A BASIS FOR SET THEORY

Bede RUNDLE

Although there is still some dispute as to whether truths of number theory enjoy the status of logical truths, it is generally thought that, like logical truths, they are in some sense necessary and non-arbitrary. What this necessity consists in is likewise a matter of dispute : for some it is merely that we do not allow them to be false, for others it is rather that, the axioms and/or rules of arithmetic being as they are, we *cannot* allow them to be false. At any rate, whatever the sense and source of this necessity, it is commonly ascribed to arithmetical truths, and almost as commonly denied of propositions of set theory, where there appears to be neither a logical nor even a psychological obstacle to denying even the least controversial postulates. For instance, the number-theoretic proposition, "if a divides b and b divides c, then a divides c" appears on little reflexion to be inescapable and non-arbitrary in a way not emulated even by a set-theoretic principle as elementary as the union axiom — "for any two sets there is a set containing just those elements which are in either" — but confronted with this axiom it is natural to react much as one would to a fiat or legislative pronouncement : "you can say there is (to be) such a set if you wish (provided it coheres with the other postulates accepted), but nothing forces its adoption upon us". Such a reaction is justified, given the customary presentations of set theory, and, in particular, the current elucidations of the notion of *set*. However, it is possible to arrive at a conception of sets in terms of which set-theoretic propositions may appear just as inescapable as those of number theory, a conception which is to some extent implicit in the actual practice of logicians, though not evident in their informal explanations of set-theoretic terms.

Before indicating how I believe set theory should be presented, I shall make a few obvious remarks about arithmetic. In order to understand even the simplest arithmetical truths, it is necessary to have some grasp of the series of natural numbers, which, in terms of the series commonly used, means being able to recite 0, 1, 2, 3, ..., preferably indefinitely, and certainly in the right order. Any truth of arithmetic is a truth concerning the members of this series, from the truths expressed in elementary computations to the general propositions of number theory proper, and both the intelligibility and the necessity of these various truths depend in an obvious way on the series of numbers: the series confers intelligibility, just in the sense that it provides the specific constants and a determinate range for the variables occurring in equations and general laws, whilst necessity, in one of its senses, is a consequence of the fixity of the series. That is, the rule for generating the numbers prescribes their ordering in an unambiguous fashion, fixing once and for all the relative positions of the numbers and therewith their individual and relational properties. Given that this rule leaves no room for repetitions, omissions, or other re-arrangements which might vie with the customary order, there remains no factor allowing of variation which could lead to a change in the truth-value of a proposition once established as true (or false). By way of analogy, consider a system based on the alphabet in its usual order, and featuring a three-place predicate, Bxyz, meaning "x is between y and z". In this system we could formultate such truths as Bgck, $\neg$ Bthp, Bxyz $\rightarrow \neg$ Byzx, $\forall$x(Bxpr $\rightarrow$ x $=$ q), $\exists$x$\exists$y (Bxdi & Bydi & x $\neq$ y), — propositions quite without interest, but nonetheless necessary: as long as the alphabet is regarded as fixed in its familiar order, competing alternatives to the above truths are precluded. Though the two systems agree in this respect, there is a difference between the series of numbers and the series of letters in the alphabet, in that the former proceed by rule rather than by rote: there is nothing comparable to the uniform formation of successors in the ordering of the alphabet, but the order of the individual letters must be committed to memory as brute fact. Indeed, it is the availability of a generating rule prescribing what is to count as a number which is what is fundamental to number theory; the actual generation of numbers according to this rule generally being of importance only at the level of explicit calculations.

Turning now to set theory, we find that at first sight the situation is rather different from that which obtains in arithmetic. Thus historically, set-theoretic axioms were not suggested by reflexion upon the properties and relations holding for members of a given cequence of sets. Again, pedagogically, it is not generally felt nesessary to specify such a sequence with the aim of justifying set-theoretic theses by indicating how they hold when interpreted with respect to this sequence. Failing the provision of a sequence standing to the axioms of set theory in the way that the natural numbers provide the rationale for the axioms of number theory, we are left with a set or sets of axioms most of which are likely to impress by their arbitrariness rather than by their inescapability. Now there is at least one reason why this unfortunate situation may be held to be inevitable, for if we think of sets as (primarily) determined by properties, then, since no listing of properties is ready to hand, no corresponding generation of sets appears to be available. Moreover, Russell's paradox, and others, may make us think twice about the security of this procedure, even if it should prove feasible.

