QUANTIFIERS AND UNIVERSAL VALIDITY

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In this paper I will consider those features of predicate calculus which make it possible to prove that there is at least one individual. The limitation on the validity of normal systems of predicate calculus (i.e., their limitation to non-empty domains) will be traced to the failure to distinguish different uses of individual variables. Both Mostowski (in [2]) and Hailperin ([1]) have presented systems of predicate calculus whose theorems are universally valid, and I will make use of their results (1). Instead of obtaining new results, I want to achieve a new understanding of old results.

The following

$$(x)f(x) \supset (\exists x)f(x)$$

is a characteristic law of those systems of predicate calculus which are valid only for non-empty domains. It seems strange that this law has been readily accepted, while in the analogous case of the interpretation of sentences whose form is

All S are P.,

great deal of energy has been expended in arguing that the non-existential interpretation is best. Students of elementary logic are commonly taught that Boolean inclusion is more useful for logical purposes than Aristotelian (traditional) inclusion. But when we are talking about all things instead of all Greeks or all men, then we agree that there must be at least one. It appears that this assumption can also be justified in terms of utility. Professor Quine writes,

> The following fact is demonstrable regarding quantificational schemata : those which turn out valid for all choices

(1) Hailperin does not present a system of predicate calculus. But his schemata can be used as the basis for a system of predicate calculus.

> of universe of a given size also turn out valid for all smaller universes, except for the empty one....
> It behooves us therefore to put aside the one relatively inutile case of the empty universe, so as not to cut ourselves off from laws applicable in all other cases (2).

The argument for accepting systems valid only for non-empty domains on grounds of utility is a weak one. For if we distinguish the theorems of predicate calculus valid for all domains from those valid only for non-empty domains, the first class will contain most of the important theorems. And the result that a formula valid for any domain of a given size is valid for all smaller non-empty domains can be obtained only because first order predicate calculus does not possess sufficient resources to state that there is more than one individual. If it were possible to say that there are at least two individuals, then there would be a formula valid for a two-element domain but not for a one-element domain.

Even if systems valid only for non-empty domains were more useful than systems valid for all domains, it would be worth while to distinguish those features that produce universally valid results from those which yield results of more limited validity. It is an advantage of formalized languages that they assist us in getting clear about the concepts we employ. If there is a system valid for all domains and an axiom which makes the system valid for all non-empty domains, this system will provide a clearer understanding of predicate calculus and the conditions of its validity than does an ordinary system of predicate calculus.

I think that the fundamental reason why standard systems of predicate calculus are so formulated as to yield

$$(x)f(x) \supset (\exists x)f(x)$$

and related formulas is that the uses of free individual variables have not been sufficiently understood. In discussing this point, I will make use of the system of predicate calculus presented by

(2) W.V. QUINE, «Meaning and Inference,» *From a Logical Point of View*, p. 161.

Hilbert and Ackermann, in *Mathematical Logic*. The axioms of this system are

$$(x)f(x) \supset f(y)$$
$$f(y) \supset (\exists x)f(x) \quad (x)(3),$$

as well as axioms common to propositional calculus. These axioms are valid for all domains : if the free variables are replaced by significant expressions, the resulting statements will always be true. However, for an empty domain, there will be no expressions which can replace 'y'; any names which belong to the same category as the individual variables must be names which have referents. Although the axioms above are universally valid, they can be used to derive

$$(x)f(x) \supset (\exists x)f(x),$$

which is not valid for an empty domain. To obtain this result, one can substitute in the following tautology,

$$p \supset q \supset .q \supset r \supset .p \supset r \quad (4),$$

and use Modus Ponens twice.

We can better understand the features permitting the proof of

$$(x)f(x) \supset (\exists x)f(x)$$

if we revise this system of predicate calculus to eliminate free individual variables. Since the axioms above are universally valid, so are the following

$$(y).(x)f(x) \supset f(y)$$
$$(y).f(y) \supset (\exists x)f(x).$$

For to say that the axioms are universally valid means that they are true for all values of their free variables. But this is what is indicated (for individual variables) by the initial universal quantifier. These generalized versions of the axioms will be the axioms of the revised system.

