CONTINGENCY AND NON-CONTINGENCY BASES FOR NORMAL MODAL LOGICS

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Contingency and non-contingency bases for modal logics provide direct bases for various logical investigations of philosophical interest. For example: for logics of causation and causal implication, for certain theories of entailment, for syllogistic systems with only contingent propositions, and for theories of future contingents. Furthermore by taking contingency as primitive various new extensions of weak modal logics — extensions which include systems **S6** - **S8** and provide interpretations of philosophical interest for these neglected systems — are suggested. Contingency and non-contingency bases are also of some formal interest; for instance **S5** has a very simple and elegant formulation in terms of non-contingency.

In this paper we present contingency and non-contingency bases for familiar normal modal logics, specifically for **T**, **S4** and **S5**. As well as the usual symbolism we use the symbol '∇' read 'it is contingent (that)' and the symbol '$\triangle$' read 'it is not contingent (that)'. The bracketing conventions follow Church.

The systems now presented are understood as subjoined to some complete basis for classical sentential logic (SL).

System **T**

1. Non-contingency bases.

Definition $\square$	$\square A$	$=_{Df}$ $A \,\&\, \triangle A$
Rule $\triangle$	A	$\longrightarrow \triangle A$

Axioms:

For $\mathbf{T}_1$: T1. $\triangle p \equiv \triangle \sim p$

T2. $p \supset . \triangle(p \supset q) \supset . \triangle p \supset \triangle q$

For $\mathbf{T_2}$: T1'. $\Delta \sim p \supset \Delta p$
T2. $p \supset . \Delta(p \supset q) \supset . \Delta p \supset \Delta q$
T3. $\Delta p \supset \Delta \sim \sim p$

For $\mathbf{T_3}$: T1''. $\Delta p \supset \Delta \sim p$ or T1'
T4. $\Delta(p \equiv q) \supset . \Delta p \supset \Delta q$
T5. $\sim p \supset . \Delta p \supset \Delta(p \supset q)$

2. Contingency bases.

Definition $\Box$ $\Box A =_{Df} \sim(A \supset \nabla A)$
Rule ∇ $A \longrightarrow \sim \nabla A$

Axioms:

For $\mathbf{T_4}$: T1a. $\nabla p \equiv \nabla \sim p$
T2a. $p \supset . \sim \nabla(p \supset q) \supset . \nabla q \supset \nabla p$

For $\mathbf{T_5}$: T1a'. $\nabla \sim p \supset \nabla p$ or T1a''. $\nabla p \supset \nabla \sim p$
T4a. $\sim \nabla(p \equiv q) \supset . \nabla p \supset \nabla q$
or T4a'. $(\nabla p \;\&\; \sim \nabla q) \supset \nabla(p \equiv q)$
T5a. $\nabla(p \supset q) \supset . \sim \nabla p \supset p$

System $\mathbf{T_5}$ is obtained from the Lemmon-Gjertsen formulation of *S5* by omitting the last axiom and weakening the unnecessarily strong first axiom.

Extensions to S4

Some formulation of **T** plus any one of the following axioms:

S41. $\Delta p \supset \Delta \Delta p$ or S41a. $\nabla \nabla p \supset \nabla p$
or S42. $\Delta p \supset \Box \Delta p$ or S42a. $\Diamond \nabla p \supset \nabla p$
or S43. $p \;\&\; \Delta p \supset . \Delta(p \;\&\; \Delta p)$ or S43a. $\nabla(p \supset \nabla p) \supset . p \supset \nabla p$

Some formulation of **T** plus any one of the following axioms:

S51.	$\triangle\triangle p$	or S51a.	$\sim\triangledown\triangledown p$
or S52.	$\triangle\Box p$	or S52a.	$\sim\triangledown\Box p$
or S53.	$\triangle\Diamond p$	or S53a.	$\sim\triangledown\Diamond p$
or S54.	$\triangle(\triangle p \supset p)$	or S54a.	$\sim\triangledown(p \supset \triangledown p)$

We now prove the deductive equivalence of the above systems to the corresponding normal modal logics. Proofs of the contingency bases, whose axioms are distinguished by a — tags, are not given but may be adapted from the proofs for the corresponding non-contingency bases. That is, theorems about $\mathbf{T}_4$ adapt from results for $\mathbf{T}_1$, $\mathbf{T}_5$ from $\mathbf{T}_3$, $\mathbf{T}+$ $\mathbf{S4}_{1a}$, from $\mathbf{T} + \mathbf{S4}_1$, etc.

