SOME FURTHER SEMANTICS FOR DEONTIC LOGIC

M. J. CRESSWELL

1. THE SYSTEM OL1

In [1] p. 95 Saul Kripke mentions a way of studying semantically systems of deontic logic. This suggestion is developed in [3] for some systems like those of [2] and completeness is proved for a number of deontic systems.

It is the purpose of this paper to continue in this spirit and to consider the plausibility of various semantics for deontic logics. For it could be that proceeding from these to the axiomatic systems might be more productive of insight than the reverse direction. Our semantic apparatus (in the terminology of [4]) will be a model consisting of a set W of worlds $\{x_1, ..., x_i, ...\}$ and an assignment V from formulae to the truth values $\{1,0\}$ and satisfying the following: (for $\sim$ and v as primitive truth-functors with the rest introduced by the usual definitions)

V1. For propositional variable p and $x_i \varepsilon W$, $V(p\ x_i) = 1$ or 0;

V2. For wff α and $x_i \varepsilon W$, $V(\sim\alpha x_i) = 1$ iff $V(\alpha x_i) = 0$; otherwise 0;

V3. For wffs α and β, $V((\alpha \vee \beta)x_i) = 1$ iff $V(\alpha\ x_i) = 1$ or $V(\beta\ x_i) = 1$, otherwise 0.

We shall also want to introduce the necessity operator L and for simplicity we shall assume our modal system to be S5 though some of the results we shall obtain can be extended to other systems. We thus have,

V4. For wff α and $x_i \varepsilon W$, $V(L\alpha\ x_i) = 1$ iff $V(\alpha\ x_j) = 1$ for every $x_j \varepsilon W$.

So far we have not introduced any deontic notions. A basic semantical idea is that some worlds are "Good" (or permitted, or pleasant or...) and others are not. This follows [3] (p. 180) except that unlike Hanson we do not, for reasons which will appear, require all words but the real one to be good. We define the obligatory as

that which has to be the case if the world is to be good (1). Where the operator O means that some (unspecified but the same throughout the formula) person ought the bring it about that — we say that Op is true in any world iff p is true in all good worlds. (If p is not brought about, i.e. is false, then the world will be bad.)
Thus where $G \subseteq W$ (G is the set of all good worlds) we have,

V5. For wff α and $x_i \varepsilon W$, $V(O\alpha\, x_i) = 1$ iff for every $x_j \varepsilon G$ $V(\alpha\, x_j) = 1$, otherwise 0.
Def P : $P\alpha =_{df} \sim O \sim \alpha$.

A model $\langle V\ W\ G\rangle$ which satisfies V1-V5 we shall call an OL1 model. Purely deontic formulae true in all such models are just those which are theorems of Smiley's OS5+ ([6] p. 129) and Hanson's DS5 ([3], p. 178). To shew this we observe that our semantics delineates the same class of valid formulae as are theorems of Anderson's "simplification" of deontic logic by the introduction of the "sanction" (2). We follow Prior ([9] p. 138) in having a propositional constant E to mean "blame is escaped" or under our semantics, "the world is good". Clearly if we let E be true in all good worlds and only good worlds then Op will be true iff $E \prec p$ is true.

From the results of [6] the purely deontic part of OL1 (Smiley's OS5+ and Hanson's DS5) can be axiomatized by:

pC. If α is a PC tautology then $\vdash \alpha$;

O1. $O(p \supset q) \supset (Op \supset Oq)$;

O2. $O(Op \supset p)$;

O3. $\sim Op \supset O \sim Op$;

R1. Uniform substitution for propositional variables (3);

(1) This puts us squarely in the camp of those who regard deontic operators as having propositional arguments. We do not here intend to discuss this (no doubt important) topic (v.e.g. [12] pp. 43, 44). A defence of this way of approaching deontic logic is given in [5]. Familiarity with these systems is assumed. For an introduction v. [9]. Note that our use of G as the set of good worlds should not be confused with its use as the designated 'real' world in the models of [1] and [3].

(2) v. [7] (which I have not seen) and [8].

(3) In all the axiomatizations that we consider we shall assume a rule of uniform substitution for propositional variables. Without such a rule the axioms

R2. Modus Ponens $\vdash \alpha, \vdash \alpha \supset \beta \rightarrow \vdash \beta$;

R3. $\vdash \alpha \rightarrow \vdash O\,\alpha$.

