TENSE LOGIC AND THE LOGIC OF CHANGE (1)

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In his paper "And Next" (2), Professor G. H. von Wright lays down the following axioms for what he calls the T-calculus:

A0 Any set of axioms which yield the complete classical propositional calculus by means of substitution and detachment,
A1 $(p \vee q)\,T\,(r \vee s) \leftrightarrow (pTr) \vee (pTs) \vee (qTr) \vee (qTs)$,
A2 $(pTq) \,\&\, (rTs) \leftrightarrow (p \,\&\, r)T(q \,\&\, s)$,
A3 $p \leftrightarrow p\,T\,(q \vee \sim q)$,
A4 $\sim(pT(q \,\&\, \sim q))$.

The rules for the calculus are detachment, substitution (of T-expressions for variables), and extensionality (the intersubstitutability of provably equivalent T-expressions). A *T-expression* is simply a wff of the T-calculus, that is, either a wff of the propositional calculus, or two T-expressions joined by the connective T, or a truth-function of T-expressions.

The T-calculus is meant to be a formal representation of the logic of change which Professor von Wright sketched in Chapter 2 of *Norm and Action* (3), with an extension to allow representation of descriptions of chains of events leading stepwise off into the future. It is easily shown that this T-calculus has the properties which Professor von Wright had argued informally the logic of change should have (4). In the case of the extension, where any T-expression, not merely wffs of the propositional calculus, are allowed to be connected by T, there has been a change from his previous position (5), however. He once maintained that T should be associative,

(1) This paper is a revision of the first two sections of the *Rudolf Carnap Essay* for 1965, given at UCLA. The results of the original paper are sketched in the third, fourth and fifth paragraph. I wish to thank Professor David Kaplan for his guidance on the original and Professor A. N. Prior for his many valuable suggestions for this revision.

(2) *Acta Philosophica Fennica*, fasc. 18 (1965), pp. 293-304.

(3) New York, The Humanities Press, 1963, pp. 17-34.

(4) *Ibid.*

(5) As reported in a talk given before the Southern California Logic Collo-

i.e. that pT (qTr) $\leftrightarrow$ (pTq)Tr should be a theorem. In the present system, this does not hold and is replaced by (pTq) Tr $\leftrightarrow$ pT (q & r).

On the basis of the intended interpretation given with the informal presentation of the logic of change, it is possible to show that it may be represented as a definitional extension of a tense logic. A T-expression of the form $\varphi T \psi$ is to be read "φ now, and on the next occasion ψ". In the F-calculus, to be described, this is simply φ & Fψ, where F is to be read "on the next occasion". Such a definition satisfied the requirements for the logic of the book but not the extension as earlier presented. Rather than the associativity of T, it gave the theorem which Professor von Wright now employs.

The F-calculus is a modification of the logic of the future presented in *Time and Modality*, by Professor A. N. Prior (6). The language is that of the propositional calculus (translated from Professor Prior's Polish notation) augmented by the one-place connective F (for Professor Prior's F_1). The axioms are a subset of the original set:

A0 Any set of axioms which yield the complete classical propositional calculus by means of substitution and detachment,
A1 $\sim Fp \rightarrow F \sim p$,
A2 $F \sim p \rightarrow \sim Fp$,
A3 $F(p \rightarrow q) \rightarrow (Fp \rightarrow Fq)$.

The rules are detachment, substitution (of F-expressions for variables) and RF: if $\vdash \varphi$, then $\vdash F\ \varphi$.

By adding to this system the definition pTq $=_{df}$ p & Fq, the results of the informally presented logic may be reproduced for the axiomatized T-calculus, in light of the modified position on extended T-expressions which the axioms entail. However, it is now possible to show that the two logics are, in fact, equivalent, for, within the T-calculus, Fp may be represented as (pv $\sim$p) Tp. In the light of Professor von Wright's proof of the completeness of the T-cal-

quium, November 1, 1963. See also note 1, p. 297 of the article described at 2.

(6) Oxford, Oxford University Press, 1957, p. 13.

The modifications were suggested by an unpublished system of Professor Dana Scott, discussed in a talk before the Southern California Logic Colloquium, October 18, 1963.

culus ([7]), the F-calculus is also shown to be complete by this equivalence.

The two systems for the future may be mirrored for the past, yielding complete systems by the same proofs. The logics of past and future may then be combined to yield complete systems which embrace mixed expressions, referring to both the past and the future, as well expressions which go entirely in one direction.