The following abbreviations are used to indicate the justification for steps in proofs:

SL	—	Standard results in sentential logic.
Feys	—	References preceded by 'Feys' are to results established in R. Feys *Modal Logics,* Louvain (1965).
SE	—	Substitutivity of strict equivalents. This derived rule for Feys' system **T** follows from Feys 81.122, 82.123.

Theorem 1. $\mathbf{T}_1$ deductively includes Feys' system **T**, i.e. every theorem of **T** is a theorem of $\mathbf{T}_1$

Proof: We derive the postulates of **T**, namely

Axioms:	A1.	$\Box p \supset p$
	A2.	$\Box(p \supset q) \supset . \Box p \supset \Box q$
Rule T:	A	$\longrightarrow \Box A$
Definition $\triangle$:	$\triangle A$	$=_{Df} \Box A \vee \Box \sim A$

ad A1. 1. $\Box p \supset . p \,\&\, \triangle p$ — Df. $\Box$, SL

2. $\Box p \supset p$ — 1, SL

ad A2. 1. $\Box(p \supset q) \,\&\, \Box p \supset . (p \supset q) \,\&\, p \,\&\, \triangle(p \supset q) \,\&\, \triangle p$ — Df. $\Box$, SL

2. $\Box(p \supset q) \,\&\, \Box p \supset . q \,\&\, p \,\&\, \triangle p \,\&\, (p \supset . \triangle p \supset \triangle q)$ — 1, T2, SL

3. $\Box(p \supset q) \,\&\, \Box p \supset .\ q \,\&\, \triangle q$ — 2, SL
4. A2 — 3, Df. $\Box$, SL

ad Rule T.

1. $A \longrightarrow \triangle A$ — Rule $\triangle$
2. $A \longrightarrow A \,\&\, \triangle A$ — 1, SL - using $p \supset .\ q \supset p \,\&\, q$
3. $A \longrightarrow \Box A$ — 2, Df. $\Box$

ad Definition $\triangle$.

1. $\Box p \vee \Box \sim p \equiv .\ p \,\&\, \triangle p \vee .\ \sim p \,\&\, \triangle \sim p$ — Df. $\Box$, SL
2. $\Box p \vee \Box \sim p \equiv .\ p \,\&\, \triangle p \vee .\ \sim p \,\&\, \triangle p$ — 1, T1, SL
3. $\Box p \vee \Box \sim p \equiv \triangle p$ — 2, SL

Theorem 2. **T** deductively includes $\mathbf{T}_1$

Proof: Using **T** we derive the postulates of $\mathbf{T}_1$

ad T1.
1. $\triangle \sim p \equiv .\ \Box \sim p \vee \Box \sim \sim p$ — Df. $\triangle$.
2. $\triangle \sim p \equiv .\ \Box p \vee \Box \sim p$ — 1, SL, SE
3. T1 — 2, Df. $\triangle$

ad T2.
1. $\sim(\Box \sim p \,\&\, p)$ — A1, SL
2. $\triangle(p \supset q) \,\&\, \triangle p \,\&\, p \supset .$ $[\Box(p \supset q) \vee \Box(p \,\&\, \sim q)] \,\&\, (\Box p \vee \Box \sim p) \,\&\, p$ — Df. $\triangle$, SL, SE
3. $\triangle(p \supset q) \,\&\, \triangle p \,\&\, p \supset .$ $[(\Box p \supset \Box q) \vee (\Box p] \& \Box \sim q)] \,\&\, [(\Box p \,\&\, p) \vee (\Box \sim p \,\&\, p)]$ — 2, A2, SL
4. $\triangle(p \supset q) \,\&\, \triangle p \,\&\, p \supset .\ [(\Box p \supset \Box q) \vee .\ \Box p \,\&\, \Box \sim q] \,\&\, \Box p$ — 3, 1, SL
5. $\triangle(p \supset q) \,\&\, \triangle p \,\&\, p \supset .\ \Box q \vee \Box \sim q$ — 4, SL
6. $\triangle(p \supset q) \,\&\, \triangle p \,\&\, p \supset \triangle q$ — 5, Df. $\triangle$
7. T2 — 6, SL

ad Rule $\triangle$.

1.	$\Box p \supset \triangle p$	SL, Df. $\triangle$
2.	$A \longrightarrow \Box A$	Rule T
3.	$A \longrightarrow \triangle A$	1, 2, SL

ad Definition $\Box$.