Instead of trying to work with the notion of sets as determined by properties, I propose to abandon this approach as far as possible, and to use a more concrete analogy as the starting point for the elucidation of the concept of set. Texts on set theory generally make no claim to provide a non-trivial characterization of the notion of set, but the unhelpfulness of synonyms, such as "class", and the misleadingness of near-synonyms, such as "collection" and "totality", are freely acknowledged. However, the abstract notion of a set as employed by logicians seems to me to have a close analogue in the notion of a *box*. The analogy of a heap is some sort of an approximation, but a null heap is not easy to grasp, whilst an empty box gives rise to no qualms. Moreover, it is clearer how a box forms a unit, something over and above its "members", the objects in it, than a heap is something over and above the things which constitute it — take everything out of a box and you are left with an empty box, but a box nonetheless, something in its own right; remove everything constituting a heap and nothing remains, but an empty heap would appear to be a non-existent heap. Suppose, now, we were interested in developing a "theory of boxes", a theory concerned with abstractly possible nestings of boxes within boxes. A natural procedure would be to start by specifying a number of individuals, $a_1, \ldots, a_n$, thought of as eligible for containment in boxes. We might then lay down that any one or more of these could be placed in a box, thus: $\{a_1\}, \ldots, \{a_n\}, \{a_1, a_2\}, \ldots, \{a_1, a_n\}, \ldots$, the braces here symbolizing boxes. We could then allow the formation of boxes containing just one box: $\{\{a_1\}\}$, $\{a_1, \{a_2\}\}$, etc., boxes containing two boxes, three boxes, and so on for any finite number of boxes. The abstract system of boxes generated in this way most likely goes beyond the possibilities open to us in any actual construction and manipulation of physical boxes, but certain fundamental features of such boxes are preserved: for instance, no allowance is made for the containment of a box within itself, and each box contains at most finitely many other boxes. Within these natural limitations, this system of boxes enjoys a measure of completeness in its coverage of all boxes of a certain sort; or, at any rate, it could be used to give a clear sense to talk of all boxes in a way which appears intuitively satisfactory.

Let us now ask how this abstract system of boxes differs from a system of sets. It might be suggested that boxes differ from sets with respect to their criteria of identity. Thus, on the one hand, we may have two empty boxes, but there is at most one empty set. On the other hand, whilst one set may be a member of many distinct sets, one box cannot be in a number of distinct boxes. Both these apparent differences may be removed by the reasonable stipulation that boxes, just like sets, are to be accounted the same if their members are the same. In this sense, two boxes are the same if they are both empty, so vacuously have the same members, whilst the same box may be in any number of distinct boxes. However, the disanalogy is not yet entirely removed, since this explanation, though adequate to boxes, takes no account of individuals: one individual can be in a variety of distinct sets, but not in more than one box at a time. There is more than one way of overcoming this discrepancy, but I propose simply to ignore individuals altogether, and rest content with no more than boxes. For mathematical purposes there is no need to allow for individuals in set theory, since nothing of interest can be said about them within such a framework, though if their inclusion is thought desirable, what I have to say about sets calls for only slight modification to meet the case. So, corresponding to the system of box theory taking the empty box as starting point we have the theory of pure sets, sets built up solely from the null set. Not only is the parallel between pure box theory and pure (finite) set theory so close as to involve perhaps no more than a difference in wording, but the procedure which I suggested for forming all arrays of boxes within boxes coincides with the procedure for specifying what are known as "natural" models for set theory. This procedure, restricted to pure sets, can be expressed more precisely by the following conditions ("$\wedge$" symbolizes the null set):

(1) $\wedge$ is a set.

(2) If $Z_1, \ldots, Z_n$ are sets, then so is $\{Z_1, \ldots, Z_n\}$.

(3) Nothing else is a set.