The T-calculus as a definitional extension of the F-calculus

In this proof, it is unnecessary to consider A0 or the rule of detachment as they are the same for both systems. The same is true for the rule of substitution once it is noted that, under the definition, all T-expressions are F-expressions. For the proof of the remaining T-axioms and the rule of extensionality, the following theorems of the basic F-calculus will prove useful. It should be noted, for the proof sketches for these theorems, that extensionality holds everywhere except within the scope of an F, by propositional calculus.

FT1 F(pvq) → (FpvFq)

1.	F((pvq) → (~p→q))	propositonal calculus (pc), RF
2.	F(pvq) → F(~p→q)	1, A3 (p/pvq, q/~p→q), det
3.	F(~p→q) → (F~p→Fq)	A3 (p/~p)
4.	F~p ↔ ~Fp	A1, A2, pc
5.	F(~p→q) → (~Fp→Fq)	3, 4, ext
6.	F(pvq) → (~Fp→Fq)	2, 5, syl
7.	(~Fp→Fq) → (FpvFq)	pc
8.	F(pvq) → (FpvFq)	6, 7, syl

(As the remaining proofs follow the same pattern the steps will be indicated only by the annotation).

FT2 (Fp & Fq) → F(p & q)
FT1(p/~p, q/~q), pc; A1 (p/~pv~q), syl; pc, RF, A3 (p/~(~pv~q), q/p & q), pc; A1, A2, pc; A1 (p/q), A2 (p/q), pc; ext, pc.

([7]) "And Next", section 8, p. 303 f.

FT3 $F(p \& q) \rightarrow (Fp \& Fq)$
pc, RF, A3 (p/p & q, q/p); pc, RF, A3 (p/q & q); pc.

FT4 $(Fp \vee Fq) \rightarrow F(p \vee q)$
FT3 (p/~p, q/~q), pc; A1, A2, pc; A1 (p/q), A2 (p/q), pc; ext, ext; pc, syl; A1 (p/~p & ~q), syl; pc, RF, A3 (p/~(~p & ~q), q/pvq), syl.

FT5 $(Fp \rightarrow Fq) \rightarrow F(p \rightarrow q)$
FT4 (p/~p); pc, RF, A3 (p/~pvq. q/p→q); syl; A1, A2, pc, ext; pc, syl.

FT6 $F(p \leftrightarrow q) \rightarrow (Fp \leftrightarrow Fq)$
pc, RF, A3 (p/p↔q, q/(p→q) & (q→p);
FT3 (p/p→q, q/q→p), syl; A3, FT5, pc; A3 (p/q, q/p), FT5 (p/q, q/p), pc; ext, ext; pc, syl.

FT7 $(Fp \leftrightarrow Fq) \rightarrow F(p \leftrightarrow q)$
pc, RF, A3 (p/(p→q) & (q→p), q/p↔q); FT3 (p/p→q, q/q→p), syl; A3, FT5, pc; A3 (p/q, q/p), A5 (p/q, q/p), pc; ext, ext; pc, syl.

FT6, and FT7, together with RF and the principle of extensionality for the propositional calculus, justify the rule of extensionality for the F-calculus. As noted, it is only necessary to show that intersubstitutivity holds within the scope of F. Suppose, then, $\varphi \leftrightarrow \psi$ is a theorem. It follows $f(\varphi) \leftrightarrow f(\psi)$ is a theorem, where f(′) is any truth function of a single sentence. By RF, $F(f(\varphi) \leftrightarrow f(\psi))$ is a theorem. By FT6 and FT7 together, this is equivalent to $Ff(\varphi) \leftrightarrow Ff(\psi)$. This latter is, therefore, a theorem. The components may be substituted for one another by the extensionality of the propositiona calculus.

Given extensionality and the five equivalences, represented by the ten theses of the F-calculus so far stated, the T-axioms follow immediately from familiar theorems of the propositional calculus:

FT8 $(p \vee q) \& F(r \vee s) \leftrightarrow (p \& Fr) \vee (p \& Fs) \vee (q \& Rr) \vee (q \& Fs)$

which, by the definition, is TA1, follows from an instance of the thesis of distribution of & over v by the equivalence FT1-FT4;

FT9 (p & Fq)&(r & Fs) ↔ (p & r)&F(q & s), i.e., TA2 is derived from a thesis of the commutativity and associativity of & by FT2-FT3;

FT10 p ↔(p & F(q v ~q)), TA3, comes from p ↔ (p &(q v ~q)) by extensionality based on the thesis stating that (q v ~q) and F(q v ~q) are equivalent as both are theorems;

FT11 ~(p & F(q & ~q)), TA4, is from ~(p & (q & ~q)) by a procedure similar to that used for FT10, though now requiring A1-A2.

