MODELS FOR LOGICAL ENTAILMENT

D. Paul SNYDER

In several recent articles, Jaakko Hintikka describes a method employing «model sets» for the study of interpretations of formal systems in general, and systems of modal logic in particular (1). A model set is a set of formulas satisfying certain conditions, e.g. if Kp q belongs to a model set, then p belongs to it, and q does (2).

The approach is extended to modal logic by using certain sets of model sets called «model systems». A model system is a set of model sets on which an «alternativeness» relation is defined. Model sets which are «realizable» alternatives to a given set may be taken as descriptions of states of affairs which could have been realized instead of the one described by that set. The possibility of a statement is interpreted as the membership of that statement in a model set which is a realizable alternative to a fixed model set taken to be realized.

Thus a model set in Hintikka's sense may be taken to be a «world description». The fixed model set may be taken to contain only true statements or, if you like, statements which describe the real world. Alternative model sets to the set which describes the real world may be taken in the ancient sense to be descriptions of possible worlds.

In a forthcoming monograph, R.W. Binkley and R.L. Clark have developed a convenient proof-detecting procedure of the Gentzen type for the Feys-von Wright calculus M, and the Lewis

(1) Jaakko HINTIKKA «Modality and Quantification», *Theoria* (Lund) 27 (1961), pp. 119-128; «The Modes of Modality,» *Acta Philosophica Fennica*, Fasc. 16 (Helsinki 1963), pp. 66-81; «Form and Content in Quantification Theory,» *Ibid.*, Fasc. 8 (Helsinki 1955), pp. 11-63.

(2) A modified of Polish notation is used throughout. The operators are 'K' (conjunction), 'A' (alternation), 'C' (material conditional), 'L' (necessity), 'M' (possibility). Negation is represented in this notation by a bar ('—') written over the left-most sign of the negated formula. The negation of 'Apq,' for example, is written '$\bar{A}$pq'.

calculi *S4* and *S5*, plus quantification. This procedure, called «cancellation», exploits formally the semantic notion of «alternativeness» set out by Hintikka.

It is the purpose of the present paper to sketch out modifications of Hintikka's model system method which reflect certain features of logical entailment as conceived by von Wright, Anderson and Belnap, and others. Corresponding modifications in the Binkley-Clark cancellation systems will yield a manageable proof-detecting device for entailment as interpreted here.

Entailment. We take entailment to be a relation between statements which is independent of either the truth value or the modal status of the statements taken individually. The minimal requirement for the truth of «p logically entails q» is that p be relevant to q in some appropriate sense.

It should be pointed out that neither entailment thus understood nor modality in general need be taken as «intensional» in the Carnapian sense. We are not claiming that p entails q if and only if there is something about the «intension» or «meaning» of p — or some «concept» in p — that «includes» the «intension» or «meaning» of q or some «concept» in q. It is by now clear from recent work of Kripke, Hintikka, Myhill and others that semantics for standard systems of modal logic need not be done in terms of intensions, individual concepts, and such. The interpretation of formal systems of logical entailment need not in this sense be intensional either. The relevance that we require in dealing with logical entailment is formal relevance, in the sense that the conjunction of p and q is formally relevant to the alternation of p and q.

Formal conditions for assuring this relevance have been devised by others, and usually involve some restriction upon the provability of the material conditional. Von Wright, for example, offers the following definition of entailment: p entails q if, and only if, $C\,p\,q$ is demonstrable independently of demonstrating either the falsehood of p or the truth of q (3).

A criterion for entailment that meshes most naturally with the

(3) *cf.* G.H. von Wright, *Logical Studies*, London 1958, p. 182. («The Concept of Entailment»).

formal aspects of this paper is suggested by R.W. Binkley. Like those of von Wright and Anderson and Belnap, Binkley's test for the entailment of q by p requires that $C p q$ be provable under restricted conditions. The proof-detecting technique employed is a Gentzen-like reduction calculus (4).