(From R3 and O1 we can easily derive $\vdash \alpha \equiv \beta \rightarrow \vdash O\alpha \equiv O\beta$, and so have a rule for substitution of proved equivalents).

This system is the same as Hanson's DS5 ([3] p. 178) though his semantics only has one world which is not good. Where x_1 is the designated "real" world, (Hanson's "G") we have (in our terminology) $V(O\alpha\, x_1) = 1$ iff for every $x_j \neq x_1$ $V(\alpha\, x_j) = 1$. While this does not matter for purely deontic formulae, when modal operators are introduced we can verify $(Op\,.\,p) \supset Lp$, for if $V(Op\, x_1) = 1$ and $V(p\, x_1) = 1$ then for every $x_j \neq x_1$ and for $x_j = x_1$, $V(p\, x_j) = 1$, hence $V(Lp\, x_1) = 1$.

Smiley ([6] p. 131) mentions the possibility of axiomatizing the combined modal and deontic systems which result from the Andersonian simplification. We shall later prove that the following system is complete with respect to OL1 models:

OL1.S5 The axioms and rules for S5 (e.e.g.[10] pp. 31-32),

OL1.OS5. The axioms and rules for OS5 (without R3 and O3),

OL1.1 $Lp \supset Op$;

OL1.2. $Op \supset LOp$.

(The proof of R3 and O3 from this should be clear)

We shall shew the validity of OL1.1 and OL1.2 (For the remainder are clearly valid and the rules validity- preserving form the results of [6]).

We shew that when $O\alpha$ is replaced by $E \dashv \alpha$ then the formulae are valid in S5. OL1.1 is $Lp \supset (E \dashv p)$ and OL1.2 is $(E \dashv p) \supset L(E \dashv p)$. Both theses are valid in S5 and so every theorem is valid.

2. The system OL2 and OL3

One of the possibly undesirable features of OL1 is that because the obligatory is that which is true in all good worlds then the logically true, that which is true in all worlds, is obligatory. And

would need to be replaced by schemata. Completeness proofs for the systems and decision procedures are given *infra.*

$Lp \supset Op$ is one of the axioms. Reasons can be given for this and certainly its contraposed form $Pp \supset Mp$ has a respectable ancestry (4). However it could be argued that this semantics takes no account of the fact that necessary propositions will be true in the bad worlds also (where the bad worlds are those not in G). Of course many ways of fulfilling our obligations will not ensure that the world is good (they may be true in bad worlds as well) though failing to fulfil any one of them will ensure that the world is bad (for there are no good worlds in which they are false).

To ensure that the world will be good we have to fulfil *all* our obligations, (and possibly do more besides). We could thus have a supererogatory operator S such that Sp is true when and only when p is false in every bad world. (i.e. if Sp is true then any world in which p is the case must be a good one). We have:

V.6 $V(S\, \alpha\, x_i) = 1$ iff $V(\alpha\, x_j) = 0$ for every $x_j \varepsilon$ W-G. (W-G is the set of worlds in W but not in G).

With the "escaping" constant E, $S\, \alpha =_{df} \alpha \dashv E$.

By interpreting good worlds as bad and bad worlds as good $S\alpha$ becomes true when α is false in every good world, i.e. when α is forbidden, and thus formally this system is equivalent to OS5 with F and S interchanged and so we shall not investigate it futher.

A combination of these systems arises by defining obligation as that which, if done, ensures that the world is good and if not done ensures it is bad. I.e. we have:

V7. $V(O\, \alpha\, x_i) = 1$ iff $V(\alpha\, x_j) = 1$ for every $x_j \varepsilon G$ and 0 for every $x_j \varepsilon$ W-G, otherwise 0.

This will make Op true when p is the agent's total obligation for it represents *precisely* what he must do to be sure that things are good. We call this system OL2. As with OL1 it is $S5^+$;

OL2.1. $Op \supset LOp$;

OL2.2. $L(p \equiv q) \supset (Op \supset Oq)$;

OL2.3. $(Op . Oq) \supset (p \equiv q)$.

(From OL2.1 and OL2.3 we have $(Op . Oq) \supset L(p \equiv q)$.)

When $O\alpha$ is replaced by $E = \alpha$ and good worlds are just those in which E is true each axiom may be seen to be valid. Aside from the intended deontic interpretation OL2 is interesting as representing the

(4) But v.e.g. [12] pp. 47, 48 for some discussion of this.

formal properties of operators which are true only when their argument has a determined pattern of values.

