NOTES TOWARDS AN AXIOMATIZATION OF INTUITIONISTIC ANALYSIS

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The situation of intuitionistic mathematics today is not unlike the situation of set-theory at the beginning of the century; it is in process of being made an honest woman of by axiomatization. The underlying *logic* (predicate calculus with identity) has as is well-known been definitively formalized by Heyting ([5] passim, [6] pp. 57-65; and cf. Kleene [9] pp. 108-166 for a more rigorous version) so that the first-order intuitionistic theories (in particular 'Peano-Heyting' arithmetic — Heyting [6] pp. 67-71, Kleene [9] pp. 181-204) can be regarded as legitimated beyond dispute. It is with the axiomatization of higher-order logic (intuitionistic analysis, theory of free-choice sequences) that this paper is concerned.

For the sake of making it more or less self-contained, at least on an informal level, I shall make a few remarks about the intuitionistic predicate calculus before proceeding to my main topic; however, these remarks will not be formal or axiomatic, but will concern the *interpretation* of the logical connectives and quantifiers (cf. Kreisel [12] pp. 121-131). In order to follow this interpretation it is essential to bear in mind that (roughly speaking) the notion of *truth* plays no role intuitionistically, at least not at the elementary level corresponding to that at which the classical connectives are explained as *truth*-functional. The role of the notion of truth at this level is carried in intuitionism by the notion of (valid) *grounds for asserting* something. In other words, one does not regard mathematical problems as existing outside of the human mind with their correct answers predetermined, for *us* to find out. What is sometimes thought of as a confusion, in the intuitionistic writings, between 'being able to assert '*A* or *B*' 'and 'being able to assert *A* or being able to assert *B*' is no confusion at all since any sense of 'being able to assert '*A* or *B*' 'in which it does *not* reduce to being able to assert one of the two alternatives is intuitionistically unacceptable.

There are two and only two kinds of grounds for asserting a disjunction '*A* or *B*' — firstly, any ground which would justify us in asserting *A*; secondly, any ground which would justify us in asserting *B*. Thus one can only assert '*A* or not *A*' if one knows which.

A ground for asserting '*A* implies *B*' consists of two parts: first, a method μ which, applied to any ground for asserting *A*, yields a ground for asserting *B*; second, a ground for asserting that μ does indeed have this property. [One might think that μ alone was indeed as the ground: this is not so because e.g. if we knew (non-constructively) that there was no (intuitionistic) ground for asserting *A*, there would exist (even constructively) a μ of the kind described — e.g. the identity map on grounds: but this would not justify us in asserting '*A* implies *B*'] Briefly: we can assert '*A* implies *B*' if and only if we know how we could obtain a proof of *B* if we were given a proof of *A*. And here 'proof' means (not formal proof, but) grounds for asserting.

A ground for asserting 'not *A*' means simply a ground for asserting '*A* implies $1 = 0$'; from this together with the preceeding paragraph it follows that it makes no difference whether we say 'we have grounds for asserting that *A* leads to a contradiction' or 'we have grounds for asserting that *A* cannot be proved' (i.e. for asserting that there can be no grounds for asserting *A*.) [The phrase 'there are no grounds' is ambiguous — it could mean we have *at present* no grounds or that (we have grounds for asserting that) the assumption that there are grounds leads to contradiction. In the latter case, we shall usually say for clarity as above 'there *can* be no grounds': this is the *mathematical* sense of negation as opposed to the *historical* sense: we shall see, however, that the historical sense is not entirely irrelevant to mathematics.]

A ground for asserting $(\exists x)\, Fx$ consists of two things: first an entity *a* in the range of the variable *x*: secondly a ground for asserting *Fa*. Thus one is in no position to assert an existence statement without possession of an example.

A ground for asserting $(\forall x)\, Fx$ consists of two things: first a method μ which yields, when applied to any entity *a* in the range of the variable *x*, a ground for asserting (= proof of) *Fa*: second, a proof that μ does indeed have this property. [Here again as in the case of implication, one might think that μ alone would suffice: but consider the following example. If we knew (non-constructively) that there were no Fermat exponents > 2 one could express this knowledge by the formula.

$$(\forall xyzw)\,(x^{w+3} + y^{w+3} \neq z^{w+3})$$

and the method μ would exist (even constructively) — namely the method which yields, when applied to four non-negative integers x, y, z, w, calculations of $x^{w+3} + y^{w+3}$ and z^{w+3}, and a comparison of the results of these calculations. But this would not justify us in asserting that if Fermat's conjecture was true classically it was assertable intuitionistically.] Briefly: we can assert $(\forall x)\, Fx$ if and only if we know how to prove Fa for each a. Thus the statement $\neg (\forall x)\, Fx$ is weaker than $(\exists x) \neg Fx$; one might be able to prove the absence of a universal method of proving Fa without being able to produce an a for which one could prove the impossibility of proving Fa. (It is in fact the (formal) justification of a certain counter-example to the inference

$$\neg (\forall x)\, Fx \rightarrow (\exists x) \neg Fx$$

which will occupy us centrally in the body of this paper).