(i) An axiom schema is a sequent of the form '$\alpha, \bar{\alpha}, \lambda$'.
(ii) A propositional formula is provable if and only if it can be reduced entirely to axiom schemata.
(iii) p entails q if and only if $C p q$ can be reduced to axiom schemata of the form '$\alpha, \bar{\alpha}, \lambda$' such that the element 'α' is derived from the antecedent of $C p q$ and the element '$\bar{\alpha}$' from the consequent.

We superscribe 'a' or 'c' to the results of each reduction step to indicate whether they are derived from the antecedent or from the consequent of the original formula. On this account $K \mathrm{p} \bar{\mathrm{p}}$ entails p:

(4) The reduction calculus used here is that of R.L. Clark and Paul Welsh *Introduction to Logic*, Princeton 1962, pp. 100ff, adapted to the present notation. In reduction we derive from a sequent containing a truth-functionally complex member a sequent or pair of sequents which has precisely the same truth conditions as the original, but which contains members that are less complex. The rules for truth-functional reduction are as follows ('α' and'β' represent well-formed formulae, 'X' and 'Y' represent any members of a sequent not involved in the application of a given rule):

$$\frac{X, A\alpha\beta, Y}{X, \alpha, \beta, Y} \qquad\qquad \frac{X, \bar{A}\alpha\beta, Y}{X, \bar{\alpha}, Y \qquad X, \bar{\beta}, Y}$$

$$\frac{X, K\alpha\beta, Y}{X, \alpha, Y \qquad X, \beta, Y} \qquad\qquad \frac{X, \bar{K}\alpha\beta, Y}{X, \bar{\alpha}, \bar{\beta}, Y}$$

$$\frac{X, C\alpha\beta, Y}{X, \bar{\alpha}, \beta, Y} \qquad\qquad \frac{X, \bar{C}\alpha\beta, Y}{X, \alpha, Y \qquad X, \bar{\beta}, Y}$$

$$\frac{X, \bar{\bar{\alpha}}, Y}{X, \alpha, Y}$$

The reductions may be viewed in the following way: the sequent above the line has a true member if and only if the sequents below the line have a true member.

$$\begin{array}{c} CKp\bar{p}p \\ \hline \bar{K}p\bar{p}^{a}, p^{a}, p^{c} \\ \hline \bar{p}^{a}, p^{a}, p^{c} \end{array}$$

The sequent containing '$\bar{p}^{a}$' and 'p^{c}' is an axiom schema fulfilling condition (iii). A similar reduction would show that $Kp\bar{p}$ entails p, but $Kp\bar{p}$ does not on this account entail every formula. The formula $CKp\bar{p}q$ reduces to the sequent '$\bar{p}^{a}, p^{a}, q^{c}$' which, although an axiom schema, does not meet condition (iii).

But this means of determining whether or not we have an entailment involves a test that can not be carried out wholly in the object language. Our aim here is to seek an interpretation for an object language which includes the entailment relation; moreover, one in which provable entailments meet Binkleys' test.

Entailment as a modal operator. The relational character of entailment as here understood can not be emphasized too strongly. It is because entailment is a relation demanding that the entailing statement be relevant to the entailed that we must reject material and strict implication as giving adequate formal renderings of entailment. Conceived as a modal relation, entailment might be characterized as «necessitation» to underscore the difference between entailment and strict implication, thus: «*p* necessitates *q*» as distinct from «It is necessary that if *p* then *q*». This reflects the *dyadic* modal character of entailment, which may be contrasted with the familiar *monadic* modal notions of the Lewis calculi. Monadic modal operators attach to single (simple or complex) statements, forming single modal statements. Dyadic modal operators attach to pairs of statements, forming single modal statements which express modal relations between their components ([5]).

Here I think we must make proper obeisance the use-mention distinction. It is often pointed out that when we say «*p* materially implies *q*» we are mentioning *p* and *q*, whereas when we say «If *p*, then *q*» we are using *p* and *q*. It is further pointed out by those who are

([5]) *cf.* VON WRIGHT, «A New System of Modal Logic,» in *Logical Studies* careful that the latter, not the former, is an appropriate reading of the formula *C p q*.