Finally, the definition of F in terms of T, to be given in the next section of this paper, represents a valid equivalence if T were defined in terms of F:

FT12 Fp ↔ ((q v ~q) & Fp), which is merely another instance of the propositional theorem mentioned in TF10, under the commutativity of &.

The F-calculus as a definitional extension of the T-calculus

To the T-calculus is added the definition of F:
$Fp =_{df} (q \vee \sim q)\, Tp$. As in the previous section, A0, substitution, and detachment need not be considered. For the proof of the remaining axioms and the rule RF, the following theorems will be needed.

TT1 (pTq)v(pT ~q)v(~pTq)v(~pT ~q)
1. (p v ~p) pc
2. (p v ~p)T(q v ~q) 1, A3 (p/p v ~p), pc
3. (pTq)v(pT ~q)v(~pTq)v(~pT ~q)
2, A1(q/ ~p, r/q, s/ ~q), pc

TT2 ~(pTq) → (pT ~q)v(~pTq)v(~pT ~q) TT1, pc

TT3 (pT ~q) → (pTq)
1. (pTq)&(pT ~q) → (p&p)T(q& ~q) A2 (r/p, s/ ~q), pc
2. (pTq)&(pT ~q) → pT(q& ~q) 1, pc, ext
3. ~(pT(q& ~q)) A4

4. $(pT\sim q) \rightarrow \sim(pTq)$ 2, 3, pc

TT4 $pT(q\vee r) \rightarrow (pTq)\vee(pTr)$ A1 (q/p, r/q, s/r), pc, ext

TT5 $\sim((p\&\sim p)Tq)$

1. $(p\&\sim p) \leftrightarrow (p\&\sim p)T(q\vee\sim q)$ A3 (p/p& ~p)
2. $(p\&\sim p) \leftrightarrow ((p\&\sim p)Tq)\vee((p\&\sim p)T\sim q)$
   1, TT4 (p/p& ~p), pc
3. $(p\vee\sim p) \leftrightarrow \sim((p\&\sim p)Tq)\&\sim((p\&\sim p)T\sim q)$ 2, pc, ext
4. $(p\vee\sim p) \rightarrow \sim((p\&\sim p)Tq)$ 3, pc
5. $\sim((p\&\sim p)Tq)$ 4, pc

TT6 $p\&(qTr) \rightarrow (p\&q)Tr$

1. $(pT(q\vee\sim q))\&(qTr) \rightarrow (p\&q)T((q\vee\sim q)\&r)$
   A2 (q/q v ~q, r/q, s/r)
2. $p\&(qTr) \rightarrow (p\&q)Tr$ 1, A3, ext, pc, ext

The F-axioms now follow immediately.

TT7 $\sim((q\vee\sim q)Tp) \rightarrow (q\vee\sim q)T\sim p$

1. $\sim((q\vee\sim q)Tp) \rightarrow (q\vee\sim q)T\sim p)\vee(\sim(q\vee\sim q)Tp)$
   $\vee(\sim(q\vee\sim q)T\sim p)$ TT2 (p/q v ~q, q/p)
2. $\sim(\sim(q\vee\sim q)Tp)$ TT5 (p/q, q/p), pc, ext
3. $\sim(\sim(q\vee\sim q)T\sim p)$ TT5 (p/q, q/ ~p), pc, ext
4. $\sim((q\vee\sim q)Tp) \rightarrow ((q\vee\sim q)T\sim p)$ 1, 2, 3, pc

By the definition of F, this is FA1.

TT8 $((q\vee\sim q)T\sim p) \rightarrow \sim((q\vee\sim q)Tp)$ (FA2)
TT3 (p/q v ~q, q/p)

TT9 $(q\vee\sim q)T(p\rightarrow q) \rightarrow (((q\vee\sim q)Tp)\rightarrow((q\vee\sim q)Tq))$ (FA3)

1. $(q\vee\sim q)T(p\rightarrow q) \rightarrow (q\vee\sim q)T(\sim p\vee q)$ pc, ext
2. $(q\vee\sim q)T(p\rightarrow q) \rightarrow ((q\vee\sim q)T\sim p)\vee((q\vee\sim q)Tq)$
   1, TT4 (p/q v ~q, q/ ~p, r/q), pc
3. $(q\vee\sim q)T(p\rightarrow q) \rightarrow (((q\vee\sim q)Tp)\rightarrow((q\vee\sim q)Tq))$
   TT7, TT8, pc, ext, pc, ext

The rule RF is justified as follows. If φ is a theorem, then, by pc, $\varphi \leftrightarrow (p\vee\sim p)$ is a theorem. By A3 (p/q v ~q, q/p) and pc, $(q\vee\sim q)T(p\vee\sim p)$ is a theorem. Therefore, by extensionality, $(q\vee\sim q)T\varphi$ is a theorem.