1.	$\sim(p \,\&\, \Box \sim p)$	A1, SL
2.	$p \,\&\, \triangle p \equiv. p \,\&\, (\Box p \vee \Box \sim p)$	SL, Df. $\triangle$
3.	$p \,\&\, \triangle p \equiv. p \,\&\, \Box p \vee. p \,\&\, \Box \sim p$	2, SL
4.	$p \,\&\, \triangle p \equiv \Box p$	3, 1, A1, SL

Theorem 3. T_1 is deductively equivalent to **T**

By Theorems 1. and 2.

Theorem 4. $\mathbf{T_2}$ is deductively equivalent to $\mathbf{T_1}$ and to **T**

The proof is almost immediate from a consideration of axioms T1, T1′, T3, sentential logic and Theorem 3.

Theorem 5. $\mathbf{T_3}$ deductively includes **T**

Proof: We first derive certain theorems of $\mathbf{T_3}$

$T_3 1$. $\triangle(p \equiv q) \supset. \triangle p \equiv \triangle q$

For	1.	$p \equiv q \equiv. q \equiv p$	SL
	2.	$\triangle(p \equiv q \equiv. q \equiv p)$	1, Rule $\triangle$
	3.	$\triangle(p \equiv q) \supset. \triangle(q \equiv p)$	2, T4
	4.	$\triangle(q \equiv p) \supset. \triangle q \supset \triangle p$	T4
	5.	$\triangle(p \equiv q) \supset. \triangle q \supset \triangle p$	3, 4, SL
	6.	$T_3 1$	T4, 5, SL

$T_3 2$. $\triangle p \equiv \triangle \sim\sim p$

By $p \equiv \sim\sim p$, Rule $\triangle$, and $T_3 1$.

$T_3 3$. $\triangle p \equiv \triangle \sim p$

For	1.	$\triangle p \supset \triangle \sim p$	T1″.
	2.	$\triangle \sim p \supset \triangle \sim\sim p$	1
	3.	$\triangle \sim p \supset \triangle p$	2, $T_3 2$, SL
	4.	$T_3 3$	1, 3, SL

$T_3 4$. $\Box p \supset \Box(q \supset p)$

For
1. $\sim p \,\&\, \triangle p \supset. \triangle(p \supset q)$ — T5, SL
2. $p \,\&\, \triangle p \supset. \triangle(\sim p \supset \sim q)$ — 1, $T_3 3$, SL
3. $\triangle(\sim p \supset \sim q) \equiv \triangle(q \supset p)$ — SL, Rule $\triangle$, $T_3 1$
4. $\Box p \supset \triangle(q \supset p)$ — 2, 3, SL, Df. $\Box$
5. $\Box p \supset. q \supset p$ — SL, Df. $\Box$
6. $T_3 4$ — 4, 5, SL, Df. $\Box$

$T_3 5$. $\Box p \,\&\, \Box q \supset \Box(p \,\&\, q)$

For
1. $\Box p \,\&\, \Box q \supset. \Box p \,\&\, \triangle(p \supset q)$ — SL, $T_3 4$, Df. $\Box$
2. $\triangle(p \supset q) \equiv \triangle(p \equiv. p \,\&\, q)$ — SL, $T_3 1$
3. $\Box p \,\&\, \Box q \supset. \Box p \,\&\, \triangle(p \equiv. p \,\&\, q) \,\&\, (p \,\&\, q)$ — 1, 2, Df. $\Box$, SL
4. $\Box p \,\&\, \Box q \supset. \triangle p \,\&\, [\triangle p \equiv \triangle(p \,\&\, q)] \,\&\, (p \,\&\, q)$ — 3, Df. $\Box$, $T_3 1$, SL
5. $\Box p \,\&\, \Box q \supset. \triangle(p \,\&\, q) \,\&\, (p \,\&\, q)$ — 4, SL
6. $T_3 5$ — 5, Df. $\Box$