The convenient locutions «*p* entails *q*» and «*p* necessitates *q*» mention *p* and *q* without using them. To be perfectly correct in reading entailment formulae, we should express these relations in such a way that the constituent formulae are used. This might be done by withholding the «if-then-» reading from material implication formulae, as has been suggested (6), and saving it for the stronger formal relation. I think, however, that it would be less misleading to read entailment formulae in the subjunctive mood; it is, after all, such subjunctive structures as counterfactuals and theoretical conditionals that have motivated much of the work in modal relations. If we mention two statements when we say that «Today is Friday» entails «Tomorrow is Saturday,» we may use them when we say «If today should be Friday than tomorrow would be Saturday» (the theoretical conditional). To avoid tampering with the verb of the constituent statements (if this seems desirable) we may become even more verbose and say «If it should be the case that today is Friday, than it would be the case that tomorrow is Saturday».

Buf if there are no purists present, it is more convenient to sacrifice absolute correctness for the sake of brevity and use the «entails» and «necessitates» readings; so long as we know that when purists *are* present we can replace «*p* entails *q*» with the more respectable «If it should be the case that *p* then it would be the case that *q*.»

The «paradoxes». An operator that purports to give a sense of implication or entailment or a strong sense of «if-then-» we will call an *arrow operator*. The so-called «paradoxes of implication» are of course not paradoxes in the literal sense of that term. They are paradoxical, if at all, only if we give the strong «if-then-» reading to the operators that are subject to them; i.e. if we take those operators to be arrow operators. If the «conditionality» operator of truth-functional logic is interpreted weakly, or interpreted merely

(6) *cf.* R.B. ANGELL, «A Propositional Logic with Subjunctive Conditionals,» *Journal of Symbolic Logic* 27 (1962), p. 327.

as an abbreviation for expressions containing «and» and «not» or of expressions containing «or» and «not», as it has been by some logicians, then there are no paradoxes (7). The sense of entailment that we are after is intended to be paradox-free. We therefore take stock of the «paradoxes» associated with other operators that have been interpreted as arrow operators.

(i) We say that an arrow operator in a given formal system is subject to the paradoxes of material implication if the following are provable for that operator in that formal system. ('p' and 'q' represent any well-formed-formulae of the system in question.)

Antecedent paradox	$\bar{p} \rightarrow (p \rightarrow q)$
Consequent paradox	$q \rightarrow (p \rightarrow q)$
Other paradoxical forms	$A(p \rightarrow q)(q \rightarrow p)$
	$A(p \rightarrow q)(p \rightarrow \bar{q})$
	$(K\bar{p}\bar{q}) \rightarrow (p \leftrightarrow q)$

(ii) We say that an arrow operator in a given formal system is subject to the paradoxes of strict implication if the following are provable for that operator in that formal system. ('p' and 'q' as above; 'M' is a monadic modal operator interpreted «It is possible that:...»)

Antecedent paradox	$\overline{Mp} \rightarrow (p \rightarrow q)$
	$\overline{M}\bar{q} \rightarrow (p \rightarrow q)$

(iii) We say that an arrow operator in a given formal system is subject to the paradox of relative necessity if the following is provable for that operator in that formal system (8).

$$(KMp\overline{M}\bar{q}) \rightarrow (p \rightarrow q)$$

We say that a formal system is paradox-ridden if its strongest

(7) *cf.* Alan Ross ANDERSON and Nuel D. BELNAP, Jr., *Journal of Symbolic Logic* 27 (1962), especially pp. 20-21. *cf.* also Angell, *op. cit.*

(8) *cf.* VON WRIGHT, *op. cit.*, pp. 89-126.

arrow operator is subject to any of the paradoxes above for every p and q.

There is an additional «paradox» of implication that is considered by some to be less objectionable than those just described.