(3) I have not employed the symbolism of HILBERT and ACKERMANN. In their book, these axioms appear as

$$(x)F(x) \rightarrow F(y)$$
$$F(y) \rightarrow (\exists x)F(x).$$

(4) Parentheses are abbreviated according to the convention of A. CHURCH, in *Introduction to Mathematical Logic*.

Certain changes must be made in the rules, if free individual variables are to be eliminated. The definition of a (well-formed) formula can remain unchanged (5) — but not all universally valid wffs will be theorems. The rules of substitution for propositional and predicate variables must be modified so that when expressions containing free individual variables are substituted, the free variables are then bound by an initial universal quantifier. These rules can be formulated so that the quantifiers added are immediately to the right of whatever initial (universal) quantifiers are there already. These rules must also make possible the substitution of a wff containing a free variable bound in the original formula, so long as this variable is bound by an initial universal quantifier (6). There will be no rule of substitution for free individual variables, but there will be a corresponding rule for those variables bound by an initial universal quantifier. This rule must provide for deleting an initial quantifier when a distinction between individual variables has collapsed.

The rule permitting the change of a bound variable can remain unchanged (7). The two rules for quantifiers must be rewritten as

> y1 From a wff $(\alpha_1)(\alpha_2)...(\alpha_n).A \supset B(\alpha_i)$ in which the consequent contains the free variable α_i while α_i does not occur in A, the wff $(\alpha_1)..(\alpha_{i-1})(\alpha_{i+1})..(\alpha_n).A \supset (\alpha_i)B(\alpha_i)$ is obtained.
>
> y2 From a wff $(\alpha_1)(\alpha_2)...(\alpha_n).A(\alpha_i) \supset B$ in which the antecedent contains the free variable α_i while α_i does not

(5) HILBERT and ACKERMANN do not allow vacuous occurrences of quantifiers. This practice can be defended on intuitive grounds, for a vacuous quantifier is not usually given any significance. Such quantifiers are admitted in order to simplify the axioms and rules. Since my goal is to achieve a better understanding of predicate calculus rather than a simpler system, it seems convenient to follow the practice of HILBERT and ACKERMANN.

(6) The rules of substitution can also be based on the simplified rules given by David PAGER in «An Emendation of the Axiom System of Hilbert and Ackermann for the Restricted Calculus of Predicates,» *The Journal of Symbolic Logic*, Vol. 27, N°. 2, pp. 131-138.

(7) Because free variables are not allowed, the rule for changing bound variables does not need the correction that is made by PAGER in the article cited in the preceding note.

occur in B, the wff $(\alpha_1)...(\alpha_{i-1})\ (\alpha_{i+1})...(\alpha_n).(\exists\alpha_i)A(\alpha_i) \supset B$ is obtained.

To insure that this revised system yields the desired results, it is sufficient to add two more rules : the ordinary rule of Modus Ponens and the following,

> π From a wff $(\alpha_1)(\alpha_2)...(\alpha_n).\ A(\alpha_i) \supset B(\alpha_i)$ in which both the antecedent and consequent contain the free variable α_i, the wff $(\alpha_1)..(\alpha_{i-1})(\alpha_{i+1})..(\alpha_n).(\alpha_i)A(\alpha_i) \supset (\alpha_i)B(\alpha_i)$ is obtained.

Now it is possible to prove that

> (1) The theorems of this revised system are valid in all domains,
> (2) Any formula containing no free individual variables that is valid in all domains is a theorem of this revised system,
> (3) The addition of the formula
>
> $$(\exists x).f(x) \vee \sim f(x)$$
>
> as an axiom will produce a system which contains (as axioms or theorems) all those formulas valid for all non-empty domains (so long as these formulas do not contain free individual variables).