This way of providing a domain of sets related to the axioms of (finite Zermelo-Fraenkel) set theory as the natural numbers are related to the axioms of number theory makes no use of the notion of a set as given by stipulating a property which its members are to satisfy. This does not necessarily mean that the two conceptions of sets are in conflict, but the intensional approach seems to me to be of only secondary interest. It is true that sets are often introduced in purely intensional terms, as when we speak or the set (more commonly: class) of dogs or of red objects, but in its informal use this terminology is generally so much dispensable verbiage: saying that Fido is a member of the set or class of dogs is no more than a ponderous circumlocution for "Fido is a dog", a proposition which does not commit us to the existence of sets in any significant sense (1). Set terminology acquires significance when sets are ascribed a unitary character, when propositions concerning them do not reduce to instances of schemata of first-order logic, and this unitary character is in no way elucidated by appealing to an alleged correspondence between sets and properties. Admittedly, the approach via properties is not without its attractions — e.g., specification of a property provides a rationale for the grouping of objects together in a set — but by and large the present procedure appears more satisfactory. Thus, as long as we confine ourselves to the property approach, the conception of a set as something over and above its members is bound to remain mysterious, whereas the analogy with boxes serves to make intelligible the notion of set involved in the recursive specification given above. Again, the conception of sets as determined by arbitrary properties, though more comprehensive,

(1) There are, of course, everyday uses of "set" which are more than jargon, as in "a set of matching knives and forks", but such sets are not the logician's; nor are those given by collective nouns, e.g., *flock, herd, family, group.*

threatens to be comprehensive to the point of paradox, and in any case the theory of sets as a branch of mathematics makes no use of properties in any unrestricted sense of this term. It is true that we do have the Axiom of Separation, $\forall z \exists y \forall x(x \in y \leftrightarrow x \in z \vee \varphi(x))$, which is said to enable us to form, given a set z, a set y containing just those members of z which have the property φ, but the "properties" appealed to here are confined to those given by conditions expressible in terms of "$\in$" and "$=$". There is no question of using arbitrary descriptions to generate new sets, but the axiom merely allows us to form subsets of sets already secured.

If, then, we take the notion of set as prior to that of property, reverse the usual presentation of set theory by starting with the sets as given by conditions (1), (2) and (3), and consider set-theoretic propositions in the light of this interpretation, we find that the status of such propositions is, in point of necessity and non-arbitrariness, much the same as that enjoyed by those of arithmetic. In a given case it may be not easy to tell whether a proposition holds under this interpretation, but the provision of a determinate domain of sets gives a clear sense to the notion of truth-conditions for such propositions and provides an alternative to the conception of postulates as mere stipulations serving to generate sets. We are not, of course, compelled to adopt the conception of sets here presented, but there may well be acceptable alternatives based on different analogies. The point is rather that some prior specification of sets seems to be called for if there is to be any chance of judging the adequacy of suggested postulates. For the sets yielded by the above conditions a suitable axiom system would be the following:

Extensionality: $\forall x(x \in y \leftrightarrow x \in z) \rightarrow y = z$
Pairing: $\exists y \forall x(x \in y \leftrightarrow x = z \vee x = w)$
Sum: $\exists y \forall x(x \in y \leftrightarrow \exists z(x \in z \wedge z \in w))$
Power Set: $\exists y \forall x(x \in y \leftrightarrow x \subseteq z)$
Separation: $\exists y \forall x(x \in y \leftrightarrow x \in z \wedge \varphi(x))$
Regularity: $x \neq \Lambda \rightarrow \exists y(y \in x \wedge \forall z(z \in y \rightarrow \neg(z \in x)))$
Transitive Hull: $\exists y \forall x(x \in y \leftrightarrow \forall w(\text{trans } w \wedge z \in w \rightarrow x \in w))$

All but the last of these axioms are familiar from standard formulations of Zermelo-Fraenkel set theory without an axiom of infinity. The last, due to Kurt Hauschild (2), enables us to form, for a given set z, its "transitive hull": $Uz \cup UUz \cup UUUz \cup \ldots$. This axiom, along with the other six, can be shown to hold in the intended interpretation, and for our purposes it has the added attraction of enabling us to prove that any model of these axioms in which all sets are finite is isomorphic to the natural model (3). The categoricity, and hence completeness (as to consequences) of this axiom set is not thereby assured, since the isomorphism holds only between models whose sets are finite, but even this limited categoricity constitutes a strong point in favour of the adequacy of these axioms to their intended interpretation. Morevover, as indicated earlier, this interpretation itself leaves nothing to be desired in its coverage of all finite sets constructible from an initially given set.