Finally, it will be useful to have that the definition of T in terms of

F, as used in the last section, represents a valid equivalence when F is defined in terms of T.

TT10 pTq ↔ (p&((qv ~q)Tq)) TT6 (q/qv ~q, r/q), pc, ext

The completeness of the F-calculus

A T-expression is said to be *elementary* if its propositional components are several occurrences of the same variable, each with or without a prefixed ~, e.g. pT ~pTp. An important meta-theorem of the T-calculus, stated already in *Norm and Action* (8), is that every T-expression is equivalent to an expression in full disjunctive normal form, each of whose truthfunctional components is an elementary T-expression. By applications of A3, it can be brought that all of these expressions are of the same length (contain the same number of T's). A T-expression is said to be a *T-tautology* if its equivalent full disjunctive normal form is a disjunction of all possible conjunctions of elementary T-expressions, one for each variable. That is, if m is the number of distinct variables in the original formula and n the length of the longest T-expression in the original formula, the original formula is a tautology just in case its equivalent full disjunctive normal form has $(2^{n+1})^m$ disjuncts, each containing m conjuncts. Professor von Wright has shown that all T-tautologies are theorems of the T-calculus (9).

Let an F-expression be called *elementary* if it is a propositional variable or the result of prefixing an F to an elementary F-expression. An F-expression may then be said to be *in F-normal form* if all its truth-functional components are elementary F-expressions. By the first ten theses of the F-calculus, it is clear that every F-expression is equivalent to an expression in F-normal form. If m is the number of distinct elementary F-expressions in the F-normal equivalent of a given expression, that given expression is an F-tautology just in case its F-normal equivalent has the value truth on every line

(8) p. 30. see also "And Next", sect. 7, pp. 300-303 and H. N. CASTANEDA, "The Logic of Change, Action, and Norms", *Journal of Philosophy*, LXII, no. 13 (June 24, 1965), pp. 333-4.

(9) See reference at note 7.

of a 2^m line two-valued truth tabel in which the lines are determined by values assigned to the elementary F-expressions in the usual way.

Given an F-tautology in F-normal form, an equivalent expression in full disjunctive normal form may be obtained directly from the truth table. Each disjunct represents a line of the table, and within each of these each conjunct is either a column heading or its negation, depending on whether or not that expression has the value truth for that line. This full disjunctive normal form is, of course, an F-tautology. It may now be converted into a T-expression by FT12 and the definition of T. Further manipulation, using TA3, TA1, and pc, will bring this expression into the form mentioned in the first paragraph of this section.

This resulting formula is clearly a T-tautology, since only tautologies (instances of $p \vee \sim p$) have been added to a tautological formula. This formula is, therefore, a theorem of the T-calculus, by Professor von Wright's completeness results. By the reduction given above, it is, therefore, a theorem of the F-calculus. But its equivalence to the original F-expression is also a theorem of the F-calculus. Therefore, the original F-expression is a theorem of the F-calculus. Hence, the F-calculus is complete in the sense that all F-tautologies are theorems.

The theorems of the F-calculus are exactly the F-tautologies. From the way that F-tautologies are defined it is clear that each has as equivalent F-normal form an instance of a propositional tautology. The three axioms are F-tautologies by this criterion and the rules of inference preserve this property.

The logic of the past

From the T-calculus and the F-calculus it is possible to obtain analogous complete logics of the past. The most direct way is to replace F in the axioms and rules of the F-calculus by P, "on the latest (past) occasion", and similarly T by Y (for Professor von Wright's $\overleftarrow{T}$) ([10]) in the T-calculus. As the future-looking systems and

([10]) "And Next", sect. 9, p. 304.
The choice of Y was suggested by Professor Prior's report of an unpublished system of Professor Dana Scott.

the past-looking ones are formally identical, all of the results for one direction hold also for the other.