No counter-examples to this or any other valid formula of the classical functional calculus are to be found in the (conventional formalizations of) intuitionistic *number-theory*. This is obvious because such formalizations are subsystems of the classical (Peano-Hilbert) one. Matters become very different at the next level, that of intuitionistic *analysis*: this is not a subsystem of classical analysis, but contains theorems which contradict theorems of classical analysis. Assuming the formalizations of intuitionistic number-theory to be completed and known (full details are in Kleene loc. cit., and cf. [10] pp. 22-43) we now discuss the formalization of intuitionistic analysis (theory of free-choice sequences).

The two central new notions of intuitionistic analysis over and above those of intuitionistic number-theory each correspond to the classical notion of *number-theoretic function*. Classically, such a function f consists of all its values

$$f(0), f(1), f(2), f(3) \ldots$$

but the standpoint of intuitionism rejects such a completed infinite totality as this. It maintains that the unanalysed classical notion of (number-theoretic) function is a conflation of two ideas neither of which involves the actual infinite: the notion of a *rule* or *computable function*, which is finitely specifiable (e.g. the function $(\lambda n)\, n^2$ which is a rule of computation we know how to apply to each number) and the notion of a *free-choice sequence*, which involves only the potential infinite. (Whether 'computable function' coincides with 'recursive function' is a question we shall return to briefly at the very end of this paper. For the present, it is to be taken as a primitive idea.)

It is easiest to approach the notion of (ordinary) free-choice sequence via another related and simpler notion due to Gödel (cf. Kreisel [11] pp. 371-378), that of an *absolutely free choice sequence*. An absolutely free choice sequence $\{\alpha(n)\}$ has *no* properties that cannot be asserted on the basis of a finite number of values. Formally this can be expressed by the schema

$$A(\alpha) \rightarrow (\exists n)\,(\forall\beta)\,(\beta \underset{n}{=} \alpha \rightarrow A\,(\beta)) \tag{1}$$

(axiom 5.1 of [11]) where $\beta \underset{n}{=} \alpha$ is short for $(\forall x \leq n)\,(\alpha(x)=\beta(x))$, and where $A(\alpha)$ contains no variables for free-choice sequences other than α. [Remarks. I. To see the need for this restriction, take $A(\alpha)$ as $(\forall x)\,(\alpha(x)=\gamma(x))$ and get an absurd conclusion. II. Since we are only using absolutely free-choice sequences for expository purposes, we have slightly simplified the original (Gödel-Kreisel) notion. The reader who refers to [11] for further details can resolve the discrepancy by noting that our 'absolutely free choice sequences' are Kreisel's 'absolutely free choice sequences chosen from the universal spread'.]

For such sequences we can of course never assert $(\forall x)\,(\alpha(x) = 0)$. The property of being identically zero is clearly not guaranteed by any finite initial segment of values. For this reason *absolutely free choice sequences are not the ones we need in analysis.* (For there is no real number zero!) They are useful only (a) for pedagogic purposes as in the present paper (b) (perhaps, on the basis of some recent work of Kreisel) foundationally — inasmuch as they are simpler objects to understand than ordinary free-choice sequences and the latter are apparently (contextually) definable in terms of them and (c) to provide counter-examples. For example, the formula

$$\neg(\forall x)(\alpha(x) = 0)$$

we have just seen to be true of all absolutely free α, while the formula

$$(\exists x)(\alpha(x) \neq 0)$$

is assuredly not; this is a counter-example to the formula $\neg(\forall x)\, Fx \rightarrow (\exists x) \neg Fx$ mentioned above.

In order to clarify the notion of *ordinary* free-choice sequence (sometimes called Brouwer free-choice sequence), which is the central notion of intuitionistic analysis, we first introduce the idea of a *spread*, which corresponds (roughly) to (a special case of) the classical 'set' of number-theoretic functions (or of sequences, or of real numbers). A spread is a law (thus a computable function) which places a restriction on finite sequences — i.e. classifies them into admissable and inadmissable. It admits the empty sequence: if it admits a sequence it admits also all its initial segments: if it admits a sequence it admits also at least one continuation (which, by the intuitionistic reading of ∃, we can effectively find). Thus if we identify finite sequences with their indices in some fixed enumeration, and spreads with their characteristic functions:

$$\begin{aligned}\mathrm{Spd}(f) \leftrightarrow f\ \langle\ \rangle = 0\ \&\ (\forall x)(f(x) = 0 \rightarrow (\exists y)(f(x^\frown\{y\} = 0))\\ \&\ (\forall xy)(f(x^\frown\{y\}) = 0 \rightarrow f(x) = 0).\end{aligned}$$