OL2 has the possibly unwelcome consequence that there are no logically distinct obligatory propositions. This comes from requiring all worlds to be either good or bad. If we have two subsets G and B of W we could have worlds which are neither. $\langle V\ W\ B\ G \rangle$ is an OL3 model iff W is a set of worlds, $B \subset W$, $G \subset W$ and $B \cap G = 0$. V must satisfy V1-V4 and:

V8. For wff α and $x_i \varepsilon W$, $V(O\ \alpha\ x_i) = 1$ iff $V(\alpha\ x_j) = 1$ for every $x_j \varepsilon G$ and 0 for every $x_j \varepsilon B$, otherwise 0.

If S (Anderson's "Sanction") is true in all and only bad worlds and E is true in all and only good worlds (where we do not have $E = \sim S$) then $O\ \alpha =_{df} (E \prec \alpha) \vee (\alpha \prec \sim S)$

The axioms for OL3 are as follows, *S5* +

OL3.1. $Op \supset LOp$;
OL3.2. $L(p \equiv q) \supset (Op \supset Oq)$;
OL3.3. $(Op . Oq) \supset (O(p . q) . O(pvq))$;
OL3.4. $(O(p . q . r) . Op) \supset O(p . q)$.

The validity of these may be seen by using the definition given above.

Obviously every OL2 model is an OL3 model (in which B = W-G). Further OL3.3 and OL3.4 may be easily derived in OL2 since $(Op . Oq) \supset L(p \equiv q)$ and $L(p \equiv q) \supset L(p \equiv (p . q))$ and $L(p \equiv q) \supset L(p \equiv (pvq))$, thence by OL2.2 to get OL3.3. The proof of OL3.4 is similar. Further, as corollaries of OL3.3 we have,

$$(Op_1 . \ldots . Op_n) \supset O(p_1 . \ldots . p_n)$$
and $$(Op_1 . \ldots . Op_n) \supset O(p_1 v \ldots v p_n),$$

and as a corollary of OL3.4 we have,

$$(O(p_1 . \ldots . p_n) . O(p_1 v \ldots v p_n)) \supset (Op_1 . \ldots . Op_n).$$

3. The Completeness of OL1-OL3

When the modal system is S5 and we have $Op \supset LOp$ (thus giving $Op = LOp$) and $(p = q) \supset (Op \supset Oq)$ (= is strict equivalence) we can use the (S5) theorems $Lp = (Lp = (p \supset p))$ and $\sim Lp = (Lp = \sim (p \supset p))$. We have, where α is formula containing $L\beta(O\beta)$

within the scope of a modal or deontic operator and γ is α with $(p \supset p)$ replacing each occurrence of $L\beta(O\beta)$ and γ' is α with $\sim(p \supset p)$ replacing each occurence then

$$\vdash \alpha = ((L\beta . \gamma) \vee (\sim L\beta . \gamma'))$$

(or $\vdash \alpha = ((O\beta . \gamma) \vee (\sim O\beta . \gamma')).)$

By repeating this process we will eventually obtain a first-degree formula (5). Once reduced to first-degree any formula may be expressed as a conjunction of disjunctions of the form; $\alpha_1 v L\alpha_2 v \ldots v L\alpha_n v O\beta_1 v \ldots v O\beta_m v P\gamma_1 v \ldots v P\gamma_k v M\delta$ (we only need one δ from the distributivity of M over v. In OL1 we have it for P also and so here only need one γ .)

The whole formula will clearly be valid iff each disjunction is. The disjunctions will be valid (in the appropriate system) if one of the following is;

in OL1, some $\alpha_i v\delta$ $(1 \leq i \leq n)$ or some $\beta_1 v\gamma v\delta$ $(1 \leq i \leq m)$ (remember that in OL1 all the γ_i's may be collected together)

in OL2, some $\alpha_i v\gamma_1 v \ldots v\gamma_k v\delta$ or $\alpha_i v \sim\gamma_1 v \ldots v \sim\gamma_k v\delta$ or $\beta_i v\gamma_1 v \ldots v\gamma_k v\delta$, or $\sim\beta_i v \sim\gamma_1 v \ldots v \sim\gamma_k v\delta$.

In OL3 some $\alpha_i v\delta$, $\beta_i v\gamma_1 v \ldots v\gamma_k v\delta$ or $\sim\beta_i v \sim\gamma_1 v \ldots v \sim\gamma_k v\delta$.