$T_3 6$. $\Box(p \supset q) \supset. \Box p \supset \Box q$

For
1. $\Box p \,\&\, \Box(p \supset q) \supset. \Box p \,\&\, \Box(q \supset p) \,\&\, \Box(p \supset q)$ — SL, $T_3 4$
2. $\Box p \,\&\, \Box(p \supset q) \supset. \Box p \,\&\, \Box(p \equiv q)$ — 1, $T_3 5$, SL, SE
3. $\Box p \,\&\, \Box(p \supset q) \supset. p \,\&\, \triangle p \,\&\, (p \equiv q) \,\&\, \triangle(p \equiv q)$ — 2, Df. $\Box$, SL
4. $\Box p \,\&\, \Box(p \supset q) \supset. p \,\&\, \triangle p \,\&\, (p \equiv q) \,\&\, (\triangle p \equiv \triangle q)$ — 3, $T_3 1$, SL
5. $\Box p \,\&\, \Box(p \supset q) \supset. q \,\&\, \triangle q$ — 4, SL
6. $T_3 6$ — 5, Df. $\Box$

That $\mathbf{T_3}$ includes **T** now follows as in the proof of Theorem 1

Theorem 6. **T** deductively includes $\mathbf{T_3}$

Proof: We derive the postulates of $\mathbf{T_3}$ using known theorems of **T**

ad T4.
1. $\triangle(p \equiv q) \,\&\, \triangle p \supset. [\Box(p \equiv q) \vee \Box \sim(p \equiv q)] \,\&\, (\Box p \vee \Box \sim p)$ — Df. $\triangle$, SL
2. $\triangle(p \equiv q) \,\&\, \triangle p \supset. \Box(p \equiv q) \,\&\, \Box p \vee. \Box(p \equiv q) \,\&\, \Box \sim p \vee. \Box \sim(p \equiv q) \,\&\, \Box p \vee. \Box \sim(p \equiv q) \,\&\, \Box \sim p$ — 1, SL

3. $\triangle(p \equiv q) \& \triangle p \supset. \Box[p \& (p \equiv q)] \vee \Box[\sim p \& (p \equiv q)]$
$\vee \Box[p \& \sim(p \equiv q)] \vee \Box[\sim p \& \sim(p \equiv q)]$ — 2, Feys 44.3, SE

4. $\triangle(p \equiv q) \& \triangle p \supset. \Box q \vee \Box \sim q \vee \Box \sim q \vee \Box q$ — 3, SL, SE

5. $\triangle(p \equiv q) \& \triangle p \supset \triangle q$ — 4, SL, Df. $\triangle$

6. T4 — 5, SL

ad T5.

1.	$\sim p \& \triangle p \supset. \sim p \& \triangle \sim p$	SL, Tl (Theorem 2.)
2.	$p \& \triangle p \equiv \Box p$	as in Theorem 2. last line
3.	$\sim p \& \triangle \sim p \equiv \Box \sim p$	2
4.	$\sim p \& \triangle p \supset \Box \sim p$	1, 3, SL
5.	$\sim p \& \triangle p \supset \Box(p \supset q)$	4, Feys 43.2, SL
6.	$\Box(p \supset q) \supset \triangle(p \supset q)$	2, SL
7,	$\sim p \& \triangle p \supset \triangle(p \supset q)$	5, 6, SL
8.	T5	7, SL

The remaining derivations parallel those in Theorem 2.

Theorem 7. $\mathbf{T_3}$ is deductively equivalent to $\mathbf{T_1}$, $\mathbf{T_2}$, and to **T**

Proof: By Theorems 5., 6., and 4.

We call the system obtained by adding axiom S4j to a non-contingency basis for **T**, **S4j**; the system obtained by adding S5j to such a basis, **S5j**. We use the Gödel formulations of **S4** and **S5** with Definition $\triangle$ as above. As special postulates **S4** has the axiom A3. $\Box p \supset \Box\Box p$, and **S5** the axiom A4. $\sim \Box p \supset \Box \sim \Box p$.

Theorem 8. **S4** deductively includes $\mathbf{S4_1}$

Proof: S41 is a theorem of **S4**

For	1.	$\triangle p \supset. \Box p \vee \Box \sim p$	Df. $\triangle$
	3.	$\triangle p \supset. \Box\Box p \vee \Box\Box \sim p$	1, A3, SL
	3.	$\triangle p \supset. \Box(\Box p \vee \Box \sim p)$	2, Feys 41.31 SL
	4.	$\triangle p \supset \Box\triangle p$	3, Df. $\triangle$
	5.	$\triangle p \supset. \Box\triangle p \vee \Box \sim \triangle p$	4, SL
	6.	S41	5, Df. $\triangle$

The result follows by this and Theorem 7.

