ON THE TRANSFORMATION OF CLOSED SEMANTIC TABLEAUS INTO NATURAL AND AXIOMATIC DEDUCTIONS*

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1. *Preliminary remarks*

In his books and articles E. W. Beth has often remarked that it is always possible to transform a closed semantic tableau for a sequent K/Z, with one formula Z in the consequent, into a natural deduction of a sort very similar to Gentzen's N-systems. With respect to implicational logic, the rules according to which one can perform this transformation are sufficiently clear. But when we go outside of this field the system of rules needed is present in Beth's work only in a rather embryonic form. The reason for this lies in the fact that Beth never presented the method of *deductive* tableaus systematically except for implicational logic, while the conversion of a semantic tableau into a natural deduction in general will consist in adding certain formulas, so that it can be read as a deductive tableau (1). If this is done correctly and systematically, the subsequent transformation of the deductive tableau into a natural deduction is a quite trivial affair. More precisely, a closed deductive tableau *is* a natural deduction, presented in a special graphical arrangement.

It turns out that several readers of Beth's book "Formal Methods", where his semantic tableau-method is most completely explained, do not see how to obtain the natural deduction from the closed semantic tableau in the general case. There are also instances of examples and fragments of such conversions in Beth's works

* My thanks are due to professors A. Grzegorczyk, A. Heyting, R. Montague and dr. J. J. A. Mooy for talks I have had with them about the topic treated in this paper.

(1) The best discussion of deductive tableaus is perhaps found in *Logique inférentielle* etc., which may well have been written later than the corresponding chapters of *Formal Methods*; see note (4).

which, in my opinion, do not always represent the most natural way of doing it. The points on which I deviate from Beth's usage have to do with seemingly very unimportant repetitions of formulas already present in the tableau. For instance, Beth always stressed the repetition of the formula Z in his deductive tableau-rule for the exploitation of a premise U → V; his rule is as follows:

Prem		Concl	
K′		Z	
U → V			
1	2	1	2
	V	U	Z

Read: if (1) U can be deduced from the set (K′, U → V) and (2) Z can be deduced from the set (K′, U → V, V), then (see above the horizontal line!) Z can be deduced from the set (K′, U → V). The rule reads, then, from bottom to top, and states the validity of the original sequent K′, U → V/Z given the validity of the two sequents mentioned in the antecedent. It can be given the following form, which shows its close relation with Gentzen's L-rule FEA:

$$\frac{K', U \to V \vdash U \qquad\qquad K', U \to V, V \vdash Z}{K', U \to V \vdash Z}$$

But when we regard the tableau-rule as a *tactical* rule, then it may be read: In order to deduce Z from the set (K′, U → V), (1) deduce, if possible, U from (K′, U → V) and (2) Z from (K′, U → V, V). If we read it like this, we see that there is no reason for the repetition of Z in subtableau (2). This of course makes a difference to the way in which the (closed) tableau is converted into a natural deduction. When Z is repeated, this means that the formula V is taken as a hypothesis (right-hand figure above the line in the Gentzen-like formulation), and the repeated Z is deduced under this hypothesis:

K′
U → V
.
:
U
———————|
V (hyp)
.
:
Z
———————|
Z (— hyp)

If, however, we take the tableau-rules to be tactical rules and hence do not repeat the Z, then V is justified by an appeal to the *modus ponens* which Beth used in examples of natural deductions:

K′
U → V
.
:
U
V m.p.
.
:
Z

Similarly, one may choose between these two natural deduction schemas:

K′	and	K′	
(Ev)U(v)		(Ev)U(v)	
———————|			
U(p)	(hyp)	U(p)	existential instantiation
.		.	
:		:	
Z		:	
———————|			
Z	(hyp)	Z	

In the first, which is almost Gentzen's N-rule EB in another graphical arrangement, U(p) is introduced as a hypothesis ([2]). Then it should be visible in the natural deduction that Z is first deduced under this hypothesis. In Beth's examples, where U(p) *is* introduced as a hypothesis, this Z is never written down ([3]). But then one could as well take the full step and recognize the schema to the right as the form of natural deduction which corresponds to the tableau-construction. Here U(p) is not a hypothesis at all, but a conclusion following from the premise (Ev)U(v). If one wants to appeal to the schema to the left, then both Z's should preferably be visible in the tableau too, in order to make the subsequent conversion into a natural deduction a completely mechanical procedure requiring no thinking.