(iv) We say that an arrow operator in a given system is subject to the paradoxes of consistency if the following are provable for that operator in that formal system.

Antecedent paradox	$(\mathrm{Kp}\bar{\mathrm{p}}) \rightarrow \mathrm{q}$
Consequent paradox	$\mathrm{p} \rightarrow (\mathrm{Aq}\bar{\mathrm{q}})$

Arrow operators which are subject to any of the four groups of «paradoxes» will yield theorems which, it may be argued, commit a fallacy of relevance. On the present view of entailment, a satisfactory formal rendering of logical entailment must yield an arrow operator which is free of at least the paradoxes of material and strict implication and the paradox of relative necessity. It is viewed as desirable also that the paradox of consistency be eliminated. The model systems for entailment offered below provide interpretations for two alternative calculi (although others can be constructed): one in which all the mentioned paradoxes are eliminated except for the paradoxes of consistency, and another in which all are eliminated, but at a price.

Von Wright has suggested that the study of modalities (including entailment) is the study of a genus of which the logical, physical and causal modalities are *species*. It is by now commonplace that physical entailment (under whatever name) must be construed in such a way as to «support» the counterfactual conditional. The same is of course true of other species of entailment, and of any arrow operator that purports to carry law-like force.

In this connection, it is helpful to look at the paradoxes of material and strict implication as cases where the counterfactual (or «counterpossible») cases allow us to derive the arrow formula trivially. $p \rightarrow q$ can be derived from the assumption $\bar{p}$ if the arrow represents material implication, and from the assumption $\overline{Mp}$ if the arrow represents strict implication.

Model sets and model systems. The model set-model system method of interpreting formal systems has the distinct advantage that formal systems of progressive degrees of complexity can be interpreted by adding progressively more conditions on the same relatively simple model, beginning with truth-functional statement logic and progressing through uniform quantification, mixed, multiple quantification, identity, elementary modal logic, dyadic modal logic, and finally formal systems of modal logic with quantification.

It is helpful to our present purpose to consider, in terms of such a progression of models, how the various «paradoxes» are provable in the successive formal systems. We need not concern ourselves with quantification conditions here, but rather with statement logic with modal operators. Our aim will be to eliminate those features of the model by virtue of which the «paradoxes» are truths of logic.

(I) The model set for truth-functional logic may be constructed according to the following conditions (9) :

Condition —. $[\bar{p}]\varepsilon\mu$ if and only if $\sim([p]\mu)$.
Condition K. $[Kpq]\varepsilon\mu$ if and only if $[p]\varepsilon\mu$ and $[q]\varepsilon\mu$.
Condition A. $[Apq]\varepsilon\mu$ if and only if $[p]\varepsilon\mu$ or $[q]\varepsilon\mu$.

Additional conditions for the remaining truth-functional operators and for negated truth-functors may be derived in obvious ways. For our present purpose, we will give only one derived condition.

Condition C. $[Cpq]\varepsilon\mu$ if and only if $[A\bar{p}q]\varepsilon\mu$.

All of the paradoxes of material implication are truths of logic in the usual sense with respect to such a model set. We show here

(9) 'p' and 'q' represent any well-formed formulae of the object language. '[p]' represents the formula of the model assigned to the object language formula *p*. 'ε' is used to signify the membership of a formula in a model set; '$\sim$(...ε...)' to signify non-membership;'μ' is used to designate any model set. These conditions are variants of Hintikka's. Conditions for quantifiers are omitted,although they may be introduced here or in many of the following modifications of the model with little difficulty.

only the antecedent paradox of material implication; for convenience we show it in the following form : that *Cpq* can be derived form the assumption $\overline{p}$.

(1) $[\overline{p}]\varepsilon\mu$	Assumption
(2) $[A\overline{p}q]\varepsilon\mu$	Condition A
(3) $[Cpq]\varepsilon\mu$	Condition C

(II) For conventional (monadic) modal calculi of the Lewis type, a model system is employed. The model system is a set of model sets, Ω, containing the fixed model set, ω, and perhaps others. A series of model systems for conventional modal calculi may be constructed according to the following conditions ([10]).