[Notation: $\langle\ \rangle$ is the index of the empty sequence; $m^\frown n$ is the index of sequence number m followed by sequence number n; $\{y\}$ is the index of the sequence whose sole term is y.] A spread can be regarded as a promise that we make concerning a sequence of numbers that we choose otherwise freely. We do not pick the terms of a Brouwer free-choice sequence *absolutely* freely: we may decide at some (not necessarily the beginning) to pick only zeros, or to pick a monotone increasing sequence of natural numbers, or a convergent sequence of rationals with a prescribed rate of convergence. On the other case we can*not* promise to pick a sequence that will contain *some* zero (at an unspecified place) — such a promise would mean nothing because we could not tell after each choice, whether we had kept our promise 'so far'. The promise' is compatible with *any* finite initial segment of choices, and finite initial segments are all we have to go on to decide whether the promise has been kept or not. Nor, for the same reason, does it make sense to promise to choose a convergent sequence of rationals *without* specifying the rate of convergence. The species of all sequences that contain at least one zero, and that of all convergent sequences do not form spreads.

An ordinary free-choice sequence is thus gradually determined by selecting for each number $n \geq o$ its n^{th} *stage*; this consists of a *number* (the n^{th} term $\alpha(n)$) and a *spread* (the n^{th} spread S_n, or $S_n(\alpha)$ if there is a risk of ambiguity). They must satisfy the following two requirements: (1) the sequence $(\alpha(o).....,\alpha(n))$ belongs to the nth spread (2) the n^{th} spread must have been shown to be included in the n-1st (for $n > o$). It is from amongst (equivalence classes of) these somewhat complicated objects that intuitionistic analysis selects its real numbers. *Whatever can be asserted of a free-choice sequence* (the prefix 'ordinary' or 'Brouwer' will henceforth be suppressed) *can be asserted after a finite number of stages*. But after a finite number n of stages all we have done is confine it to a spread (namely the intersection of the spread S_n with the spread of all extensions of $(\alpha(o), ...,\alpha(n))$. Thus the old schema (1) gives way to

$$A(\alpha) \rightarrow (\exists f)\,(f \text{ is a spread } \&\ (\forall\beta)\,(\beta\varepsilon f \rightarrow A(\beta))\ \&\ \alpha\varepsilon f) \qquad (2)$$

(Kreisel [12] p. 135, axiom 2.521), where again $A(\alpha)$ contains no Greek letters free other than α, and where $\beta\varepsilon f$ is short for $(\forall n)$ $[f(\{\beta(o)\}^\frown\{\beta(1)\}^\frown....^\frown\{\beta(n)\})=0]$.

(For a possible restriction on (2) (to extensional A) see the very end of the present paper. This restriction will not affect the validity of any of our arguments.)

The question arises as to what axioms, other than (a possibly restricted) (2), we are to postulate for these objects. There are two axiom-systems extant, that of Kleene's book [10] and that of Kreisel's so far unpublished Stanford report (of which an outline is available in [12], pp. 133-143). (We find Heyting's axiomatization in [7] pp. 163-165 so lacking in rigor that we are loth to regard it as a system; contradictions remain even after the trivial inconsistency between axioms 12.1 and 12.12 is repaired by stipulating that q be non-empty. However, we find the whole of § 12 of [7] extremely suggestive (that was where we first encountered a clearly stated notion of 'stage' such as we used above) and some of the formalizations of analysis that we intend to construct on the basis of the present paper will be much closer in spirit to [7] than to [10] or Kreisel's Stanford report). We shall concentrate our attention on Kreisel's system, because it is more powerful than Kleene's and makes explicit the distinction between free-choice sequences and (computable) functions, which Kleene's does not. (In particular, we do not see that the notion of a spread is definable in Kleene's system: certainly his definition ([10] p. 56) does not accord with Brouwer's intention since it implies (cf. [10] p. 167, *R 14.9 with $\alpha = \beta$) that every free-choice sequence is sole member of some spread, which should be true only for free-choice sequences whose n^{th} term is given by a rule.) However (to avoid complicated formulas) we shall suppose Kreisel's system supplemented by a notation for functionals, i.e. mappings F from free-choice sequences to integers and mappings Φ from free-choice sequences to free-choice sequences.

The crucial axioms of Kreisel's system other than (2) relate to the interpretation of prefixes $(\forall x)(\exists y)$, $(\forall \alpha)(\exists x)$, $(\mathrm{A}\alpha)(\exists \beta)$ etc. Recalling our informal explication of the quantifiers, we see that we can assert $(\forall x)(\exists y)A(x,y)$ if and only if we know a method given the x to find the y. That is, there is a function or functional to compute the y from the x and we have axioms of choice:

$$(\forall x)(\exists y)A(x,y) \rightarrow (\exists f)(\forall x)A(x,f(x)) \quad (3)$$

$$(\forall \alpha)(\exists x)A(\alpha,x) \rightarrow (\exists \mathbf{F})(\forall \alpha)A(\alpha,\mathbf{F}(\alpha)) \quad (4)$$

$$(\forall \alpha)(\exists \beta)A(\alpha,\beta) \rightarrow (\exists \Phi)(\forall \alpha)A(\alpha,\Phi(\alpha)) \quad (5)$$