To shew this we prove the following rules:

DR1. (OL1 and OL3). $\vdash \alpha v\beta \rightarrow \vdash L\,\alpha v M\,\beta$.

This is a known result of S5.

DR2. (OL1). $\vdash \alpha v\beta v\gamma \rightarrow \vdash O\alpha v P\beta v M\gamma$.

Proof:

[3] p. 185, T8	(1) $(P \sim p . O \sim q) \supset P \sim (pvq)$
(1) Def P, PC	(2) $\sim (OpvPq) \supset \sim O(pvq)$
(2) contraposition	(3) $O(pvq) \supset (OpvPq)$
(3) S5	(4) $\sim Mr \supset ((pvq) = (pvqvr))$
(3) (4) Subs eq.	(5) $\sim Mr \supset (O(pvqvr) \supset (OpvPq))$

(5) For this purpose deontic operators are treated as modal operators. The definition of first-degree is that of [13] p. 144. Reduction to first-degree (as given in say [11]) is not normally carried out in this way but the present way seems to adapt more easily to a statement of the conditions under which reducibility obtains. The reduction of purely deontic formulae in $0S5^+$ to a normal form of the kind given is used in [6] p. 125.

(5) PC	(6) $O(pvqvr) \supset (OpvPqvMr)$
ex hypothesis, R3	(7) $O(\alpha v\beta v\gamma)$
(6) (7) MP	(8) $O\alpha vP\beta vM\gamma$ QED

DR3 (OL2). $\vdash (\alpha v\beta_1 v...v\beta_k v\gamma) . (\alpha v \sim \beta_1 v...v \sim \beta_k v\gamma)$
$\rightarrow \vdash L\alpha vP\beta_1 v...vP\beta_k vM\gamma$

Proof:

OL2.3	(1) $(O \sim \beta_1 . ... . O \sim \beta_k) \supset (\beta_1 = \beta_1)...(\beta_1 = \beta_k)$
ex hyp., necessitation	(2) $L((\alpha v\beta_1 v...v\beta_k v\gamma).(\alpha v \sim \beta_1 v...v \sim \beta_k v\gamma))$
(1) (2) S5	(3) $(O \sim \beta_1 ... O \sim \beta_k) \supset L((\alpha v\beta_1 v\gamma).(\alpha v \sim \beta_1 v\gamma))$
(3) S5	(4) $(O \sim \beta_1 ..... O \sim \beta_k) \supset (L\alpha vM\gamma)$
(4) PC, Def P	(5) $L\alpha vP\beta_1 v...vP\beta_k vM\gamma$ QED

DR4 (OL2 and OL3). $\vdash (\alpha v\beta_1 v...v\beta_k v\gamma).(\sim \alpha v \sim \beta_1 v...v \sim \beta_k v\gamma)$

$\rightarrow \vdash O\alpha vP\beta_1 v...vP\beta_k vM\gamma$

Proof;

ex hyp., necessitation	(1) $L((\alpha v\beta_1 v...v\beta_k v\gamma).(\sim \alpha v \sim \beta_1 v...v \sim \beta_k v\gamma))$
(1) S5	(2) $\sim M\gamma \supset ((\sim \beta_1 ... \sim \beta_k) = (\alpha . \sim \beta_1 .. \sim \beta_k))$
(1) S5	(3) $\sim M\gamma \supset ((\sim \beta_1 v...v \sim \beta_k) = (\alpha v \sim \beta_1 v .. v \sim \beta_k))$
OL3.3, (4) Subs eq.	(4) $\sim M\gamma \supset ((O \sim \beta_1 ... O \sim \beta_k) \supset ((O(\alpha . \sim \beta_1 ... \sim \beta_k) . O(\alpha v \sim \beta_1 v...v \sim \beta_k)))$
(4) (5) Syll	(6) $(\sim M\gamma . O \sim \beta_1 ... O \sim \beta_k) \supset O\alpha$
(6) PC, Def P	(7) $O\alpha vP\beta_1 v ... vP\beta_k vM\gamma$ QED.

Hence if one of the conditions holds the whole disjunction is provable in the appropriate system and hence is valid. (Since α_1 — δ are all PC formulae they will be theorems iff they are valid.) If none of the conditions hold we may construct a falsifying model. For OL1 given a disjunction:

$\alpha_1 vL\alpha_2 v...vL\alpha_n vO\beta_1 v...vO\beta_n vP\gamma vM\delta$

let $W = \{x_1, ..., x_{m+n}\}$, $G = \{x_{n+1}, ..., x_{m+n}\}$.