Coming now to negation, we find that for purposes of converting a closed tableau into a natural deduction — and in general, if we want to *see* what really happens in the deductive tableau — certain repetitions are desirable which Beth mostly omitted ([4]). This concerns the rule for the exploitation of the form of a potential conclusion $\bar{U}$, as well as the rule — valid only from the classical point of view — that a formula U in the right-hand column can be "conserved" in negative form to the left. The ensuing natural deductions in both cases represent a *reductio ad absurdum* (this is the simplest analysis). Hence the corresponding tactical rules will be:

Prem	Concl
K	$\bar{U}$
insert: U	$\bar{U}$

Prem	Concl
K	U
insert: $\bar{U}$	U

The U under "Prem" in the tableau to the left is a hypothesis, and $\bar{U}$ should be deduced under this hypothesis; then the hypothesis may be withdrawn and $\bar{U}$ asserted unconditionally (not counting

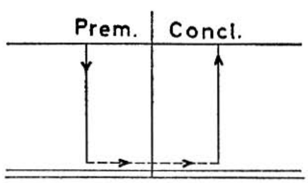
([2]) But in Gentzen's rule EB the K′ is missing; in his L-rule EEA, Γ needs not contain ExFx.

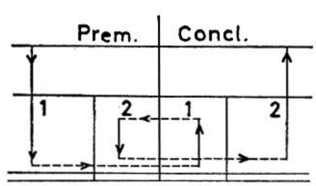
([3]) See for instance *Formal Methods*, p. 141f.

([4]) *Loc. cit.* — To my knowledge, such repetitions are only found in the natural deduction on p. 21 of *Logique inférentielle* etc., but even there they are not taken up in an explicit statement of the tableau-rules.

earlier hypotheses). The repetition of the formula in the right column should therefore not be omitted if one wants a smoothly running natural deduction.

Consistency in the treatment of formulas which are deduced under a hypothesis will make pratical examples easier to make and to follow. Another important point is the notation one chooses for the introduction and withdrawal of hypotheses. I shall use the signs: ————|, ————||, ————|||, and so forth, to be inserted in the natural deduction immediately before the introduction of the first, second, third, ..., hypothesis, and the signs: ————ᴵ, ————ᴵᴵ, ————ᴵᴵᴵ, and so forth, to indicate the withdrawal of the first, second, third, ..., hypothesis. Thereby I think the natural deductions become easier to read. In order to facilitate the conversion of a closed deductive tableau into a natural deduction, these hypothesis-signs will be shown also in the tactical rules for the development of the tableau, see below. If we also take up in these tactical rules the name of the natural rule by which the potential conclusion is eventually justified, then a closed deductive tableau is converted into a completely clear natural deduction by the simple procedure of writing the formulas in the tableau in a vertical arrangement in the order in which they appear along the arrow:





The principles of natural deduction we shall use will these be called:

		Abbreviation	Gentzen
(i)^{D}	the trivial deduction	triv	
(ii^{a})^{D}	modus ponens	mp	FB
(ii^{b})^{D}	conditionalization	cond	FE
(iii^{a})^{D}	distinction of cases	dc	OB
(iii^{b})^{D}	disjunctive weakening	dw	OE
(iv^{a})^{D}	specification of terms	st	UB

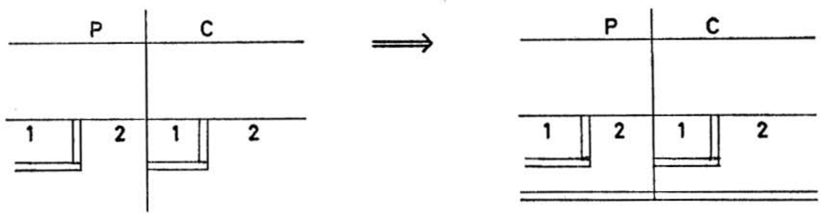
$(iv^b)^D$	conjunctive enumeration	ce	UE
$(v^a)^D$	ex falso sequitur quodlibet	efsq	
$(v^b)^D$	reductio ad absurdum	raa	
$(v^c)^D$	classical (non-intuitionistic) reductio ad absurdum	cl.raa	
$(vi^a)^D$	universal instantiation	ui	AB
$(vi^b)^D$	universal generalization	ug	AE
$(vii^a)^D$	existential instantiation (exposition)	ei	
$(vii^b)^D$	existential generalization	eg	EE

Apart from $(v^c)^D$, the form of the *reductio ad absurdum* which is forbidden from the intuitionistic standpoint, all these principles are valid in intuitionistic logic, as is well known.

In order to obtain complete accordance with Gentzen's N-system in the sentential calculus, we would have to introduce a sign for the absurd proposition (see below, under the treatment of Minimal Calculus). We shall not do that at the moment.