where f is a variable for (computable) functions, and where A contains no additional free-choice variables as parameters. (There is a like axiom for the prefix $(\forall \alpha)(\exists f)$: cf. Brouwer [1], p. 253, second paragraph: we shall not discuss it further because it is entirely analogous to (4) and does not raise the problems connected with (5) which will be crucial below.) Our central question is what further we can say about the **F** and Φ of (4) and (5). This hinges on what is meant by being 'given' a free-choice sequence. We clearly never are 'given' a *whole* free-choice sequence, but only an initial segment ($\alpha(0),...,\alpha(n)$) together with the spreads $S_0,...,S_n$; thus the x in (4) and the β in (5) must depend only on these. This requires that in a suitable sense **F** and Φ must be continuous. We maintain that Kleene and Kreisel have slightly misunderstood the sense of 'continuous' which is appropriate, and that their misunderstanding has rendered it impossible for them to formulate certain arguments of Brouwer (cf. Heyting [8] pp. 114-120, Kleene [10] pp. 174-176, and references therein given). (A hint of where we are going is contained in the observation that Heyting's continuity axiom 12.22 ([7] p. 164) requires (roughly) x in (4) to depend continuously on the *stages* of α (including the S_i) whereas the corresponding theorem in Kleene ([10] p. 73, *27.2) and axiom in Kreisel ([12] p. 140, 2.6211) requires continuous dependence only on the *values* $\alpha(i)$. We shall argue that for *extensional A* (which are the only kind occurring in the systems of [10] and [12]) the misunderstanding makes no difference and (4) with an **F** continuous in the ordinary (Kreisel) sense can be justified anyhow: but that even for extensional A the corresponding strengthening of (5) (Φ continuous in the product topology) cannot be made.)

Kreisel defines the species K of continuous functionals F inductively as follows ([12] p. 140, 2.621):

$$\text{All constant functionals } (\lambda\alpha)n \text{ are continuous.} \tag{6}$$

If F has the property that for each number n we can find a continuous functional $\mathbf{F}_n$ such that

$$(\forall\alpha)[\mathrm{F}(\{n\}^\frown\alpha) = \mathbf{F}_n(\alpha)] \tag{7}$$

then F is continuous.

F is continuous only as required by (6)-(7)

$$[\{n\}^\frown\alpha \text{ in (7) is defined by } \{n\}^\frown\alpha(0) = n, \{n\}^\frown\alpha(x+1) = \alpha(x).] \tag{8}$$

Classically this is equivalent to $\mathbf{F}(\alpha)$ depending on a finite number of values of α. Intuitionistically it gives the added condition that $\mathbf{F}(\alpha)$ can always be calculated in a finite number of steps, which is just what we need (cf. Brouwer [3], p.13 at the top) for formalizing the proof of e.g. the bar theorem. (We regard Kreisel's deduction (in the unpublished Stanford report) of the bar theorem from (4') (=(4) with the added condition $\mathbf{F}\varepsilon K$), even though it is not at all technically difficult, as a truly major advance in the clarification of Brouwer's thought: it transforms his metamathematical argument into a mathematical one, and the most we expect from the present line of thought is that it may perhaps ultimately accomplish the same for the controversial 'historical' proofs begun in [2] and discussed in Chp. VIII of [8] and IV of [10].)

Let us consider whether the functional $\mathbf{F}$ of (4) has to be continuous (in the strong sense of (6)-(8)). There is a distinction here according as A is or is not extensional in α — i.e. whether α appears in A only through its values $\alpha(n)$ or whether the spreads S_n also occur. In the latter (intensional) case there is certainly no need for $\mathbf{F}$ to be continuous: for example, let $A(\alpha, x)$ be

$$\begin{array}{l}\{0\}\varepsilon S_0(\alpha)\ \&\ n = 0 \\ \{0\}\varepsilon S_0(\alpha)\ \&\ n = 1\end{array} \tag{9}$$

where $\{0\}\varepsilon S_0(\alpha)$ means that the spreadlaw of S_0 admits the one-termed sequence whose only term is 0: then the functional $\mathbf{F}$ of (4) is unambiguously defined and non-extensional, i.e. it takes different values for extensionally equal values of the argument; on the other hand, it is easy to prove by induction, using the definition (6)-(8), that every $\mathbf{F}\varepsilon K$ is extensional.

Thus the stronger formula

$$(\forall\alpha)(\exists x)A(\alpha,x) \Rightarrow (\exists\mathbf{F}\varepsilon K)(\forall\alpha)A(\alpha,\mathbf{F}(\alpha)) \tag{4'}$$

(essentially 2.6211 of [12], p. 140) fails for intensional A; however, it is not possible to express such intensional conditions as (9) in the Kreisel (or Kleene) formalism; there free-choice sequence variables α occur only in the context $\alpha(\mathfrak{T}_1)=\mathfrak{T}_2$ so that

$$A(\alpha,n)\ \&\ \alpha=\beta \Rightarrow A(\beta,n) \tag{10}$$

is a provable schema (cf. [10] p. 16, lemma 4.2; here $\alpha=\beta$ abbreviates $(\forall x)(\alpha(x) = \beta(x))$. It consequently becomes important to verify (4′) for extensional A, i.e. A satisfying (10).