The proof of these results follows very closely the proof given by Mostowski in [2], and will not be given here ([8]).

([8]) In [2], Mostowski is dealing with the system of predicate calculus found in Church's *Introduction to Mathematical Logic*. Mostowski retains the axioms and rules, except for Modus Ponens, for which he substitutes,

> If A, B are wffs such that all individual variables free in A are also free in B, if A, $A \supset B$ are theorems, then B is a theorem.

An analogue to this rule for the revised system of Hilbert and Ackermann would be

> μ If A is a wff containing distinct free individual variables $\alpha_1, \alpha_2, ..., \alpha_n$ and no other free individual variables, B is a wff containing distinct free individual variables $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_m$ and no others if $(\alpha_1)(\alpha_2)...(\alpha_n)A$, $(\alpha_1)(\alpha_2)...(\alpha_n)(\beta_1)(\beta_2)...(\beta_m).\ A \supset B$ are theorems then $(\alpha_1)...(\alpha_n)(\beta_1)...(\beta_m)$ B is a theorem.

By eliminating free individual variables and changing the rules as required by this elimination, it is possible to reconstruct predicate calculus so that its theorems are valid in all domains. Now consider a derivation of

$$(x)f(x) \supset (\exists x)f(x)$$

in the original system. Substitution in the tautology given earlier yields

$$(x)f(x) \supset f(y) \supset .f(y) \supset (\exists x)f(x) \supset .(x)f(x) \supset (\exists x)f(x).$$

By using Modus Ponens twice, the formula above is obtained. Suppose a parallel maneuver is made in the revised system. Substitution in the same tautology yields

$$(y).(x)f(x) \supset f(y) \supset .f(y) \supset (\exists x)f(x) \supset .(x)f(x) \supset (\exists x)f(x).$$

By π,

$$(y)[(x)f(x) \supset f(y)] \supset (y).f(y) \supset (\exists x)f(x) \supset .(x)f(x) \supset (\exists x)f(x).$$

Then by Modus Ponens,

$$(y).f(y) \supset (\exists x)f(x) \supset .(x)f(x) \supset (\exists x)f(x).$$

It is not possible to go farther. This formula is equivalent to

$$(\exists y)[f(y) \supset (\exists x)f(x)] \supset .(x)f(x) \supset (\exists x)f(x),$$

but the antecedent of this last formula is not universally valid and cannot be proved. The occurrence of free variables in the original system, and the possibility of substituting 'f(y)' for a propositional variable, make it possible to derive

$$(x)f(x) \supset (\exists x)f(x).$$

It is a simple matter to prove that μ is equivalent to Modus Ponens plus π. This equivalence makes it possible to utilize MOSTOWSKI's proof for the revised system of HILBERT and ACKERMANN.

The elimination of free variables has also eliminated this result.

If we consider the axioms of the original system :

$$(x)f(x) \supset f(y)$$
$$f(y) \supset (\exists x)f(x),$$

it is natural to construe the variable 'y' as bound by an initial universal quantifier. But in the two formulas

$$(y).(x)f(x) \supset f(y)$$
$$(y).f(y) \supset (\exists x)f(x),$$

the variable 'y' is playing two different roles. Just what the difference is can be understood if we consider

$$(x)f(x) \supset (y)f(y)$$
$$(\exists y)f(y) \supset (\exists x)f(x),$$

which are equivalent to the formulas above. 'f(y)' in the consequent amounts to the same thing as '(y)f(y),' but in the antecedent it is equivalent to '($\exists$y)f(y).' In the original system, the distinct roles of 'f(y)' are confused — this confusion is made all the easier to overlook by the presence of free individual variables.