> *Conditions —, K, and A* above are taken to apply to every μ in Ω.
> *Condition M.* $[Mp]\varepsilon\mu\varepsilon\Omega$ if and only if for some model set λ such that $H\lambda\mu$, $[p]\varepsilon\mu$.
> *Condition L.* $[Lp]\varepsilon\mu\varepsilon\Omega$ if and only if for every model set λ such that $H\lambda\mu$, $[p]\varepsilon\lambda$.

These conditions yield a model system for the modal calculi *Mn* of von Wright, in which all basic subformulas in a given formula fall under the same number of modal operators ([11]). Modifications, consisting of certain assumptions about the H-relation, yield model systems for the calculi *M*, *S4*, and *S5*. Since the «paradoxical» theorems involving strict implication occur in all the monadic modal calculi mentioned, we will consider the model system in its simplest form.

The strict implication formula *L C p q* may be derived from the assumption $L\overline{p}$ in the following way.

([10]) We adopt the following further vocabulary for describing the model.
'Ω' designates the model system.
'ω' designates the fixed model set («the set of true statements»).
'μ' and 'λ' are variables ranging over model sets.
H^2 is the alternativeness relation defined on Ω. 'Hλμ' is read «λ is an H-alternative to μ (and λεΩ and μεΩ)».

([11]) G.H. von Wright, *An Essay in Modal Logic*, Amsterdam 1951, p. 61.

(1) $[L\bar{p}]\varepsilon\omega$ — Assumption
(2) $[\bar{p}]\varepsilon\mu$ for every μ such that $H\mu\omega$ — Condition L
(3) $[A\bar{p}q]\varepsilon\mu$ for every μ such that $H\mu\omega$. — Condition A
(4) $[Cpq]\varepsilon\mu$ for every μ such that $H\mu\omega$. — Condition C
(5) $[LCpq]\varepsilon\omega$. — Condition L

A more informal approach: «*Supposition*» *in M*, *S4*, *S5*. We have mentioned the theoretical conditional and other subjunctive forms of ordinary English which carry the force of entailment. In cases where the entailing statement is known to be false, we sometimes use yet another construction : «Suppose *p* were the case...» and consider what else would be the case under such a supposition, as a consequence of *p*.

The model for truth-functional logic may be regarded as the set of true statements. On such a model, the supposition of something which is false is permitted only at the price of trivializing any consideration of consequences.

The models for monadic modal systems, on the other hand, allow us to suppose something which is false by conjuring up model sets which are reasonable (i.e. possible) alternatives to the set which contains only true statements. But we are not permitted to suppose anything impossible and consider its consequences. To use an example of von Wright's, we would not be permitted in these systems to suppose that there are 26 prime numbers under 100, and to consider the genuine consequences of such a supposition.

The following model systems will allow us to «suppose» that *p* is the case when in fact *p* is impossible or, in the final modification, when *p* is self-contradictory.

(III) *The first entailment calculus, YS.* We introduce into the object language the dyadic modal operator 'Y', intended as an arrow

(12) A model system for von Wright's calculus *M* is obtained by adding the following condition to the model system for *Mn*:

Condition Ref. For every μ such that $\mu\varepsilon\Omega$, $H\mu\mu$.

A model system for *S4* is obtained by adding to the model system for *M* the condition that the H-relation is transitive, and a model system for *S5* by adding the further condition that the H-relation is symmetrical.

operator in the sense described above. The result of prefixing 'Y' to a pair of well-formed formulae of the object language is a well-formed formula of the object language.

We make the following addition to the vocabulary for describing the model system: S^2 is a second alternativeness relation defined on Ω. '$S\mu\lambda$' is read «μ is an S-alternative to λ (and $\mu\varepsilon\Omega$ and $\lambda\varepsilon\Omega$).»