Since no $\alpha_1 v\delta$ is valid and no $\beta_i v\gamma v\delta$ is valid then we can find a PC assignment which falsifies each. Let $V(p\ x_i) = 1$ or 0 (for propositional variable p) according as the PC assignment falsifying $\alpha_i v\delta$ gives it 1 or 0 and let $V(p\ x_{n+i}) = 1$ or 0 according as the PC assignment which falsifies $\beta_i v\gamma v\delta$ gives it 1 or 0. Thus each α_i will be false in some world (hence $V(L\alpha_i x_1) = 0$) and each β_i will be false in some good world (hence $V(O\beta_i x_1) = 0$) and δ will be false in every world (hence $V(M\delta\ x_1) = 0$) and γ will be false in every good world. (Hence $V(P\gamma\ x_1) = 0$) Thus the whole disjunction is falsified.

To ensure the non-emptiness of G we can always ensure that there are some $O\beta_i$'s since from OL1.1 and O4 we have $Mp \supset (OpvMp)$ and so can add an $O\delta$ to the disjunction.

For OL2 and OL3 we have to shew how to falsify a disjunction of the form

$\alpha_1 vL\alpha_2 v...vL\alpha_n vO\beta_1 v...vO\beta_m vP\gamma_1 v...vP\gamma_k vM\delta$.

Let $W = \{x_1, ..., x_{n+m}\}$.

Now (for each $1 \leq i \leq n$) one of

$(\alpha_i v\gamma_1 v...v\gamma_k v\delta)$ or $(\alpha_i v \sim\gamma_1 v...v \sim\gamma_k v\delta)$

is not valid (if they were both valid they would be theorems and here (by DR2) the whole disjunction would be a theorem). If the former is not valid then let $x_i \varepsilon G$. If $(\alpha_i v \sim\gamma_1 v...v \sim\gamma_k v\delta)$ is not valid then let not $(x_1 \varepsilon G)$. In either case let $V(px_i) = 1$ or 0 according as the falsifying assignment gives it 1 or 0.

Now $(\beta_i v\gamma_1 v...v\gamma_k v\delta)$ or $(\sim\beta_i v \sim\gamma_1 v...v \sim\gamma_k v\delta)$ are not both valid. (If they were the whole disjunction would be a theorem by DR4) If the former is not valid then let $x_{n+1} \varepsilon G$ If $(\sim\beta_i v \sim\gamma_1 v...v \sim\gamma_k v\delta)$ is not valid then let not $(x_{n+1} \varepsilon G)$. In either case $V(px_{n+1}) = 1$ or 0 according as the falsifying assignment gives it 1 or 0.

An argument similar to that used for OL1 will show that the whole disjunction is false in some OL2 model. For OL3 we let $x_1, ..., x_n$ be neither good nor bad while $x_{n+1}, ..., x_{n+m}$ are made good or bad by the method for OL2.

If B and G are required to be non-empty we would need to be sure that there is always an $O\beta$ such that one of $(\beta v\gamma_1 v...v\gamma_k v\delta)$ and $(\sim\beta v \sim\gamma_1 v...v \sim\gamma_k v\delta)$ is not valid. If $Op \supset Mp$ and $Op \supset M \sim p$ are both added as axioms then we may replace $M\delta$ by $(O\delta vO \sim \delta vM\delta)$.

These two axioms are valid on the assumption that G and B are not empty.

4. THE SYSTEM OL4

All the systems so far considered contain $Op \supset LOp$ as valid. The presence of this has sometimes been thought objectionable in a deontic logic (v.e.g. [13]). While this can be rationaliezd when Op is interpreted as saying that p is entailed by the agent's moral code ([6]), it might plausibly be claimed that what a person's duties are in one situation (i.e. state of affairs or world) may be different from what they are in another. If we want to incorporate this into an account which says that one's duties are what one must do if the world is to be good, we could say that it is only worlds which we have the power to bring about in which this can be so. I.e. our duties are what is true in all good worlds open to us. If we let xRy mean "The situation x can be brought about by the agent when in situation y", then $\langle VWRG \rangle$ is an OL4 model iff W is a set of worlds, R is reflexive over W, $G \subseteq W$ and V satisfies V1-V4 and:

V9. $V(O\alpha x_i) = 1$ iff for every $x_j R x_i$, such that $x_j \varepsilon G$, $V(\alpha x_j) = 1$, otherwise 0.

