimen assumptions. But neither solution is in the spirit of Mackie's system, since they re-introduce a feature of such systems as Copi's and Fitch's which Mackie was concerned to eliminate.

(2) This proviso is slightly stronger than need be, but more mechanism in the syntactical metalanguage is needed to state a weaker form.

The third solution is to introduce a fourth style of variable, using letters of the alphabet such as r, s, t... In effect, as in Mackie's system, the style of variable will provide an inbuilt memory of their mode of introduction into a derivation. We set up the rules so that r-type variables are like a-type variables for the purposes of U.G., like k-type variables for the purposes of E.I., and play their own special role for the purposes of C.P. (arb.). The full set of rules now reads

U.I.:	$(x)\varphi x;\ \therefore\ \varphi a$
	$\varphi k;\ \therefore\ \varphi a$
U.I. (arb.):	$(x)\varphi x;\ \therefore\ \varphi k$
U.I. (spec.):	$(x)\varphi x;\ \therefore\ \varphi r$
U.G.:	$\varphi k;\ \therefore\ (x)\varphi x$
E.G.:	$\varphi a;\ \therefore\ (\exists x)\varphi x$
	$\varphi k;\ \therefore\ (\exists x)\varphi x$
	$\varphi r;\ \therefore\ (\exists x)\varphi x$
E.G. (arb.):	$\varphi(k,a_k);\ \therefore\ (\exists x)\varphi(k,x)$
E.G. (spec.):	$\varphi(r,a_r);\ \therefore\ (\exists x)\varphi(r,x)$
E.I.:	$(\exists x)\varphi x,\ \{a^*,b^*,c^*,d,e,f\};\ \therefore\ \varphi d$
E.I. (arb.):	$(\exists x)\varphi(k,x),\ \{a^*,b^*,c^*,d,e,f\};\ \therefore\ \varphi(k,d_k)$
	$(\exists x)(k,l,x),\ \{a^*,b^*,c^*,d,e,f\};\ \therefore\ \varphi(k,l,d_{kl})$
E.I. (spec.):	$(\exists x)\varphi(r,x),\ \{a^*,b^*,c^*,d,e,f\};\ \therefore\ \varphi(r,d_r)$
	$(\exists x)\varphi(r,s,x),\ \{a^*,b^*,c^*,d,e,f\};\ \therefore\ \varphi(r,s,d_{rs})$ &c.
C.P.	$\ulcorner\ A \rightarrow B;\ \therefore\ A \supset B,$

where A does not contain k
C.P. (arb.):

$$\ulcorner\ \varphi r \rightarrow \psi r;\ \therefore\ \varphi k \supset \psi k$$
$$\ulcorner\ \varphi r \rightarrow B;\ \therefore\ \varphi k \supset B.$$

As in Mackie's original rules, we must read EI with an "applicable by default" proviso; since $(\exists x)\varphi(k,x)$ and $(\exists x)\varphi(r,x)$ are cases of $(\exists x)\varphi x$, we require a decision to be made between applying EI (arb.) or EI (spec.) or plain E.I. We give E.I. the lowest priority; it is applicable only if neither of E.I. (arb.) or E.I. (spec.) is applicable.

In the rule C.P. (arb.) we need no clause such as

$$\{r^*,s,t\},\ \ulcorner\ \varphi s \rightarrow \psi s;\ \therefore\ \varphi k \supset \psi k,$$

i.e. we may use a letter in a specimen assumption which has previously been used. This is so since if the new assumption is outside the scope of any others, all others will have been cashed in favour of a k-type variable, and if the new assumption is inside the scope of another, then it amounts to just one assumption of the conjunction of the two. As for k-type variables, r-type variables may not be used in the symbolization of the premisses of an argument, nor may they be introduced (except as subscripts) by any form of E.I.

Under the amended rules, the original derivation will become

1″.	$\lceil(\exists x)Frx$	Spec.
2″.	$\lfloor Fra_r$	1″, E.I. (spec.)
3″.	$(\exists x)Fkx . \supset . Fka_k$	C.P. (arb.)

Line 3″, like 3′, may have U.G. applied to it, but the result may not have E.G. applied to it, and the invalid line 5 thus may not be derived, as in the first and second solutions.

The rules have been designed so that all valid arguments using a-type variables in specimen assumptions will still be valid when the a-type variables are replaced by r-type variables, so that nothing which is wanted has been lost from Mackie's system. There is now a longer list of rules, although C.P. (arb.) has been simplified, but Mackie's requirements on elegance continue to be satisfied. As in Mackie's original system, a certain amount of informal explanation is needed in order to make clear the status of such expressions as '$\exists$'(x)φx' in the rules, but short of a full-blown syntactical metalanguage this is unavoidable.

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