2. *Rules for constructing deductive tableaus*

In the complete list of deductive tableau-rules that follows here, we shall assume that the sets of formulas in *both* columns are ordered. Z, then, always indicates the last formula in the right-hand column, when the form of this formula is unspecified in the rule. We shall shorten "Prem" and "Concl" to "P" and "C". C is the column of formulas which we successively want to obtain as (intermediate) conclusions. P is the column for the premises, hypotheses, and already deduced formulas. K is the class of all formulas under "P" at the moment when we apply the rule, and L the class of all those under "C". K′ and K″ are sub-classes of K, and L′, L″ are subclasses of L; each of these classes may be empty. Ø is the empty class of formulas. The rules are given the form of transformation rules in order to make them easier to read; the double arrow should be read as follows: "A tableau of this form (left-hand tableau-fragment) may be enriched as follows (right-hand tableau-fragment)." On the extreme right is added the ensuing natural deduction, which is valid provided the dotted vertical lines can be filled in validly. We assume the formulas in L′ to be justified already, on the same condition. The double vertical bars between subtableaus (1) and (2) in rule (iii^b)^D indicate that only one of these subtableaus need be closed, then the whole tableau counts as closed:



(i)^D

P	C		P	C		
K′	L′	⇒	K′	L′		K′
Z	Z		Z	Z	triv	Z
K″			K″			K″
						Z triv
						L′

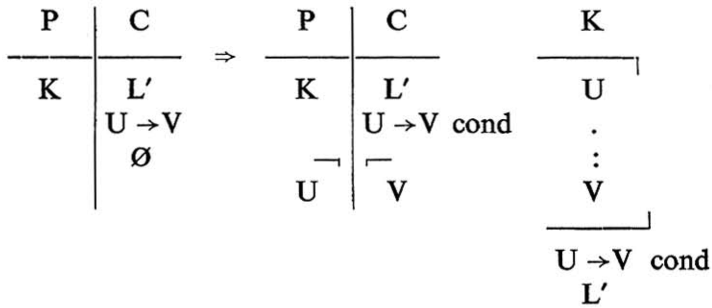
(ii^a)^D

P	C		P	C
K′	L′	⇒	K′	L′
U → V	Z		U → V	Z
K″			K″	

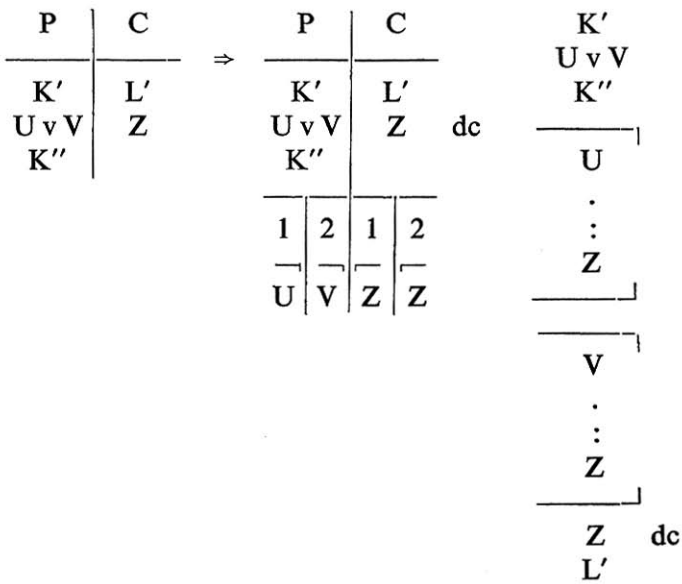
1	2	1	2
	U	Vmp	

K′
U → V
K″
.
:
U
V mp
.
:
Z
L′

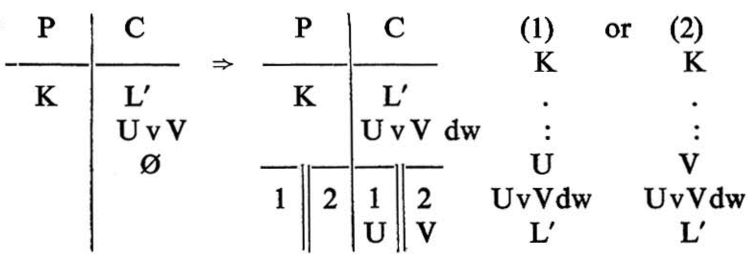
$(ii^b)^D$



$(iii^a)^D$



$(iii^b)^D$



(ivᵃ)ᴰ

P	C
K′	L′
U&V	Z
K″	

⇒

P	C
K′	L′
U&V	Z
K″	
U st	
V st	

K′
U & V
K″
U st
V st
.
:
Z
L′

(ivᵇ)ᴰ

P	C
K	L′
	U&V
	Ø

⇒

P		C	
K		L′	
		U&V ce	
1	2	1	2
		U	V

K
.
:
U
.
:
V
U & V ce
L′

(vᵃ)ᴰ

P	C
K′	L′
$\bar{U}$	Z
K″	

⇒

P	C
K′	L′
$\bar{U}$	Z efsq
K″	
	U

K′
$\bar{U}$
K″
.
:
U
Z efsq
L′

(vᵇ)ᴰ

P	C
K	L′
	$\bar{U}$
	Ø

⇒

P	C
K	L′
	$\bar{U}$ raa
U	$\bar{U}$

K

U
.
:
$\bar{U}$

$\bar{U}$ raa
L′

```
(vᶜ)ᴰ     P | C           P | C              K
        ----+----   ⇒   ----+----          ------┐
          K | L′          K | L′              Ū
            | U             | U   cl.raa      .
            | Ø          ─┐ | ┌─              :
            |             Ū | U               U
                                           ------┘
                                              U    cl.raa
                                              L′
```

```
(viᵃ)ᴰ      P | C               P | C             K′
        ------+----   ⇒   -------+----       (v)U(v)
           K′ | L′             K′ | L′           K″
      (v)U(v) | Z         (v)U(v) | Z           U(p)  ui
           K″ |                K″ |              .
              |           U(p)ui  |              :
                                                 Z
                                                 L′
```

```
(viᵇ)ᴰ    P | C               P | C                 K
        ----+--------   ⇒   ----+------------       .
          K | L′              K | L′                :
            | (v)U(v)           | (v)U(v) ug       U(p)
            | Ø                 | U(p)          (v)U(v) ug
                                                    L′
```