The model for the first calculus involving the Y-operator is obtained by adding the following conditions to the conditions for the systems *Mn* (13):

Condition Y. $[Ypq]\varepsilon\mu\varepsilon\Omega$ if and only if for every model set λ such that $S\lambda\mu$, $[A\bar{p}q]\varepsilon\lambda$.
Condition S. If $H\lambda\mu$, then $S\lambda\mu$.

The various paradoxes of strict implication are not truths of logic with respect to such a model if they are rewritten substituting the 'Y' operator for the arrow (e.g. 'YLpYpq'). The conditions on the model do not warrant the derivation in the proof in (II) above of:

(4) $[Ypq]\varepsilon\omega$.

This could be derived using *Condition Y* only if $[A\bar{p}q]\varepsilon\mu$ for every μ such that $S\mu\omega$, which is not the case.

For the calculus YS, dyadic modal formulae are interpreted by means of an alternativeness relation that allows for model sets which contain impossible statements. «Supposing» that there are 26 prime numbers under 100 is analogous, I suggest, to constructing such a model set and considering what else must be in-

(13) The conditions which yield models for Y-calculi which are extensions of *M*, *S4*, and *S5* may easily be constructed. For *M*, we add the following conditions:

Condition Ref. S. For every μ such that $\mu\varepsilon\Omega$, $S\mu\mu$.
Condition Ref. ω. $H\omega\omega$.
Condition Ref. H. For every μ and λ such that $\mu\varepsilon\Omega$, and $\lambda\varepsilon\Omega$, if $H\mu\lambda$, then $H\mu\mu$.

For the *S4* extension, transitivity of the S- and H-relations is assumed, and for the *S5* extension, symmetry of the two relations is assumed.

cluded in the model set as a non-trivial consequence of the inclusion of the impossible statement (not as a trivial consequence of the statement's impossibility).

The progression of model systems thus far may be summarized as follows:

Model Set. A single set of statements, which contains only true statements.

H-model system. A set of model sets, of which one (ω) contains only true statements, while others may contain statements which are contingently false (but neither impossible nor self-contradictory). With respect to such a model system, the «supposition» of a contingently false statement is permissible.

S-model system. A set of model sets, containing all the ingredients of an H-model system, with additional model sets permitted (those which are S-alternatives to ω but not H-alternatives) which may contain statements that are necessarily false (but not self-contradictory). In such a model system, the «supposition» of a necessarily false statement is permissible.

The following «desirable» logical entailments are truths of logic with respect to an S-model system:

1. *Ypq* and *Yqp*, where *p* and *q* are truth-functionally equivalent by De Morgan's laws, double negation, or associativity, commutativity or distribution of truth-functional operators.
2. *YKpqp* and *YKpqq*
3. *YpApq* and *YqApq*
4. *YYpqYMpMq*
5. *YYpqYLpLq*

Although the «paradoxes» of strict implication are not truths of logic with respect to an S-model system, the «paradoxes» of consistency, $YKp\bar{p}q$ and $YpAq\bar{q}$ are truths of logic with respect to such a model system.

The characteristic of the remaining model system in the progression is obvious, then:

S′-model system. A set of model sets containing all the ingredients of an H-model system, with additional model sets permitted (those which are S′-alternatives to ω but not H-alternatives) which may contain at least one contradictory pair of statements, *but which do thereby contain all statements.*

(IV) *The second entailment calculus, YS′.* The model for the second calculus containing the Y-operator is obtained by making the following changes and additions in the model for the system *Mn*:

> (The relation S^2 of the previous system is replaced by the relation S'^2.)
>
> *Condition* — is replaced by the following condition:
>
> *Condition H—.* If for some λ, Hμλ, then $[\bar{p}]\varepsilon\mu$ if and only if $\sim([p]\varepsilon\mu)$.
>
> Conditions *A*, *K*, *M*, *L*, and *C* remain as before.
>
> The following two conditions are added:
>
> *Condition Y′.* $[Ypq]\varepsilon\mu\varepsilon\Omega$ if and only if for every model set λ such that $S'\lambda\mu$, either $\sim([p]\varepsilon\mu)$ or $[q]\varepsilon\lambda$.
>
> *Condition S′*, If Hλμ, then $S'\lambda\mu$