Given such a G and R we can always define an R′ such that xR′y iff xRy and $x \varepsilon G$ and so this semantics boils down to that for $OM^+(DM)$ whose axioms are those of $OS5^+$ without O3. The completeness of the deontic fragment of this is proved in [3] pp. 186-188. When modality is added we get the axiomatic OL4 by adding to OM^+:

OL4.1. $Lp \supset Op$.

Although formally this semantics is easier to set out when R means "x is a good world open to the agent in world y" there seem interpretational advantages in letting it mean simply "x is open to the agent in y" and having our subset G of W. This means that we can introduce a deontic necessity operator say Hp (p *has* to be brought about) and Cp (p *can* be brought about). This should not be confused with logical necessity which in our present modelling is truth in all worlds but formally they are clearly the necessity opera-

([6]) Where the content of this code is fixed in advance. For a discussion of the adequacy of such a defence v. [14].

tors of the system T and so would reflect the Anderson simplification when T is its basic logic. The valid $Hp \supset Op$ and its corollaries might then be more sensible interpretations of the Kantian "ought implies can".

This semantics enables a solution of the *robber's paradox* of [13] p. 294. This results from the theorem, $L(p \supset q) \supset (Fp \supset Fq)$, i.e. if p entails q then if q is forbidden ($Fp = O \sim p$) p is forbidden. But where p is "x helps y whom he has robbed" and q is "x robs y" then since x cannot help someone whom he has robbed without having robbed him, and since robbing y is forbidden (to x (7)) then helping y whom he has robbed is forbidden. In our present system what is forbidden to a person can vary depending on the circumstances. For when x is in the state of having robbed y he is powerless to bring about a world in which y is not robbed by him. Thus x is not forbidden to have robbed y (on the ground that he cannot now prevent it). He is of course still forbidden to *rob* y. Note that $L(p \supset q) \supset (Fp \supset Fq)$ is still a theorem but its interpretation becomes innocuous. All this becomes more plausible if we think of states of affairs temporally and speak about possible futures in the manner of [15] pp. 5-8 (8).

The imposition of transitivity would reflect the view that whatever it is in our power to put within our power is already within our power. Whether this is true or not will probably depend of subtle meanings of "within our power". E.g. it is within my power to put it into my power that I should understand Greek (for I could learn). There does seem a sense in which a knowledge is within my power but also a sense in which it is not (for I know no Greek). At any rate we have a means of giving formal expression to this. We shall not investigate systems in which R is transitive. They will be S4 counterparts to our S5 and T systems.

The imposition of reflexiveness ensures that the agent always has the power to do nothing (though the world may not be good if he does.)

(7) We are assuming that there is only one agent who is the subject of the moral judgement. For a discussion of the 'Good Samaritan' paradox which arises when other agents are involved v. [9] and [13].

(8) Indeed the combination of tense logic and deontic logic and its semantical study may well enable the formalization of many interesting ethical statements.

5. The Completeness of OL4

To prove the completeness of OL4 we shall use an adaptation of Henkin's [16] device of maximal consistent sets in [17]. This will prove simpler than e.g. the tableau method though it will not provide a decision procedure. The purely deontic fragment of OL4 is proved complete by the tableau method in [3]. We define a set Γ of formulae to be *consistent* iff it contains no finite subset $\{\alpha_1, ..., \alpha_n\}$ such that $\vdash \sim(\alpha_1 ... \alpha_n)$. Γ is maximal iff for every wff α either $\alpha \in \Gamma$ or $\sim\alpha \in \Gamma$.