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(viiᵃ)ᴰ      P | C               P | C              K′
        -------+----   ⇒   -------+----       (Ev)U(v)
            K′ | L′             K′ | L′           K″
      (Ev)U(v) | Z        (Ev)U(v) | Z           U(p)  ei
            K″ |                K″ |              .
               |           U(p)ei  |              :
                                                  Z
                                                  L′
```

$(vii^b)^D$	P	C	⇒	P	C	K
	K	L′		K	L′	.
		(Ev)U(v)			(Ev)U(v) eg	:
		Ø			U(p)	U(p)
						(Ev)U(v) eg
						L′

Conditions for the individual constants p:

in $(vi^a)^D$: none. Any number of formulas $U(p_1)$, $U(p_2)$, ..., may be inserted;

in $(vi^b)^D$: p must not occur in K, L′ or in (v)U(v). Only one formula U(p) may be inserted;

in $(vii^a)^D$: p must not occur in K′, (Ev)U(v), K″, L′ or Z. Only one formula U(p) may be inserted (this is sufficient, and also the most natural, but it is not a strictly necessary condition. Also the corresponding semantic rule $(vii^a)^S$ has this clause);

in $(vii^b)^D$: none. Only one formula U(p) may be inserted.

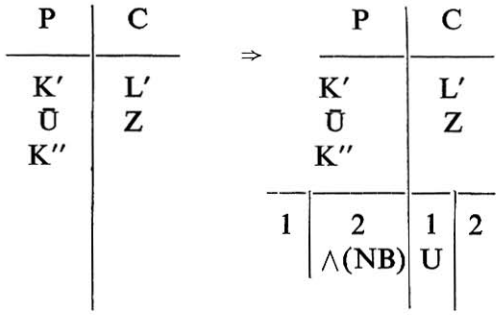
3. *The deductive tableau-form of Minimal Calculus*

Such tactical rules can also be given for the theory of negation of I. Johansson called Minimal Calculus. The form these rules were given by Beth [5] is not entirely adequate for the general form of the problems to be answered by these rules, namely: (a) How do I exploit a premise $\bar{U}$? and (b) How do I exploit the fact that the formula to be deduced has the form $\bar{U}$?

Let $\wedge$ be a propositional constant standing for "the false", "the absurd", or "contradiction". Then the tactical rules for Minimal Calculus will be the following:

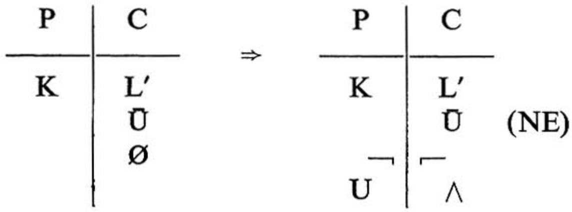
[5] *Formal Methods*, p. 128. In general, when P_1 is a premise and we want to exploit a natural deduction-rule $\frac{P_1, P_2}{Q}$, the tableau-rule will recommend a splitting of the tableau: (1) try first to deduce P_2, and (2) thereafter to deduce Z from K and Q.

$(v^a)^{DM}$



Here "NB" refers to the Gentzen-rule by that name, which clearly justifies our tactics, and which is valid in Minimal logic: from U and Ū we can infer ⋀. So if Ū is among the premises, one should try to deduce U; then ⋀ follows immediately. Thereafter one must try to deduce Z from K′, Ū, K″, and ⋀.

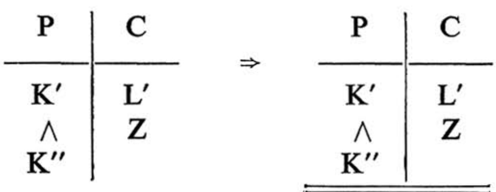
$(v^b)^{DM}$



Here "NE" refers to Gentzen's rule that if from U we can infer ⋀, then Ū may be asserted.

The intuitionistic principle which is excluded by minimal calculus is that from the absurd one can infer anything. As a tactical rule it can be formulated as follows, as a new closure rule:

$(v)^{DI}$



(Gentzen did not give any name to the rule by which we justify Z in this case. It seems reasonable to say that this is the "real" *ex falso.*)

It is easy to see that if this rule is added to $(v^a)^{DM}$, then sub-tableau (2) of the latter closes, and sub-tableau (1) corresponds to the original intuitionistic rule $(v^a)^D$ as we stated it in Par. 2.

Following Beth (6), we have in Par. 2 construed the difference between intuitionistic and classical logic as consisting in the rule $(v^c)^D$, corresponding to the "classical *reductio ad absurdum*". As the basic non-intuitionistic principle for negation one can of course also take: from a double negation $\overline{\overline{U}}$, infer U. As a deductive tableau-rule we then allow:

$$(v)^{DK} \qquad \begin{array}{c|c} P & C \\ \hline K & L' \\ & U \\ & \emptyset \end{array} \quad \Rightarrow \quad \begin{array}{c|c} P & C \\ \hline K & L' \\ & \underline{U} \\ & \overline{\overline{U}} \end{array}$$

By applying the intuitionistically valid $(v^b)^D$ to this latter tableau-fragment, it is easy to see that any "supplantment" of the formula U by a later formula under "C" can be rendered ineffective.

The following well-known axiom-schemas will eliminate applications of these rules:

$$\begin{array}{ll} (v^a)^{DM}: & \overline{U} \rightarrow (U \rightarrow \wedge) \\ (v^b)^{DM}: & (U \rightarrow \wedge) \rightarrow \overline{U} \\ (v)^{DI}\ : & \wedge \rightarrow Z \\ (v)^{DK}\ : & \overline{\overline{U}} \rightarrow U \end{array}$$

(6) In *Formal Methods*, p. 144, and *Logique inférentielle* etc., p. 20. Peirce's Law, which he uses more often, does not fit very well into a system of natural deduction, and the Law of the Excluded Middle as well as the inference from $\overline{\overline{U}}$ to U lead to more complicated tableaus than $(v^c)^D$ does, as one can easily verify.

4. *From semantic tableau to deductive tableau*

Most of the rules Beth gave for developing semantic tableaus (S-rules) resemble the D-rules of Par. 2 for the same logical sign very strongly, although their motivation is different (7). But there are some differences, such as the repetition of certain formulas in the D-rules, which of course has no function in the S-rules. Furthermore we have added certain class-signs K″ in the left-hand column of the rules $(—^{a})^{D}$ and in $(i)^{D}$; but these should also be added to Beth's $(—^{a})^{S}$ - rules in order to facilitate the comparison. The L in the right column of an $(—^{a})^{S}$-rule must of course be understood as the (L′, Z) of the corresponding D-rule. It is important that where the D-rule has the sign for the empty class, Ø, one should insert an L″ in the corresponding S-rule. Then the classes of formulas in the two columns of a semantic tableau may be considered as ordered. Disregarding the signs in the D-rules for the introduction and withdrawal of hypotheses and the names of the rules with which the conclusions are justified, all of which can easily be added to the completed semantic tableau, there is yet the conspicuous difference between $(iii^{b})^{D}$ and $(iii^{b})^{S}$, and between the clauses on $(vii^{b})^{D}$ and $(vii^{b})^{S}$; we shall come back to this presently.

Let us first consider the general case of a formula U in the right-hand column which produces, when treated with the appropriate S-rule, new formulas in the right column, and assume L″ non-empty.

True	False
K	L′
	U
	L″

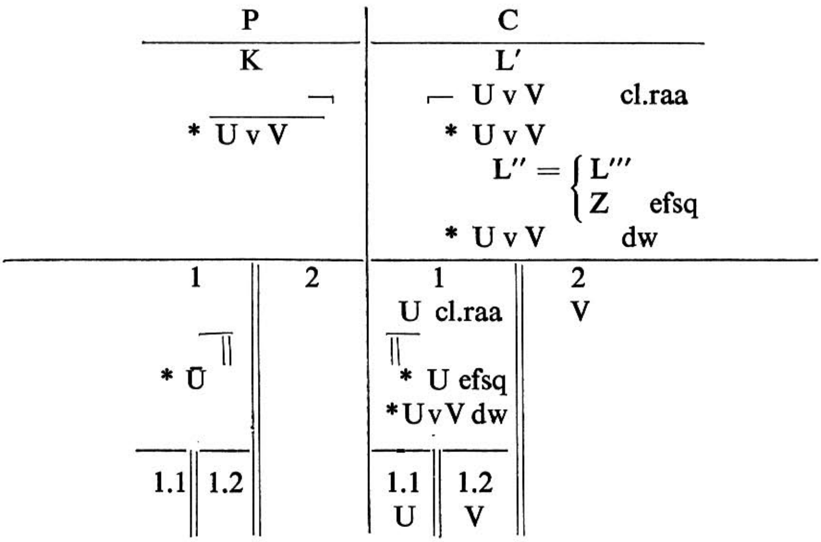
If we "treat" U, according to the rule $(—^{b})^{S}$ for the main connective in U, *after* the introduction of the formulas in L″, then it will be necessary to transform this fragment of the semantic tableau into a fragment of a deductive tableau in the following manner:

(7) *Formal Methods*, pp. 11, 40-44, 49, 55.

	P	C	
	K	L′	
		U	cl.raa
$(v^c)^D$:	$\overline{U}$	U	
		L″ = { L	
		Z	efsq
$(v^a)^D$:		U	

before we "treat" the last formula according to the rule $(-^b)^D$. If we do not do this, the possibility of defending U in the natural deduction we are about to construct is not guaranteed. But if one follows this advice, one can be certain that every application of a rule that leads to the introduction of a new formula in the right-hand column, at the same time will inform us about how to defend the formula just preceding it. That is, assuming the formulas under C to be $Z_1, Z_2, Z_3, \ldots$, the insertion in the tableau of Z_{n+1} goes together with the defense or justification of Z_n.

Some special attention must be paid to the difference between the two rules $(iii^b)^D$ and $(iii^b)^S$ for disjunction. It is clear that, with the aid of the classical principle $(v^c)^D$, we can so to speak construct the classical disjunction-principle $(iii^b)^S$, starting from the intuitionistic principle $(iii^b)^D$:



The sub-tableau (1.2) contains both formulas U and V in the right column, just like the rule $(\text{iii}^b)^S$, and U is "conserved" in the left column (as $\bar{U}$), for later use.

It is now obvious that the following fragment of a (closed) semantic tableau wherein $(\text{iii}^b)^S$ has been applied:

True	False
K	L′
	U v V
	L″
	U
	V

may be transformed into a fragment of a (closed) deductive tableau by adding the formulas in sub-tableau (1.2) above which are preceded by an asterisk.