What is obvious on our interpretation of quantifiers and of what it means to be 'given' α is this:

If $(\forall\alpha)(\exists x)A(\alpha,x)$, there exists a uniform method μ by which we can compute such an x from any given α using only finitely many values $\alpha(0),\ldots,\alpha(n)$ and finitely many spreads $S_0,\ldots,S_n$.

What we need to show for extensional A is that we do not need to use spreads at all; i.e. there exists a uniform method μ' such that for each α there is a number n' (possibly $> n$) such that we can compute a suitable x using only $\alpha(0),\ldots,\alpha(n')$; such a μ would correspond to an $\mathbf{F}\varepsilon K$. (Also, we must not assume that A extensional in (4) guarantees that $\mathbf{F}$ can be chosen extensional, though this will turn out to be in fact the case: extensionality of A in (5) does *not* guarantee extensionality of Φ, as we shall show by a counter-example later.)

The general idea of the proof is this. We are given a sequence

$$\begin{array}{l} \langle \alpha(0), S_0 \rangle \\ \langle \alpha(1), S_1 \rangle \\ \langle \alpha(2), S_2 \rangle \\ \vdots \\ \vdots \\ \vdots \end{array}$$

and a method μ to obtain an x with $A(\alpha,x)$. Let U be the universal spread (admitting all finite sequences, i.e. the constant function $(\lambda x)0$). Operate with μ on the sequence

$$\begin{array}{l} < \alpha(0), U > \\ < \alpha(1), U > \\ < \alpha(2), U > \\ \vdots \\ \vdots \\ \vdots \end{array}$$

then eventually (after n' steps) we will get a value x'. We know however not yet that $A(\alpha,x')$ but only that $A(\alpha',x')$, where α' is like α except that each of $S_0,\ldots,S_n$, have been replaced by U. If however A is extensional, by (10)$A(\alpha,x')$, q.e.d.

There is some difficulty in adequately formalizing the above argument which can be taken care of only by rigorously developing a theory of 'stages'; this we shall do in a forthcoming continuation of this paper. (A hint as to the difficulty: It is reasonable to accept as an axiom (cf. axiom 12.21 of [7], p. 164. and note that the 'stages' are explicitly mentioned) the existence of a free choice sequence with any prescribed beginning $(\beta(0),S_0,\ldots..,\beta(n),S_n)$; in particular this would enable us, given α and n', to construct an α' with $\alpha'(i) = \alpha(i)$, $S_i(\alpha') = U$ for $i=0,\ldots,n'$. Unfortunately we are *not* given n', but have to compute n' from α' itself!) The remedy is roughly to prove using intensional analogues of (4′) and (6)-(8) that the algorithm μ' will terminate: thus μ' is to be directly defined in terms of μ. This is not hard if we are careful: We first define the species K^* of *continuous intensional algorithms* (c.i.a's) by which we permit the x to be obtained from the sequence $\{<\alpha(i),S_i>\}$; this will contain all constant functionals $(\lambda\alpha)n$ and also all functionals $\mathbf{F}$ such that for each n_0 and S_0 we can find a c.i.a. $\mathbf{F}_{n0,s0}$ for which always

$$\mathbf{F}(<n_0,S_0>{}^\frown\alpha) = \mathbf{F}_{n0,s0}(\alpha)$$

By induction using the clauses (6′)-(8′) (corresponding respectively to (6)-(8) above) in the definition of K^* we fill in the missing step in the above proof, namely (I): For every c.i.a. $\mathbf{F}$, there is another c.i.a. $\mathbf{F}'$ such that $\mathbf{F}'(\alpha)=\mathbf{F}(\alpha'_k)$, where α'_k is like α except that in a certain initial segment $(<\alpha(0),S_0>,\ldots..,<\alpha(k),S_k>)$ the spreads S_i are replaced by U, and the computation of $\mathbf{F}(\alpha'_k)$ uses only the values $\alpha(0),\ldots..,\alpha(n)$, and the spreads $S'_0,\ldots,S'_k$ all $= U$; further, this $\mathbf{F}'$ is continuous on Kreisel's definition.