Theorem 9. **S4**$_1$ deductively includes **S4**

Proof: A3 is a theorem of **S4**$_1$

For	1. $\Box p \vee \Box \sim p \equiv \triangle p$	as in Theorem 2., last line
	2. $\Box p \vee \Box \sim p \supset \triangle\triangle p$	1, S41, SL
	3. $\Box p \supset \triangle\triangle p$	2, SL
	4. $\Box p \supset . \Box\triangle p \vee \Box \sim \triangle p$	3, 1, SL
	5. $\Box p \supset \triangle p$	1, SL
	6. $\triangle p \supset \sim \Box \sim \triangle p$	A1, SL (1)
	7. $\Box p \supset \sim \Box \sim \triangle p$	5, 6, SL
	8. $\Box p \supset \Box\triangle p$	4, 7, SL
	9. $\Box\triangle p \equiv \Box(\Box p \vee \Box \sim p)$	Rule T, 1, Feys 46.22
	10. $\Box p \supset \Box(\Box p \vee \Box \sim p)$	8, 9, SL
	11. $\Box p \supset . \Box p \,\&\, \Box(\Box p \vee \Box \sim p)$	10, SL
	12. $\Box p \supset \Box[p \,\&\, (\Box p \vee \Box \sim p)]$	9, Feys 44.1, SL
	13. $p \,\&\, (\Box p \vee \Box \sim p) \supset . p \,\&\, \Box p \vee . p \,\&\, \Box \sim p$	SL
	14. $\sim(p \,\&\, \Box \sim p)$	A1, SL
	15. $p \,\&\, (\Box p \vee \Box \sim p) \supset . \Box p$	13, 14, SL
	16. $\Box[p \,\&\, (\Box p \vee \Box \sim p)] \supset . \Box\Box p$	15, Rule T, A2, SL
	17. A3	12, 16, SL

Theorem 10. **S4**$_1$ is deductively equivalent to **S4**

Proof: By Theorems 8. and 9.

Theorem 11. **S4**$_2$ is deductively equivalent to **S4** and to **S4**$_1$

Proof: It suffices to show the equivalence of **S4**$_2$ with **S4**
1. S42 is a theorem of **S4** by SL and line 4. of Theorem 8.
2. A3 is a theorem of **S4**$_2$

For	1. $\Box p \supset \triangle p$	as for line 5. Theorem 9.
	2. $\Box\triangle p \supset \triangle\triangle p$	1
	3. $\triangle p \supset \triangle\triangle p$	S42, 2
	4. A3	as in Theorem 9.

(1) By Theorem 7. and the fact that the extensions of **T** under consideration contain no new primitive connectives, it follows that the theorems, rules of **T** hold for all these systems.

Those who accept **S4** but object to **S5** may be interested to compare the $\mathbf{S4}_{2a}$ basis for **S4** with the Gödel basis for **S5**, that is:

the **S4** basis	**T**	+	$\Diamond\nabla p \supset \nabla p$,	and
the **S5** basis	**T**	+	$\Diamond\Box p \supset \Box p$.	

Theorem 12. $\mathbf{S4}_3$ is deductively equivalent to each of **S4**, $\mathbf{S4}_1$ and $\mathbf{S4}_2$

Proof: It suffices to show the equivalence of $\mathbf{S4}_3$ with **S4**

1. S43 is a theorem of **S4**

For	1. $\Box p \supset .\Box\Box p \vee \Box \sim \Box p$	A3, SL
	2. $\Box p \supset . \triangle\Box p$	1, Df. $\triangle$
	3. $p \,\&\, \triangle p \equiv \Box p$	as in Theorem 2., last line
	4. S43	2, 3, SE

2. A3 is a theorem of $\mathbf{S4}_3$

For	1. $\Box p \supset \triangle\Box p$	S43, Df. $\Box$
	2. $\triangle\Box p \supset . \Box\Box p \vee \Box \sim \Box p$	as in Theorem 9., line 1
	3. $\Box p \supset . \Box\Box p \vee \Box \sim \Box p$	1, 2, SL
	4. $\Box p \supset \sim\Box\sim\Box p$	A1, SL
	5. A3	3, 4, SL

Theorem 13. $\mathbf{S5}_1$ is deductively equivalent to **S5**

Proof: It suffices to show that S51 is a theorem of **S5** and that A4 is a theorem of $\mathbf{S5}_1$