It is possible to formulate a system of predicate calculus whose theorems are valid for all domains, where some of these theorems contain free individual variables. The system formulated by Mostowski is such a system. But the most natural way to construct a system valid for all domains seems to be by way of eliminating free individual variables. For if free individual variables are allowed, they must be treated as if they were bound by an initial universal quantifier. The occurrence of free individual variables in an ordinary system of predicate calculus makes it easy to overlook the introduction of existential claims. When all the variables are bound, it is less difficult to detect the assumptions which guarantee the existence of at least one individual.

There are systems which contain no free individual variables, in which it is possible to prove that there is at least one individual. The outline of such a system is presented by Quine, in *Mathematical*

Logic (9). A system of predicate calculus based on Professor Quine's schemata would be much like the system which results from Hilbert and Ackermann's system if the free individual variables in the axioms and theorems are bound but no other changes are made (i.e. such a system contains the same essential results as Hilbert and Ackermann's system). The principal difference between a system based on Quine's schemata and one containing the closures of the axioms and theorems of Hilbert and Ackermann's system is that the former contains vacuous quantifiers (and in Quine's system, it is possible for a quantifier to occur within the scope of another quantifier which contains the same quantified variable). A system with no free individual variables which is valid only in non-empty domains is one whose rules justify quantificational replacements that are not justified in dealing with an empty domain — this will be shown below.

Hailperin, in [1], has revised Quine's schemata *100-*105 to produce a system which is universally valid. However, in doing this, he has made important use of vacuous quantifiers. Hailperin replaces

> *104 If Φ' is like Φ except for containing free occurrences of α' where Φ contains free occurrences of α, then $\vdash[(\alpha)\Phi \supset \Phi']$.

by

> QE4 If Φ' is like Φ except for containing free occurrences of α' wherever Φ contains free occurrences of α, then $\vdash(\alpha)\Phi \supset (\alpha')\Phi'$ (10).

And he adds

> QE5 If α is not free in Φ, $\vdash\Phi \supset (\alpha)\,\psi \supset .(\alpha)(\Phi \supset \psi)$.

(9) In the first edition of *Mathematical Logic*, schemata *100-*105 are found on p. 88. In later editions, the number of schemata is reduced by elimination of

$*101 \vdash[(\alpha)(\beta)\Phi \supset (\beta)(\alpha)\Phi]$.

However, in [1], Hailperin refers to *100-*105, and I shall follow him in this.

I have said that schemata provide the *outline* of a system of predicate calculus, because Professor QUINE does not himself use these schemata as a basis for a system of predicate calculus.

(10) The expression '$\vdash\Phi$' is used to designate the (universal) closure of 'Φ'.

In Hailperin's treatment, a formula bound by an initial universal quantifier, even a vacuous quantifier, is valid in the empty domain. In a system based on schemata *100-*105, it is possible to prove.

$$(x)[p \& \sim p] \supset p \& \sim p.$$

But this cannot be obtained in Hailperin's system; for in an empty domain the antecedent will be valid and the consequent invalid ([11]).

In [3], Professor Quine argues that Hailperin's treatment of vacuous quantifiers is a natural one. He claims that we must equate

$$(x)p$$

with

$$(x)[p \& [f(x) \supset f(x)]],$$

because

$$p \equiv .p \& [f(x) \supset f(x)]$$

is tautologous. However, his argument does not succeed in establishing its conclusion. For the argument depends on our accepting schema *102,

$$\vdash [(\alpha)(\Phi \supset \psi) \supset .(\alpha)\Phi \supset (\alpha)\psi].$$

When quantifiers are interpreted in such a way that vacuous quantifiers do not contribute to the meaning of the formulas in which they occur (I take this to be the normal interpretation of vacuous quantifiers), then schema *102 is not universally valid. Accepting this schema requires one to accept Hailperin's treatment of vacuous quantifiers, but nothing requires us to accept this schema.