We construct (following [17]) given a consistent formula $\sim\eta$ (i.e. given η is not a theorem) a series of maximal consistent sets which together define a model in which η is false. Let Γ_1 be a maximal consistent set of wffs containing $\sim\eta$. Given a maximal consistent set Γ_i construct a maximal consistent subordinate Γ_j as follows.

a) For every α such that $M\alpha \in \Gamma_i$ let Γ_j be a (non-good) set containing α and every β such that $L\beta \in \Gamma_i$. Suppose this were inconsistent. Then it would contain some $\{\beta_1, ..., \beta_n\}$ such that $\vdash \sim(\alpha . \beta_1 . ... . \beta_n)$ hence (from DR1) $\vdash \sim(M\alpha . L\beta_1 ... L\beta_n)$ contrary to the consistency of Γ_i. Then increase Γ_j to a maximal consistent set (Note that DR1 holds in OL4).

b) For every α such that $P\alpha \in \Gamma_i$ let α_j be a good set containing α and every β such that $O\beta \in \Gamma_i$ and every γ such that $L\gamma \in \Gamma_i$. The consistency of this follows from DR2 (provable in OL4). To ensure that there are some good sets we need to ensure that there is some $P\alpha$ in every Γ_i. This is so in virtue of $Lp \supset Pp$ (from OL4.1 and O4) since for any theorem α $L\alpha$ will be in Γ_j and so $P\alpha$ will. Every such Γ_j is called a subordinate of Γ_i.

We construct a model which verifies $\sim\eta$. Assume a set W of worlds and let each $x_i \in W$ be associated with a maximal consistent set Γ_i. Let $x_i \in G$ iff Γ_i is a good set. Let x_jRx_i iff Γ_j is Γ_i or is a subordinate of Γ_i. For variables let $V(p\, x_i) = 1$ or 0 according as $p \in \Gamma_i$.

Lemma. $(V(\alpha\, x_i) = 1$ iff $\alpha \in \Gamma_i$, otherwise 0.

Proof by induction on the construction of α.

By definition the lemma holds for propositional variables. From the maximal consistency of each Γ_i it clearly holds for truth functors. For L it is sufficient to shew that if $L\alpha \in \Gamma_i$ then $\alpha \in \Gamma_i$ and $L\alpha$ is in every subordinate of Γ_i and in the set of which Γ_i is a subordinate and if not ($L\alpha \varepsilon \Gamma$), then α is not in some subordinate (since all sets are connected by subordination this will shew from the induction hypothesis that the lemma holds for L). By S5 if $L\alpha \varepsilon \Gamma_i$ then $\alpha \varepsilon \Gamma_i$ and $LL\alpha \varepsilon \Gamma_i$ hence $L\alpha$ is in every subordinate. Suppose not ($L\alpha \varepsilon \Gamma_k$) where Γ_k is the set of which Γ_i is a subordinate, then $\sim L\alpha \varepsilon \Gamma_k$ (Γ_k maximal) hence $L \sim L\alpha \varepsilon \Gamma_k$ hence $\sim L\alpha \varepsilon \Gamma_i$ hence not ($L\alpha \varepsilon \Gamma_i$) contrary to reductio hypothesis. If not ($L\alpha \varepsilon \Gamma_i$) then $M \sim \alpha \varepsilon \Gamma_i$ hence for some subordinate Γ_j, $\sim \alpha \varepsilon \Gamma_j$. Hence the induction holds for L.

Suppose $O\alpha \varepsilon \Gamma_i$ then α will be in every good subordinate and if Γ_j is good then it will be the subordinate of some Γ_k and (because Γ_k contains $O(Op \supset p)$ which is a theorem) if Γ_i contains $O\alpha$ then it contains α. Hence (induction hypothesis) $V(\alpha\, x_j) = 1$ for every good $x_j R x_i$, hence $V(O\alpha\, x_i) = 1$. If $O\alpha \varepsilon \Gamma_i$ then $P \sim \alpha \varepsilon \Gamma_i$ hence $\sim \alpha$ is in some good subordinate Γ_j hence $V(\sim \alpha\, x_j) = 1$ for some good $x_j R x_i$. Hence $V(O\alpha\, x_i) = 0$. Hence the lemma holds and in particular $V(\sim \eta\, x_1) = 1$ hence $V(\eta\, x_1) = 0$. I.e. if η is not a theorem then it is not valid, hence if it is a theorem it is valid. QED.

If OL4 had been based on another modal system than S5 the proof would have been more complicated since the relation R of modal necessity might have been different from the R of deontic necessity. Where they are the same we have a semantics for the "H" system discussed above. Our completeness proof can be easily modified to cover the O/H/L system based on $OL4 + Lp \supset Hp$, $Hp \supset Op$, $Hp \supset p$, $H(p \supset q) \supset (Hp \supset Hq)$.

Systems analogous to OL2 and OL3 but with the relation R can be studied but they become very complicated.

Victoria University of Wellington M. J. CRESSWELL

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