The «price» that must be paid in constructing this last model system is the giving up of the «disjunctive syllogism». Consider, for example, an S′ model set that contains both of $\bar{p}$ and *p*. By condition *A* on model sets, such a model set would contain *Apq* if it contains *p*. If *q* could be derived from *Apq* and $\bar{p}$, then it could be demonstrated that such a model set contains *every* formula.

In the Anderson-Belnap «Pure Calculus of Entailment,» $KApq\bar{p} \rightarrow q$ is not a theorem, and this purported entailment does not meet Binkley's test described above. $YKApq\bar{p}q$ is not a truth of logic with respect to an S′ model system. Anderson and Belnap offer an «independent proof» that the disjunctive syllogism commits a fallacy of relevance, primarily because it depends upon the entailment $K\bar{p}q \rightarrow q$ (which is not fallacious) *and* the purported entailement $K\bar{p}p \rightarrow q$ (which, on this account, *is* fallacious) (14).

(14) Alan Ross ANDERSON and Nuel D. BELNAP, Jr., «Tautological Entailments,» *Philosophical Studies*, XIII (1962), p. 19.

As was the case with the preceding model systems, the S′ model system may be modified to produce models for extensions of the monadic modal calculi *M*, *S4*, and *S5* (15). *YS′M*, the dyadic extension of the Feys- von Wright calculus *M*, is of particular interest The model system is constructed by adding the following conditions to those for the S′ model system :

> *Condition Ref. S′*. For every μ such that μεΩ, S′μμ.
> *Condition Ref.* ω. Hωω.
> *Condition Ref. H.* If for some λ, Hμλ, then Hμμ.

YS′M as a calculus of logical entailment. The dyadic modal calculus *YS′M*, whose model we have just described, is not wholly paradox-free. Certain intuitively undesirable formulae involving iterations of modal operators are truths of logic with respect to the S′ model system. However, it is paradox-free within the limitations described below, and has the following features :

> 1. Where *p* and *q* do not themselves contain modal operators, none of the «paradox» formulae are truths of logic with respect to the S′ model system if they are written with ’Y’ as the arrow operator (16).
> 2. Where *p* and *q* do not themselves contain modal operators, *Ypq* is a truth of logic with respect to the S′ model system *if and only if p* entails *q* according to Binkley’s test.

(15) It should be noted that there are a number of alternative ways of combining quantification with the various monadic and dyadic modal calculi. Such quantified modal logics are outside the scope of the present paper, however.

(16) Some «second-generation» paradoxes are provable, however. Although $YKp\bar{p}q$ is not a truth of logic with respect to the S′ model system for every *p* and *q*, $YKLp\overline{L}pq$ is a truth of logic with respect to the S′ model system. We may therefore view the model as only partially satisfactory, or else do one of the following (the author favors the first) : (a) We may eliminate the monadic operators from the vocabulary of the system and limit the number of iterations of the ’Y′ operator to avoid other higher-order paradoxes. Since the original purpose for the (Lewis) monadic modal systems was to give a formal rendering of entailment, we might properly view these operators as superseded by ’Y’.

3. The five «desirable» logical entailments listed above in the discussion of S-model systems are truths of logic with respect to the S′ model system.

4. The Y-relation interpreted on the S′ model system has in addition the following features which are viewed as desirable in a formal rendering of logical entailment.

(a) *Limited «Contraposition».* $YYpqY\bar{q}\bar{p}$ is not a truth of logic, but $YKKMpMqYpqY\bar{q}\bar{p}$ is a truth of logic: «If *p* and *q* are both possible and *p* entails *q*, then $\bar{q}$ entails $\bar{p}$» ([17]).

(b) *Reflexivity.* *Ypp* is a truth of logic.