Finally we must consider the difference between the clauses on $(\text{vii}^b)^D$ and $(\text{vii}^b)^S$. When the latter rule is applied, any number of formulas $U(p_1)$, $U(p_2)$, …, may be inserted (8), since they all have to

(8) This is clearly the intention, considering the equivalence between (Ev)U(v) and $(v)\overline{U(v)}$ in two-valued logic.

be false if (Ev)U(v) is to be false, but the former rule allows for the nisertion of only one such formula. The way in which we transform a semantic tableau-fragment where $(vii^b)^S$ has been used in a deductive tableau-fragment where $(vii^b)^D$ has been employed, is of course quite analogous to the way in which we transform an application of $(iii^b)^S$ into applications of $(iii^b)^D$:

	True	False		P	C
	K	L′		K	L′
		(Ev)U(v)			(Ev)U(v) cl.raa
		L′			
$(vii^b)^S$:		$U(p_1)$	$(v^c)^D$:	*(Ev)U(v)	*(Ev)U(v)
		$U(p_2)$			L″ = { L‴
					Zefsq
			$(v^a)^D$:		*(Ev)U(v) eg
			$(vii^b)^D$:		$U(p_1)$ cl.raa
			$(v^c)^D$:	*$U(p_1)$	* $U(p_1)$ efsq
			$(v^a)^D$:		(Ev)U(v) eg
			$(vii^b)^D$:		$U(p_2)$

Of course, if one of the $U(p_i)$'s in the semantic fragment [or: one of the formulas U and V in the fragment where $(iii^b)^S$ is used] has not contributed to the closure of the semantic tableau, we may start by crossing it out in the semantic fragment, thereby simplifying the construction to the right.

However, in practice one will not as a rule develop a semantic tableau if a natural deduction is wanted, it will be more natural to construct a deductive tableau directly. In order to avoid unnecessary applications of the non-intuitionistic $(v^c)^D$, it is wise to make it a rule always to "treat" Z first with the appropriate rule for the main logical sign in Z, and then such formulas under "P" as do not lead to the introduction of any new formulas under "C". Applications of $(ii^a)^D$ and $(iii^a)^D$ should be postponed as long as possible in order not to complicate the construction unnecessarily.

5. *From deductive tableau to axiomatic deduction*

Beth has shown how a closed semantic tableau can be converted into an axiomatic deduction ([9]). He does this by giving, for each semantic tableau-rule, one or more axiom-schemas which will allow us to perform the deductive step to which the tableau-rule corresponds. Thereby he makes the tableau-rules superfluous one by one, until we are left with an axiomatic system employing only the two inference rules *modus ponens* (henceforth: R1) and the universal generalization of theses (R2). But some of the axioms Beth uses for this purpose are not intuitionistically valid, and for that reason they are not as intimately related to the tableau-rules as one may wish them to be. For the sentential calculus he does list also an intuitionistic axiom system ([10]), which does the same job of making the tableau rules superfluous as the non-intuitionistic system. But the heuristic by which he professes to obtain it is unnecessarily indirect, for he seems to arrive at these intuitionistic axioms by performing certain simplifying manipulations on the more complex axioms he already has constructed for the elimination of his semantic tableau-rules. But the intuitionistic axioms are of course quickly written down once the *deductive* tableau-rules are explicitly stated, and they are all of them well known from several axiomatizations of inferential logic.

For quantificational logic, however, Beth gives us only one axiom system, consisting of four axiom schemas, two of which are not intuitionistically valid ([11]). He proves, first, that if there is a closed semantic tableau for a sequent Ø/Z, then Z is a thesis of his axiomatization of elementary logic. Thereafter he proves that if there is a closed semantic tableau for a sequent K/Z, then there is an axiomatic deduction of Z from K. This proof makes use of the former theorem.