It would be useful to combine the F- and the P-calculi and the T- and the Y-calculi. Again, the most direct way is to add to the F-calculus and the T-calculus a mirror image rule (MI) (11): if φ is a theorem of the PF-(YT) calculus, then the result of replacing F (T) by P (Y), or vice versa, throughout φ is a theorem of the PF- (YT-) calculus.

This approach requires also a change in the formation rules to allow expressions which contain both F and P (T and Y). When this is done, a problem of interpretation arises. Within the YT-calculus it can be shown that the following is a theorem, by A1, MI, pc, A3, and extensionality:

$$(pT(pYp))v(pT(pY\sim p))v(pT(\sim pYp))v(pT(\sim pT\sim p))v$$
$$(\sim pT(pYp))v(\sim pT(pY\sim p))v(\sim pT(\sim pYp))v(\sim pT(\sim pY\sim p))$$

In this, the second, fourth, fifth, and seventh disjuncts go against the intended interpretation. The second, for example, says that now p is true and on the last occasion before the next occasion ~p is true. But the last occasion before the next occasion is just now. It is not possible to prove within the YT-calculus, however, that p T(pY ~p) implies a contradiction. Similarly, in the PF-calculus, p and PFp are independent of one another. To retain the interpretation, a new axiom must be added to each system.

The choice of an axiom for the PF-calculus is easy:

A4 $p \rightarrow PFp$

For the YT-calculus several possibilities present themselves. Only one of these, however, is similar in form to any of Professor von Wright's original axioms, the others being only conditionals:

A5 $\sim(pT(qY\sim p))$.

The two new systems are still equivalent. To show this, it is sufficient to show that the new axioms are equivalent as the results for the other axioms and the rules are not affected by the additions.

Within the PF-calculus, the converse of the new axiom is a theorem (the initial 27 theorems are FT1-12, their mirrors images and the mirror images of the axioms):

(11) This rule was suggested by its use in several papers of Professor Prior.

PT28 $PFp \rightarrow p$

1. $\sim p \rightarrow PF \sim p$ A4 ($p/\sim p$)
2. $\sim PF \sim p \rightarrow p$ 1, pc, ext
3. $PFp \rightarrow p$ 2, A1, A2, pc, ext, A1, A2, pc, MI, (p/Fq), ext, pc

The equivalent of TA5, under the definition, follows:

PT29 $\sim(p \& F(q \& P \sim p))$

1. $(p \& Fq \& FP \sim p) \rightarrow (p \& Fq \& \sim p)$ pc, A4, PT28, pc, MI, ext
2. $\sim(p \& Fq \& FP \sim p)$ 1, pc
3. $\sim(p \& F(q \& P \sim p))$ 2, PT2, PT3, pc, ext

For the reduction of the PF-calculus to the YT-calculus, the following theorems are required (as before, the first 24 theorems are TT1-10, their mirror images and the mirror images of the axioms):

YT25 $pTq \rightarrow p$

1. $p \leftrightarrow pT(q \vee \sim q)$ A3
2. $p \leftrightarrow (pTq) \vee (pT \sim q)$ 1, TT4 ($r/\sim q$), pc
3. $pT\,q \rightarrow p$ 3, pc

YT26 $pTq \rightarrow (r \vee \sim r)Tq$

1. $(p \& (r \vee \sim r))T(q \& q) \rightarrow (pTq) \& ((r \vee \sim r)Tq)$ A2 ($q/r \vee \sim r$, r/q, s/q), pc
2. $pTq \rightarrow (r \vee \sim r)Tq$ 1, pc, pc, ext

YT27 $pT(qYr) \rightarrow r$

1. $pT(qYr) \rightarrow (p \& (r \vee \sim r))T(qYr)$ pc, pc, ext
2. $pT(qYr) \rightarrow ((pT(qYr)) \& (rT(qYr))) \vee ((pT(qYr)) \& (\sim rT(qYr)))$ 1, pc, ext, A1 ($p/p \& r$, $q/p \& \sim r$, r/qYr, s/qYr) ext, A2 (q/r, r/qYr, s/qYr) ext, A2 ($q/\sim r$, r/qYr, s/qYr) ext
3. $\sim(\sim rT(qYr))$ A5 ($p/\sim r$), pc, ext
4. $pT(qYr) \rightarrow rT(qYr)$ 2, 3, pc, pc
5. $pT(qYr) \rightarrow r$ 4, YT25 (p/r, q/qYr), pc