1. S51 is a theorem of **S5**

For	1. $\triangle p \vee \sim\triangle p$	SL
	2. $\triangle p \supset \Box\triangle p$	as in Theorem 8., line 4.
	3. $\sim\triangle p \equiv . \sim\Box p \,\&\, \sim\Box\sim p$	Df. $\triangle$, SL
	4. $\sim\Box p \supset \Box\sim\Box p$	A4
	5. $\sim\Box\sim p \supset \Box\sim\Box\sim p$	A4
	6. $\sim\triangle p \supset . \Box\sim\Box p \,\&\, \Box\sim\Box\sim p$	3, 4, 5, SL
	7. $\sim\triangle p \supset \Box(\sim\Box p \,\&\, \sim\Box\sim p)$	6, Feys 44.3, SE
	8. $\sim\triangle p \supset \Box\sim\triangle p$	7, 3, SE

9. $\Box\triangle p \vee \Box\sim\triangle p$	1, 2, 8, SL
10. S51	9, Df. $\triangle$

2. A4 is a theorem of $S5_1$

Eor	1. $\triangle p \equiv . \Box p \vee \Box\sim p$	as in Theorem 1., last line
	2. $\triangle\triangle p \supset . \Box\triangle p \vee \Box\sim\triangle p$	1, SL
	3. $\Box\triangle p \supset \triangle p$	A1
	4. $\Box\triangle p \supset . \Box p \vee \sim\Diamond p$	3, 1, SL
	5. $\Box p \supset \Box\Diamond p$	Feys 36.0, A2, SL
	6. $\Box\triangle p \supset . \sim\Diamond p \vee \Box\Diamond p$	4, 5, S2
	7. $\Box\sim\triangle p \supset \Box(\sim\Box p \,\&\, \Diamond p)$	1, SL, Rule T, A2
	8. $\Box\sim\triangle p \supset . \Box\sim\Box p \,\&\, \Box\Diamond p$	7, Feys 44.3, SE
	9. $\Box\sim\triangle p \supset \Box\Diamond p$	8, SL
	10. $\Box\sim\triangle p \supset . \Diamond p \supset \Box\Diamond p$	9, SL
	11. $\triangle\triangle p \supset . \Diamond p \supset \Box\Diamond p$	2, 6, 10, SL
	12. A4	11, S51, SL

The **S5** axiom $\triangle\triangle p$ reveals especially clearly the interpretation of the modalities of **S5**

Theorem 14. Systems **S5**, **S5**$_1$, **S5**$_2$, **S5**$_3$ and **S5**$_4$ are deductively equivalent

Proof: By Theorem 13. and the following results

1. S52 yields S53 and conversely

For	1. $\triangle\Box p$	$\leftrightarrow$	$\triangle\sim\Box p$	T1
	2. $\triangle\Box p$	$\leftrightarrow$	$\triangle\sim\Box\sim p$	1, SL, SE
	3. $\triangle\Box A$	$\leftrightarrow$	$\triangle\Diamond A$	2

2. $\triangle(\triangle p \supset p) \equiv \triangle\Diamond p$ is a theorem of **T**

For	1. $\triangle p \supset p \equiv \sim(\triangle p \,\&\, \sim p)$	SL
	2. $\triangle p \supset p \equiv \sim(\sim p \,\&\, \triangle\sim p)$	1, T1, SE
	3. $\triangle p \supset p \equiv \sim\Box\sim p$	2, Theorem 2. last line, SE
	4. $\triangle(\triangle p \supset p) \equiv \triangle\Diamond p$	3, Rule $\triangle$, $T_3 1$, SL

3. S52 is a theorem of T5

1.	$\Box p \vee \sim \Box p$	SL
2.	$\Box\Box p \vee \Box \sim \Box p$	1, A3, A4, SL
3.	$\triangle \Box p$	2, Df. $\triangle$

4. $\triangle \Diamond p \supset . \Diamond p \supset \Box \Diamond p$ is a theorem of **T**

1.	$\triangle \Diamond p \supset . \Box \Diamond p \vee \Box \sim \Diamond p$	from Theorem 1., last line
2.	$\Box \Diamond p \supset . \Diamond p \supset \Box \Diamond p$	SL
3.	$\Box \sim \Diamond p \supset \sim \Diamond p$	A1
4.	$\Box \sim \Diamond p \supset . \Diamond p \supset \Box \Diamond p$	3, SL
5.	$\triangle \Diamond p \supset . \Diamond p \supset \Box \Diamond p$	1, 2, 4, SL

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