If one adopts an ordinary interpretation of vacuous quantifiers, schema *102 fails to be universally valid. To see this more clearly, consider the following proof of

$$(x)f(x) \supset (\exists x)f(x).$$

([11]) Professor Quine, in [3], has shown that Hailperin's system can be simplified by eliminating QE5 and replacing QE4 by

> QE4' If α occurs free in Φ, and Φ' is like Φ except for containing free occurrences of α' wherever Φ contains free occurrences of α, then $\vdash (\alpha)\Phi \supset \Phi'$.

i $(y).(x)f(x) \supset f(y)$ *104

ii $(y).(x)f(x) \supset f(y) \supset .f(y) \supset (\exists x)f(x) \supset .(x)f(x) \supset (\exists x)f(x)$ *100

iii $[ii] \supset .(y)[(x)f(x) \supset f(y)] \supset (y).f(y) \supset (\exists x)f(x) \supset . (x)f(x) \supset (\exists x)f(x)$ *102

iv $(y)[(x)f(x) \supset f(y)] \supset (y).f(y) \supset (\exists x)f(x) \supset .(x)f(x) \supset (\exists x)f(x)$ iii, ii, *105

v $(y).f(y) \supset (\exists x)f(x) \supset .(x)f(x) \supset (\exists x)f(x)$ iv, i, *105

vi $[v] \supset .(y)[f(y) \supset (\exists x)f(x)] \supset (y).(x)f(x) \supset (\exists x)f(x)$ *102

vii $(y)[f(y) \supset (\exists x)f(x)] \supset (y).(x)f(x) \supset (\exists x)f(x)$ vi,v,*105

viii $(y).f(y) \supset (\exists x)f(x)$ will not be proved

ix $(y).(x)f(x) \supset (\exists x)f(x)$ vii, viii, *105

x $[ix] \supset .(x)f(x) \supset (\exists x)f(x)$ *104

xi $(x)f(x) \supset (\exists x)f(x)$ x,ix, *105

The formula at step vi is the first one in the proof that is not universally valid. This is the key step in the proof, for it makes possible the transition from formulas that are universally valid to the formula valid only for non-empty domains. *102 makes it possible to begin with

$$(y).f(y) \supset (\exists x)f(x)$$

and derive

$$(y)f(y) \supset (\exists x)f(x).$$

But the original formula is equivalent to

$$(\exists y)f(y) \supset (\exists x)f(x).$$

In effect, *102 permits the replacement of a particular quantifier in the antecedent by a universal quantifier. But this move is only justified if there is at least one individual in the domain being considered — this is the point where existential presuppositions are involved. To modify schemata *100-*105 so that they yield universally results, it is sufficient to replace *102 by

*102′ If ψ contains free occurrence of α,
$\vdash[(\alpha)(\Phi \supset \Psi) \supset .(\alpha)\Phi \supset (\alpha)\Psi]$ (12).

In the Hilbert and Ackermann system, it is possible to begin with universally valid formulas and derive formulas valid only in non-empty domains, because all occurrences of a free variable are treated in the same way. But a variable occurring free in the antecedent and not in the consequent is being used in a different way than a variable free in the consequent and not free in the antecedent. The difference becomes clear when free variables are eliminated in favor of variables bound by initial universal quantifiers. There are also systems containing only bound variables in which the distinction between the use of 'y' in

$$(y).(x)f(x) \supset f(y)$$

and

$$(y).f(y) \supset (Ex)f(x)$$

is ignored. Ignoring this distinction leads to adoption of a system valid only for non-empty domains. It may be the case that systems whose theorems are universally valid are less important than systems valid only for non-empty domains. But there is surely some value to understanding the difference between the two kinds of systems. And there is also a value in making assumptions explicit, even if these are only the assumptions inherent in a formal system. By formulating a universally valid system, and then considering the consequences of assuming that there is at least one individual, it is possible to realize these values (13).

(12) This will not be proved here. A system based on the revised schemata will be essentially like the revised system of HILBERT and ACKERMANN, except for the occurrence of vacuous quantifiers. The proof that the system based on the revised schemata is universally valid is basically the same as the proof presented by MOSTOWSKI in [2].

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