For (I) to make α'_k must exist; i.e. we require (II). For each free-choice sequence α and each number k there is a free-choice sequence α'_k obtained by replacing each of $S_0(\alpha),\ldots,S_k(\alpha)$ by U. Then

$$\begin{aligned}(\forall\alpha)(\exists x)A(\alpha,x) &\rightarrow (\exists\mathbf{F}\varepsilon K^*)(\forall\alpha)A(\alpha,\mathbf{F}(\alpha))\\ &\rightarrow (\exists\mathbf{F}\varepsilon K^*)(\forall\alpha)(\exists k)A(\alpha'_k,\mathbf{F}(\alpha^{k'}))\\ &\rightarrow (\exists\mathbf{F}'\varepsilon K)(\forall\alpha)(\exists k)A(\alpha'_k,\mathbf{F}'(\alpha))\end{aligned}$$

But if A is extensional $\alpha'_k=\alpha\rightarrow[A(\alpha'_k,\mathbf{F}'(\alpha))\rightarrow A(\alpha,\mathbf{F}'(\alpha))$ and so

$$(\forall\alpha)(\exists x)A(\alpha,x) \rightarrow (\exists \mathbf{F}'\varepsilon K)(\forall\alpha)A(\alpha,\mathbf{F}'(\alpha))$$

which proves (4′), q.e.d. All this requires detailed verification in a formal system.

Briefly: There always exists an **F** satisfying (4) but possibly using the S_i; thus $\mathbf{F}\varepsilon K^*$; if in addition A is extensional, $\mathbf{F}\varepsilon K$, i.e. **F** is continuous in the ordinary (Kreisel) sense (and hence, using (6)-(8), is itself extensional). Since only extensional contexts occur in the systems of Kreisel and Kleene, the presence of the continuity axiom 2.6211 ([12], p. 140) in the former and the continuity theorem *27.2([10], p. 73; also the *bar theorem*, axiom ${}^{\times}$26.3, pp. 54-55) in the latter are justified.

Matters are much subtler with (5). Both Kleene ([10]. p. 73, axiom ${}^{\times}$27.1) and Kreisel ([12], p. 135, axiom 2.511) assert (5) for a Φ continuous in the produce topology, i.e., such that

$$(\forall n)[(\lambda\alpha)(\Phi\alpha)(n)\varepsilon K)].$$

We denote the species of such Φ by K^{∞}.

Evidently this is false for intensional A, but Kleene and Kreisel are not concerned with these. What we claim is that *even for extensional A we do not have*

$$(\forall\alpha)(\exists\alpha)A(\alpha,\beta) \rightarrow (\exists\Phi\varepsilon K^{\infty})(\forall\alpha)A(\alpha,\Phi(\alpha)) \qquad (5')$$

In other words, A may be extensional but there may be no extensional Φ which always computes a β with $A(\alpha,\beta)$. (If there were, an argument to be given later in this paper would prove that it must lie in K^{∞}.)

Let us see first why the 'obvious' method for obtaining from a possibly intensional Φ satisfying (5) an extensional and hence continuous one with the same property does not go through. We are given a free-choice sequence

$$\begin{array}{l} <\alpha(0),S_0> \\ <\alpha(1),S_1> \\ <\alpha(2),S_2> \\ \quad\vdots \\ \quad\vdots \\ \quad\vdots \end{array}$$

and want to compute successively the terms $\beta(0)$, $\beta(1)$,... of a sequence β such that $A(\alpha,\beta)$. Proceeding as above (the argument would claim) we first compute $\beta(0)$ by applying the (algorithm for computing the) possibly intensional Φ to (sufficiently many terms of) the 'purified' sequence α', i.e.

$$\begin{array}{l} <\alpha(0),U> \\ <\alpha(1),U> \\ <\alpha(2),U> \\ \quad\vdots \\ \quad\vdots \\ \quad\vdots \end{array}$$

then we compute $\beta(1)$ by applying Φ to perhaps more terms of this sequence, and so on. But $A(\alpha',\Phi(\alpha'))$; $(\lambda\alpha)(\Phi\alpha')\varepsilon K^{\infty}$; and extensionality of A yields

$$\alpha'=\alpha \rightarrow [A(\alpha,\Phi(\alpha))\leftrightarrow A(\alpha',\Phi(\alpha'))]$$

q.e.d.(?).

This argument is not valid because there is no such (Brouwer) *free-choice sequence as* α'. Principle (II) above yields only: For each α and n, there is a sequence α'_n, differing from α only in that each of $S_0,...,S_n$ is replaced by U. Consequently we *can* assert: If $(\forall\alpha)(\exists\beta)A(\alpha,\beta)$, and if A is extensional, there exists $\Phi\varepsilon K$ such that

$$(\forall n)(\exists\beta_n)[A(\alpha,\beta_n)\ \&\ \beta_n=\Phi(\alpha)]$$

This by no means implies $A(\alpha,\Phi(\alpha))$ (trivially, taking $A(\alpha,\beta)$ to be e.g. $(\exists x)(\beta(x)\neq 0)$ and $\Phi\alpha$ to be $\lambda x0$).

Returning to our remark (the crux really of this whole paper) that there is no such (Brouwer) free-choice sequence as α', we recall that whatever can be said about a (Brouwer) free-choice sequence α can be said on the basis of a finite initial segment $\{<\alpha(i), Si(\alpha)>\}$ $i\leq n$. So we can *never* say of such a sequence «All the S_i are $= U$» (i.e. «No restriction will ever be imposed»). Only for absolutely free α can this be asserted; and they do not fall within the range of the quantifier $(\forall\alpha)$ in (5). (In fact we can verify that (5′) holds if the quantifiers are restricted to *only* absolutely free choice sequences;

but again, not if we restrict them to the union of the two species of absolutely free ones and ordinary ones.)