One can prove the latter theorem directly, for any K, without making the roundabout *via* a sequent with K = Ø, for an axiom

([9]) *Formal Methods,* pp. 32f, 41-42-45, 50-52, 55; see also: Semantics as a Theory of Reference, p. 70, pp. 77-81.

([10]) *Formal Methods,* p. 127f.

([11]) *Formal Methods,* p. 50 (*Ax. XIV*) and p. 55 (*Ax. XVI*). Also the axiomatization in "Semantics as a Theory of Reference" contains two non-intuitionistic axioms, but different ones.

system which contains only one non-intuitionistic principle, while using the same very simple inference rules R1 and R2. As the non-intuitionistic principle we may take the axiom shcema $(\bar{U} \rightarrow U) \rightarrow U$, corresponding to the "classical *reductio ad absurdum*", but Peirce's Law, the Law of the Excluded Middle or $\bar{\bar{U}} \rightarrow U$ would of course also be possible choices. — All the axiom schemas in this system are intimately connected with the rules for constructing deductive tableaus (and therefore also with the semantic tableau-rules). We list them along with the numbers of the deductive tableau-rules (D-rules) to which they correspond. The part of the system belonging to sentential calculus is the intuitionistic system referred to above which Beth also mentions, with the additon of $(\bar{U} \rightarrow U) \rightarrow U$.

$$(ii^{b})^{D}: \begin{cases} U \rightarrow (V \rightarrow U) \\ [U \rightarrow (V \rightarrow W)] \rightarrow [(U \rightarrow V) \rightarrow (U \rightarrow W)] \end{cases}$$

$$(iii^{a})^{D}: (U \rightarrow Z) \rightarrow [(V \rightarrow Z) \rightarrow [(U \vee V) \rightarrow Z]]$$

$$(iii^{b})^{D}: \begin{cases} U \rightarrow (U \vee V) & (1) \\ V \rightarrow (U \vee V) & (2) \end{cases}$$

$$(iv^{a})^{D}: \begin{cases} (U \,\&\, V) \rightarrow U \\ (U \,\&\, V) \rightarrow V \end{cases}$$

$$(iv^{b})^{D}: U \rightarrow [(V \rightarrow (U \,\&\, V)]$$

$$(v^{a})^{D}: \bar{U} \rightarrow (U \rightarrow Z)$$

$$(v^{b})^{D}: (U \rightarrow \bar{U}) \rightarrow \bar{U}$$

$$(v^{c})^{D}: (\bar{U} \rightarrow U) \rightarrow U$$

$(vi^{a})^{D}$: $(v)U(v) \rightarrow U(p)$ for all individual constants p

$(vi^{b})^{D}$: $(v)[W \rightarrow U(v)] \rightarrow [W \rightarrow (v)U(v)]$
whereby applications of $(vi^{b})^{D}$ are reduced to applications of R2

$(vii^{a})^{D}$: $(v)[U(v) \rightarrow Z] \rightarrow [(Ev)U(v) \rightarrow Z]$
whereby applications of $(vii^{a})^{D}$ are reduced to applications of $(vi^{b})^{D}$

$(vii^{b})^{D}$: $U(p) \rightarrow (Ev)U(v)$ for all individual constants p

The second and the third from the bottom replace Beth's two non-intuitionistic quantificational axiom schemas in *Formal Methods*. *Proof* that the axiom schema $(v)[U(v) \rightarrow Z] \rightarrow [(Ev)U(v) \rightarrow Z]$ reduces applications of $(vii^{a})^{D}$ to applications of $(vi^{b})^{D}$ (not counting applications of the rules for sentential connectives):

	P			C		
	K′ (Ev)U(v) K″ $(v)[U(v) \rightarrow Z] \rightarrow [(Ev)U(v) \rightarrow Z]$			L′ Z triv		
	1	2		1	2	
$(ii^{a})^{D}$:		$(Ev)U(v) \rightarrow Z$ mp		$(v)[U(v) \rightarrow Z]$ ug		
		2.1	2.2		2.1	2.2
$(ii^{a})^{D}$, $(i)^{D}$:			Z mp		(Ev)U(v) triv	
$(vi^{b})^{D}$:				$U(p) \rightarrow Z$ cond		
$(ii^{b})^{D}$:	U(p)			Z		

The formula U(p) occurs in the left column and the last formula in the right column is Z, just as after an application of $(vii^{a})^{D}$.

Proof that the axiom schema $(v)[W \rightarrow U(v)] \rightarrow [W \rightarrow (v)U(v)]$ reduces applications of $(vi^{b})^{D}$ to applications of R2.

Let W_K be the conjunction of all the formulas in K. Then a closed deductive tableau where $(vi^{b})^{D}$, $(i)^{D}$, $(ii^{a,b})^{D}$, and no other tableau-rules have been used may be enriched as follows for each application of $(vi^{b})^{D}$ (with the formulas that are predeced by an asterisk):

	P			C		
	K $*(v)[W_K \to U(v)] \to [W_K \to (v)U(v)]$			L′ $(v)U(v)$ triv		
	1	2		1	2	
$(ii^a)^D$:		$*W_K \to (v)U(v)$ mp		$*(v)[W_K \to U(v)]$ R2		
		2.1	2.2		2.1	2.2
$(ii^a)^D$, $(i)^D$:			$*(v)U(v)$ mp		$*W_K$ triv	