(c) *Restriction on Angell's Axiom* ([18]). $YYpq\overline{Y}p\overline{q}$ is not a truth of logic, but $YKMpYpq\overline{Y}p\overline{q}$ is a truth of logic: «If *p* is possible and *p* entails *q*, then *p* does not entail $\bar{q}$.»

(d) *Entailment of tautology by contradiction.* $YKp\bar{p}Aq\bar{q}$ is not a truth of logic ,but $YKp\bar{p}Ap\bar{p}$ is a truth of logic.

(e) *Link between entailment and necessity.* $YY\bar{p}pLp$ is a truth of logic. «A statement entailed by its own negation is a necessary statement» ([19]).

(f) *Modus Ponens.* *YKpCpqq* is not a truth of logic (because of the «paradoxes» surrounding 'C'; see the discussion of the disjunctive syllogism above), but *YKpYpqq* is a truth of logic ([20]).

(g) *Transitivity of entailment.* *YKYpqYqrYpr* is a truth of logic.

Temple University, Philadelphia D. Paul SNYDER

(b) We may regard the monadic and dyadic modalities as independent, and drop *Condition S* on the model. (c) We may accept the paradoxes, and even try to justify them as reasonable, as has been done at one time or another with respect to each of the paradoxes of implication mentioned in this paper.

([17]) *cf.* VON WRIGHT, «A New System of Modal Logic,» Theorem 25; Anderson and Belnap, «The Pure Calculus of Entailment,» Axiom E-13.

([18]) *cf.* R.B. ANGELL, «A Propositional Logic with Subjunctive Conditionals,» Axiom 10.

([19]) *cf.* VON WRIGHT, *op. cit.*, Theorem 21.

([20]) This feature is shared with Anderson-Belnap «Pure Calculus of Entailment.»

BIBLIOGRAPHY

[1] ANDERSON, Alan Ross, «Some Open Problems Concerning the System E of Entailment,» *Acta Philosophica Fennica*, XVI (1963), 7-18.

[2] ANDERSON, Alan Ross, and Nuel D. BELNAP, Jr. «A Simple Treatment of Truth Functions,» *The Journal of Symbolic Logic*, XXIV, (1959), 301-302.

[3] ———., «The Pure Calculus of Entailment,» *The Journal of Symbolic Logic*, XXVII (1962), 19-52.

[4] ———., «Tautological Entailments», *Philosophical Studies*, XIII (1962), 9-24.

[5] ANGELL, R.B. «A Propositional Logic with Subjunctive Conditionals,» *The Journal of Symbolic Logic*, XXVII (1962), 327-343.

[6] BELNAP, Nuel D., Jr. «Entailment and Relevance,» *The Journal of Symbolic Logic*, XXV (1960), 144-46.

[7] CLARK, Romane, and Paul WELSH. *Introduction to Logic*. Princeton, 1962.

[8] HINTIKKA, Jaakko. «Modality and Quantification,» *Theoria* (Lund), XXVII (1961), 119-128.

[9] ———., «The Modes of Modality,» *Acta Philosophica Fennica*, XVI (1963), 65-81.

[10] ———., «Form and Content in Quantification Theory,» *Acta Philosophica Fennica*, VIII (1955), 7-55.

[11] KRIPKE, Saul. «Semantical Considerations in Modal Logic,» *Acta Philosophica Fennica*, XVI (1962), 83-94.

[12] ——., «A Completeness Theorem in Modal Logic,» *The Journal of Symbolic Logic*, XXIV (1959), 1-15.

[13] SNYDER, Donald Paul. *Causation, Predication, and Modal* Logic. (Dissertation) University Microfilms, Ann Arbor, 1964.

[14] VON WRIGHT, Georg H. *An Essay in Modal Logic*. Amsterdam, 1951.

[15] ———., *Logical Studies*. London, 1957. [«A New System of Modal Logic,» pp. 89-126; «The Concept of Entailment,» pp. 166-191].