To sum up, we have

$$(\forall\alpha)(\exists x)A(\alpha,x) \rightarrow (\exists \mathbf{F})(\forall\alpha)A(\alpha,\mathbf{F}(\alpha))$$
$$\mathbf{F} \text{ extensional} \rightarrow \mathbf{F} \text{ continuous}$$
$$A \text{ extensional} \rightarrow \mathbf{F} \text{ extensional}$$
$$(\forall\alpha)(\exists\beta)A(\alpha,\beta) \rightarrow (\exists\Phi)(\forall\alpha)A(\alpha,\Phi(\alpha))$$
$$\Phi \text{ extensional} \rightarrow \Phi \text{ continuous (i.e. } \varepsilon K^{\infty})$$

[Proof. (4′) which we have justified implies (cf. [10] p. 89, *27.1′) $(\forall\alpha)(\exists!\beta)A(\alpha,\beta) \rightarrow (\exists\Phi\varepsilon K^{\infty})(\forall\alpha)A(\alpha,\Phi(\alpha))$ (for extensional A). Take $A(\alpha,\beta)$as $\Phi(\alpha)=\beta$, q.e.d. Kleene observes (loc.cit. p. 73) that he knows no statement of (5′) in Brouwer's writings; and van Heijenoort, who has an encyclopedic knowledge of the Dutch papers, has assured me that no such exists; this paper claims to offer a reason why. The lack of (5′) does not impoverish 'classical' intuitionistic mathematics (real function theory) because all we need in practice is the $(\forall\alpha)(\exists!\beta)$ continuity schema.] Finally

A extensional does *not* imply Φ extensional; more precisely we have not yet *established* that A extensional$\rightarrow\Phi$ extensional; the argument that seemed to establish it turned out invalid. Now we proceed to exhibit an actual *counter-example* to

A extensional $\rightarrow \Phi$ extensional
(i.e. to (5′)).

In fact it will be more than a pedantic correction to the axioms of Kleene (×27.1, [11], p. 73) and Kreisel (2.511, [12], p. 135); it will turn out to be essentially connected with the notorious 'historical arguments' of Brouwer ([2]; [8] Ch. VIII; [10] Ch. IV), which have prove so recalcitrant to axiomatization. [Kripke has given an axiom which enables us to formalize them. We shall explain and adopt it later. The (or a) reason it has not been usually adopted is that it contradicts (5′); people had been reluctant to give up (5′) because they felt that if they did they would have to give up (4′) too. Our argument above showed that the reasons which compel us to accept (4′) do not extend to (5′).)

First let us give a typical Brouwer historical argument; the construction of a counter-example to $\neg(\forall x)Fx \rightarrow (\exists x)\neg Fx$ and more specifically the proof (cf. [8], p. 117, Theorem 1) of

$$\neg(\forall\alpha)\,[\neg(\forall x)\,(\alpha(x)=0) \rightarrow (\exists x)\,(\alpha(x)=0)] \tag{11}$$

For every (canonical) real number (generator) $\alpha\varepsilon[0,1]$ define $\Phi(\alpha)$ as follows: $\Phi(\alpha)(n)$ is [computed immediately after $\alpha(n)$ is known; it is] to be 0 until α has been proved to be either rational or irrational; after that it is to be 1. (The words between [ ] seem mysterious and irrelevant in the present context; they will receive their formal justification in my detailed formalized article «Systems of Intuitionistic Analysis» which was referred to above.) We claim that

$$(\forall\alpha)(\alpha\varepsilon[0,1] \rightarrow \neg(\forall x)\,[\Phi(\alpha)(x) = 0]) \tag{12}$$

For we could only assert $(\forall x)[\Phi(\alpha)(x)=0]$ if we *knew* (a) that we could never show α to be rational and (b) that we could never show α to be irrational. But we could only know this if we knew that α was neither rational nor irrational; contradiction.

On the other hand

$$(\forall\alpha)\,(\alpha\varepsilon[0,1] \rightarrow (\exists x)[\Phi(\alpha)(x) = 1]) \tag{13}$$

is certainly false. If there were such an x by the fan theorem there would be a fixed finite number k such that any two (canonical) real numbers (generators) whose first k terms (as the first k terms of two free-choice sequences) agreed would give the same x. Then all we would have to do to determine whether $\alpha\varepsilon[0,1]$ was rational or not would be to compute this first k terms (and the associated terms of $\Phi(\alpha)$). By this time we would have proved «α is rational or irrational» which we could not do without knowing (or at least having discovered a method of finding out) which it was. [This part of the argument again differs from Brouwer's in ways which will be analysed in the forthcoming more formal paper.[So (12) is true and (13) is false; this proves (11).