				$*\ W_K \to U(p)$ $U(p)$	thesis (deduction theorem)	

When this tableau is re-written as if it were a natural deduction from K and this axiom, generalization is applied only to the thesis $W_K \to U(p)$. W_K does not contain p, for p is the same new constant which originally was introduced under $(vi^b)^D$; therefore generalization of this formula with respect to p yields $(v)[W_K \to U(v)]$.

Note that the axiom-schemas which we have used to replace $(vi^b)^D$ and $(vii^a)^D$ are descriptions of those slightly more complicated versions of the D-rules which imply the introduction and withdrawal of a hypothesis, namely:

	P	C
	K	L′ $(v)U(v)$
$(vi^b)^{D'}$:	W_K	$U(p)$

	P	C
	K′ $(Ev)U(v)$ K″	L′ Z
$(vii^a)^{D'}$:	$U(p)$	Z

— in both cases for an "arbitrary" p, i.e., the deduction of Z must be possible for every p. This condition is expressed by the universal quantifier in the antecedent of each axiom. If we use the "hypothetical" versions of $(vi^a)^D$ and $(vii^b)^D$, we are led to these axiom-schemas:

$$[U(p) \to Z] \to [(v)U(v) \to Z]$$
$$[W \to U(p)] \to [W \to (Ev)U(v)]$$

where again there are no clauses on the choice of constants p.

The same sort of considerations can be made about D-rules and axiom-schemas for the sentential calculus. But whether we use the simpler or the more complicated, hypothetical version of a D-rule, both the simple and the hypothetical version of the corresponding axiom-schema can be used to eliminate it. So whichever version we choose, we can now prove the following.

Theorem [12]. A closed deductive tableau for the sequent K/Z, with[out] applications of $(v^c)^D$, can always be converted into a classically [intuitionistically] valid axiomatic deduction of Z from K which may [not] contain applications of the axiom-schema $(\bar{U} \rightarrow U) \rightarrow U$.

Proof. Construct a new deductive tableau for the sequent K^+/Z which one obtains by adding to K the axioms corresponding to each application in the first tableau of the rules $(iii^{a,b})^D$ — $(vii^{a,b})^D$ including $(v^c)^D$, but excepting $(vi^b)^D$. Each axiom should be "treated" (with the rules $(ii^a)^D$, $(ii^b)^D$, and $(i)_D$) at the same level in the tableau on which the corresponding tableau-rule originally was applied. In this new tableau only the rules $(i)^D$, $(ii^{a,b})^D$, and $(vi^b)^D$ are used. Enrich this second tableau with the appropriate axioms $(v)[W_K \rightarrow U(v)] \rightarrow [W_K \rightarrow (v)U(v)]$ and other formulas as explained above, for each application of $(vi^b)^D$. Add the last-mentioned axioms to K^+, the result will be called K^{++}. Now write this enriched second tableau on the vertical form, as if it were a natural deduction of Z from K^{++} (the repetitions resulting from applications of $(i)^D$ can of course be omitted, but only these). In this deduction no other rules than R1, R2, and the conditionalization principle are used. Finally eliminate the applications of the latter, by using appropriate applications of the axiom-schemas listed under the number $(ii^b)^D$, in the usual manner.

— Since every closed semantic tableau can be converted into a closed deductive tableau for the same sequent, it follows that we can always transform a closed semantic tableau into an axiomatic deduction, in such a way that if the semantic tableau shows no "supplantment" of any formula Z_n under "False" by a later for-

(12) See *Formal Methods* pp. 50-52, Theorems 17-18.

mula Z_{n+k}, then the axiomatic deduction makes no use of non-intuitionistic axioms.

(A formula Z_n is supplanted by Z_{n+k}, from the inferential point of view, if the former is treated after the latter in the semantic tableau.)

Therefore, this axiomatization is better suited to show "the fundamental unity of the three methods" (13) — the semantic, the axiomatic and the method of natural deductions.

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(13) *Op. cit.*, Preface, p. XII.

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