YT28 $pT(qYr) \rightarrow pTq$

1. $q \leftrightarrow qY(r \vee \sim r)$ A3 (p/q, q/r), MI
2. $pTq \leftrightarrow pT(qY(r \vee \sim r))$ pc, 1, ext

3. $pTq \leftrightarrow (pT(qYr))v(pT(qY \sim r))$ 2,TT4 (p/q, q/r, r/~r), MI, ext, TT4(q/qYr, r/qY~r) ext.
4. $(pT(qYr)) \rightarrow pTq$ 3, pc

YT29 $(pT(qYr)) \leftrightarrow (p\&r)Tq$

1. $pT(qYr) \rightarrow r$ YT27
2. $pT(qYr) \rightarrow pTq$ YT28
3. $pT(qYr) \rightarrow (p\&r)Tq$ 1, 2, pc, TT6 (p/r, q/p, r/q), pc
4. $(p\&r)Tq \rightarrow r\&(pTq)$ A2 (q/r, r/q, s/q), pc, YT25 (p/r), pc
5. $pTq \rightarrow (pT(qYr))v(pT(qY \sim r))$ pc, A3 (p/q, q/r), MI, ext, A1 (p/q, s/~r), pc, MI, ext TT4 (q/qYr, r/qY ~r), pc
6. $pT(qY \sim r) \rightarrow \sim r$ YT27 (r/~r)
7. $(p\&r)Tq \rightarrow r\&((pT(qYr))v \sim r)$ 5, 6, pc, 4, pc
8. $(p\&r)Tq \rightarrow pT(qYr)$ 7, pc
9. $(pT(qYr), \leftrightarrow (p\&r)Tq$ 3, 8, pc

PA4 follows immediately by substitution (p/qv ~q, q/qv ~q, r/p) in YT29; with the help of pc and ext:

YT30 $p \rightarrow (qv \sim q)T((qv \sim q)Yp)$

Both of these systems are complete. This is most easily seen in the case of the PF-calculus. By virtue of A4 and its converse, every formula in which an F occurs within the scope of P is equivalent to one without this overlay. By MI, the same holds for formulae with P within the scope of F. Formulae without overlay divide into two parts, one purely P-expressions, the other purely F. To make this explicit, the formula may be reduced to full disjunctive normal form and the separation made within this form. Clearly, the whole will be a tautology just in case the F disjunct is or the P disjunct is. Whichever disjunct is a tautology, it is a theorem of the PF-calculus since it is a theorem of the P-calculus or of the F-calculus, by the completeness results for these separately. Within the PF-calculus, this theorem implies the entire full disjunctive normal form, hence also the original mixed formula equivalent to it.

To show the completeness of the YT-calculus, the followign theorems will be needed:

YT31 (pTq)Yr ↔ (pYr)Tq

1.	(pTq)Yr → pTq	YT25, MI, (p/pTq, q/r)
2.	(pTq)Yr → (pv ~p)Yr	YT26, MI, (p/pTq, q/r)
3.	(pTq)Yr → p	1, YT25, syl
4.	(pTq)Yr → (pv ~p)Tq	1, YT26 (r/p), syl
5.	(pTq)Yr → pYr	2, 3, pc, TT6 (q/pv ~p), MI, pc, pc, ext
6.	(pTq)Yr → (pYr)Tq	4, 5, pc, TT6 (p/pYr, q/pv ~p, r/q), pc, pc, ext

The converse is proved in a similar manner.

YT32 (pTq)Yr ↔ (pTq)&(pYr)

1.	(pTq)Yr → pTq	YT25, MI, (p/pTq, q/r)
2.	(pTq)Yr → (pYr)Tq	YT31
3.	(pTq)Yr → pYr	2, YT25 (p/pYr), syl
4.	(pTq)Yr → (pTq)&(pYr)	1, 3, pc
5.	(pTq)&(pYr) → pYr	pc
6.	(pTq)&(pYr) → (pv ~p)Yr	5, YT26 (q/r, MI, syl
7.	(pTq)&(pYr) → (pTq)	pc
8,	(pTq)&(pYr) → (pTq)Yr	6, 7, pc, TT6, MI, (p/pTq, q/pv ~p), pc
9.	(pTq)Yr ↔ (pTq)&(pYr)	4, 8, pc

Every occurrence of a Y within the scope of a T, or vice versa, must be either on the right or on the left. If it is on the right, the formula may be reduced by YT29, or its mirror image, to a purely T, or Y, expression. If it is on the left, YT32 performs a similar function. The remainder of the argument is essentially the same as that for the PF-calculus.

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