The initial shock of this odd-looking argument (the prototype of many sophisticated ones in Brouwer's later writings — references in [8], p. 115) is mitigated by the isolation of the underlying principle called *Kripke's schema*; this reads

$$(\exists\beta)_B[(\forall x)(\beta(x)=0)\leftrightarrow\neg A \\ \& \\ (\exists x)(\beta(x)=1)\rightarrow A)], \tag{14}$$

(B restricting β to the complete binary spread), where A may contain parameters. β is obtained by setting $\beta(x)=0$ until a proof of A is obtained, and then $=1$. [Parenthetical remark: Brouwer in [3], p. 4, penultimate paragraph, asserts [14] rather unambiguously with a *bi*conditional in the second conjunct: i.e. if A is true it must eventually be proved (and (16) below can be strengthened to $A\leftrightarrow(\exists n)(\vdash_n A)$). This strengthening (which is not used in those historical arguments of Brouwer which *I* have examined — certainly not in those cited in Heyting [8] or Kleene [10]) raises problems of interpretation. Kleene and Moschovakis in conversation with the writer suggested that this presupposes a *linear ordering of all possible proofs*, so that the apparent reference to time is really a reference to Gödel-numbers of (informal) proofs. All the creating subject has to do while the free-choice parameters of A flow by him is (not to experience the truth of A but) to apply the algorithm of (15) below to all successive strings of symbols (checking references to those parameters against the available constantly-growing information about their stages). I cannot accept this interpretation of $A\rightarrow(\exists x)(\beta(x)=1)$ firstly because it does not accord with Brouwer's language ('experience the truth of A' is his phrase) and secondly because it savors of von Dantzig's attempted rescue of the historical arguments in [4] ,which it seems to me has been conclusively refuted by Kleene himself ([10], pp. 175-176). On the other hand I *can* offer an alternative interpretation of $A\rightarrow(\exists x)(\beta(x)=1)$ (in view of the intuitionistic interpretation of implication this means 'given a proof of A one can find a stage at which it will have been proved') not open to these objections. If proofs are given with so to speak a date attached, we have only to look at this date to find the x! — it depends what constitutes being 'given' a proof. Further examination of this curious question seems idle until more of Brouwer's historical arguments have been analysed: for those cited by Kleene and Heyting only $\rightarrow$ is necessary in the second conjunct of (14), and we shall so state it.] In the historical argument just given (proof of (11)) A is 'α is rational or α is irrational'. As long as we have a sufficiently clear idea of the meaning of 'A has been proved by the n^{th} stage' (written $\vdash_n A$) to justify the three following propositions

$$\vdash_n A \vee \neg \vdash_n A \tag{15}$$

$$(\vdash_n A) \rightarrow A \tag{16}$$

$$(\neg(\exists n) \vdash_n A) \rightarrow \neg A \tag{17}$$

derivation of (14) is a simple exercise. ((17) is interesting: it is called by Kreisel the 'axiom of Christian charity' because it says the only grounds we could have for asserting that a proposition would never be proved are that we already know it to be absurd — and not e.g. that people are too stupid.)

However, when we have gotten over the shock of the 'historical' definition of a free-choice sequence (even of a function, if A in (14) contains no free-choice parameters), there still remains to trouble us, the moment after constructing the highly discontinuous function(al) Φ of (12)-13), Brouwer's appeal to the fan theorem — i.e. to continuity. However *Φ is a functional from free choice sequences to free-choice sequences*, so since we no longer have (5′) we do not require it to be continuous; the functional from α to x, assumed for reductio ad absurdum to exist in (13), is on the contrary an integer-valued function, continuous by (4′). Thus *we have iustified* (4′) *in such a way that Kripke's schema* (14) *can be kept* and the result (not quite the method — that must await our future formalization) of Brouwer's argument remains intact. That is the central result of this paper.

Concluding remarks. (a) For the *formalization* of the resulting theory we can take as a first step either Kreisel's axioms, replacing (5′) by (14); or if we wish a deeper analysis we replace (5′) by (Kreisel's own) axioms (15)-(17) for the 'creating subject' and put into (4′) the condition that A must be extensional. (Also in a few other axioms: either in (2) (axiom 2.521 of [12]) A must be required to be extensional, or else we extend the formalism by the notation $S_x(\alpha)$ with suitable axioms and replace (2) by

$$A(\alpha) \rightarrow (\exists n)(\forall \beta)[(Ax \leq n)(\beta(x) = \alpha(x) \,\&\, S_x(\beta) \equiv S_x(\alpha)) \rightarrow A(\beta)]$$

where $\equiv$ denotes intensional identity.)

(b) Kripke's schema has as a consequence that the *species of computable functions cannot be enumerated by a formula*, i.e. for any formula A (n,x,y) with only the three (numerical) free variables indicated

$$(\exists f) \neg(\exists n)(\forall x)(\forall y)(f(x)=y \leftrightarrow A(n,x,y))$$

For suitable A, this is sometimes called *the negation of* Church's thesis: but this is ill-advised since this formula f ranges over *historically* as well as mathematically defined functions.

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