While there is no difficulty in providing an underlying basis of sets for a theory encompassing individuals, the extension to transfinite sets stretches to breaking-point the analogy of sets with boxes. This is not surprising, but it might well be argued that to treat an infinite totality as a determinate unit, as something more than an indefinite or arbitrarily large totality, just *is* to misapply an analogy with finite sets, however we conceive of these. Even if we do find the notion of an infinite set intelligible, there is still the question of deciding just how far into the transfinite we are going to go, a decision which is not taken merely by agreeing to allow for more than finite sets. Thus, using the functions φ defined

$$\varphi(0) = \wedge,$$
$$\varphi(\alpha + 1) = \text{the power set of } \varphi(\alpha).$$

For λ a limit ordinal, $\varphi(\lambda)$ = the union of the sets $\varphi(\beta)$ for $\beta < \lambda$, we can continually extend our natural model by appropriate choice of λ, adjusting our axiom set to the successive domains by means of suitable axioms of infinity. There is an appareent difficulty here, in that the machinery necessary for generating the requisite ordinals is not generally available until the theory of sets has itself been sufficiently developed, though, just as there is no necessity to present

(2) Kurt HAUSCHILD, "Modelle der Mengenlehre, die aus endlichen Mengen bestehen", *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, Band 9, Heft 1, (1963), pp. 7-12, "trans w" abbreviates "$\forall u \forall v (u \in v \wedge v \in w \rightarrow u \in w)$".

(3) HAUSCHILD, *op. cit.*, pp. 11-12.

arithmetic in a set-theoretic framework, so ordinal number theory allows of an independent development (4). Supposing, then, that we have succeeded in specifying a domain of sets, I wish to emphasize that the necessity of a corresponding set of axioms is in no way impugned by the arbitrariness of the stopping-point of the progression of sets into the transfinite. The necessity for postulating an axiom of infinity has been taken to show the breakdown of attempts to reduce mathematics to logic, but while I do not wish to maintain that such an axiom is within the province of logic, I do not believe that its status is significantly different from that of the other axioms of set theory. It may be possible to represent set-theoretical theses as truths of logic, but the conception of sets which I have been advocating suggests a more immediate parallel with geometry and other systems of pure mathematics. Thus, in setting up a mathematical theory we may proceed by considering certain physical entities and operations in a more or less abstract way, in the sense that we prescind from certain features of the actual or possible phenomena being envisaged. In geometry, for instance, we set aside the physical difficulties which might actually be met in constructing a figure of a given shape and size. In the theory of combinations and permutations we disregard the possibility that certain re-arrangements might be too numerous for anyone to accomplish. For the purposes of theory, possibilities which in practice might be quite real — that objects should be mislaid or decay, that anyone should have the time, energy or ability to perform a certain action, — are ignored. This elimination of possibilities in the idealized theory is the source of the necessity of its propositions, in that the exclusion of certain possibilities means the exclusion of falsifying conditions. It is worth emphasizing that the immunity to falsification enjoyed by such propositions depends on this restriction of possibilities and not on the creation of a new domain of abstract entities. Abstraction leaves the original entities intact, bringing with it a change in viewpoint but not in subject matter. There is no call for additional entities; indeed, no call for any actual entities at all, but it suffices that we should be able to understand the supposition that there should be entities of a

(4) Cf. Gaisi Takeuti, "A Formalization of the Theory of Ordinal Numbers", *The Journal of Symbolic Logic*, vol. 30, (1965), pp. 295-317.

certain kind, and that we should be able to prescind from certain of their features. Indeed, it is perhaps misleading to speak of the conditions defining natural models as providing a means of *generating* sets, since this suggests that some actual construction of sets, whether on paper, in the mind, or elsewhere, is a necessary preliminary. These conditions specify what is to be admitted as a set just as a rule for generating numbers specified what was to count as a number. Any actual application of the rules to construct sets is of only secondary importance, like the use of diagrams in geometry as an aid to understanding. However, this fact does little to modify the problematic character of infinite sets, since there is still the question of whether there *could* be sets of this sort. In the case of finite sets, the ideographical representation given by the brace notation may offer some sort of reassurance to the nominalistically inclined, but no representation of an actual infinite can be provided in this, or it would appear, in any other way. Still, granted that we can attach a coherent sense to the conception of an infinite set, the justification of an axiom of infinity is straightforward : as with the other axioms adopted, it is justified by its holding in any domain conforming to the initial prescription of allowable sets; it functions not as a creative postulate, but as a description of the contents of any such domain. To a large extent it may be arbitrary just what sets we allow beyond finite sets, but this arbitrariness does not extend to the propositions which are true of a domain once fixed.

Trinity College, Oxford Bede RUNDLE