DIALECTIC LOGIC

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1. *Dialectic as a chapter of mathematical logic*

The dialectic method of reasoning has been introduced in many ways by many authors, sometimes without mention of the word «dialectic». In some of these formulations dialectic has violated the law of contradiction: for example, «The Tathagata is not to be recognized by the thirty-two marks, because what one said to be the thirty-two marks are told by the Tathagata to be no-marks and therefore to be the thirty-two marks.» (Suzuki [8, p. 45].) The content may differ, but this same «syllogism» is also used by authors who explicitely call such reasoning «dialectic». In these usages the dialectic principle plays the role of rule of inference and the logical contradictions involved block formalization of dialectic within the framework of classical mathematical logic, although a formalization is possible in some types of inconsistent logic. (See, for example, [1].)

Other authors hold that dialectic does not involve rejection of the law of contradiction. Hegel belongs to this group. His argument is that violation of the law of contradiction makes it impossible to disprove any proposition at all: it is impossible to assert anything because statements become indifferent to proof, so to speak [7, p. 9]. Although this is true of classical mathematical logic, one should distinguish between the semantic and syntactic meanings of consistency. A formal system is semantically consistent if some true formulas are provable but their negations are not; a formal system is syntactically (intrinsically) consistent if not all formulas are provable. Thus, a system can be semantically inconsistent (with respect to a given interpretation) while still syntactically consistent [1]. In other words, violation of the law of contradiction does not necessarily imply that statements become indifferent to proof (all provable), although the logical sacrifices one must make to achieve such an end may be considered too high a price to pay.

We shall be concerned here with Hegel's dialectic as developed in the *Encyclopedia* [3] and the *Science of Logic* [4]. (See also [6] and [7].) It is our thesis that, in contrast to the reasoning of the forementioned dialecticians, Hegelian reasoning does not use the dialectic principle as a rule of inference but as a rule of formation. This paper will outline a formalization of such dialectic by means of an applied higher order predicate calculus in which the dialectic principle is conveyed by two «rules of involvement» (see rules 6 and 7, next section). This procedure will show that dialectic — in Hegel's form, at least — is far from constituting that traditional boundary beyond which mathematical logic presumably cannot trespass.

The calculus used has only a finite number of «synthesizing» predicate letters, in order to be in accord with Hegel's contention that the dialectic process is finite in the sense that there are synthetic logical categories that have no negation and hence cannot be further synthesized. The inclusion of an infinite number of synthesizing predicate letters offers no problem, of course. Additionally, since every predicate calculus leads to some kind of formal number theory, a «dialectic» number system is introduced as an extension of the proposed formalization.

2. *Formation rules for a dialectic predicate calculus*

Let us consider a predicate calculus that includes (a) individual variables x_1, x_2, ...; (b) individual constants R_1, R_2, ...; (c) function letters f^1_1, f^1_2, ..., f^1_j, ...; (d) predicate constants A^1_1, A^1_2, ..., ε, S_2, S_3, ..., S_k, A^2_{k+1}, ..., A^3_1, A^3_2, ...; (e) predicate variables A^1_j, B^1_j, ...; (f) propositional connectives $\sim$, $\supset$; (g) quantifiers $\forall$, $\exists$; (h) parentheses and commas.

Formation rules

(1) Individual variables are terms.

(2) If f^1_j is a function letter and $t_1, t_2, ..., t_i$ are terms, then $f^1_j\ (t_1, t_2, ..., t_i)$ is a term.

(3) If t is a term and R_i is an individual constant, then $t\ \varepsilon\ R_i$ is an atomic formula.

(4) If $\mathbf{A}^1_j$ is a predicate constant or a predicate variable whose arguments are (exclusively) terms, and $t_1, t_2, ..., t_i$ are terms, then $\mathbf{A}^1_j\ (t_1, t_2, ..., t_i)$ is an atomic formula.

(5) Every atomic formula is a well-formed formula (*wf*).

(6) If t_1 and t_2 are terms and R_i is an individual constant, then

S_2 $(t_1 \varepsilon R_i, \sim t_2 \varepsilon R_i)$ is a *wf* — abbreviated S_2 (t_1, t_2, R_i).

(7) If P_j is a *wf* involving only predicate letters ε, S_2, ..., S_j $(j<k)$, then S_{j+1} $(P_j, \sim P'_j)$ is a *wf*, where the prime in P'_j indicates that terms and, in particular, individual variables in P'_j are different from those of P_j, although P_j and P'_j have the same individual constant, S_{j+1} $(P_j, \sim P'_j)$ will be abbreviated S_{j+1} (P_j).

(8) If Q^1_m, ..., T^r_s are *i* wfs or terms of types $a_1, a_2, ..., a_i$, respectively, and $\mathbf{A}^1_j$ is a predicate constant or predicate variable of type $(a_1, a_2, \quad, a_i)$, then $\mathbf{A}^1_j$ $(Q^1_m, ..., T^r_s)$ is a wf (for definition of type, see [5, pp. 152-153]).

(9) If A and B are wfs and x is an individual variable, then $\sim A$, $A \supset B$, and $(\forall x)A$ are wfs.

(10) If A is a wf and A^1_j is a predicate variable, then $(\forall A^1_j)A$ is a wf.

(11) These are all the formation rules.

Wfs may be abbreviated as follows: S_3 (S_2) stands for

$$S^3 (S_2(x,y,R_i), \sim S_2(z,w,R_i)).$$

Recursively, S_{j+1} (S_j) stands for

$$S_{j+1} (S_j(...(S_2(x,y,R_i))...)), \sim S_j(...(S_2(z,w,R_i))...))),$$

an expression with N(j) different individual variables, x, y, z, w included — $N_{(j+1)} = 2N_{(j)}$ with N(2) = 2. Wfs that involve other predicate letters may be written using the preceding abbreviations, as follows. $A^4_m(A^1_p(S_2(t_1, t_2, R_i), (\forall A^1_j)A^1_j\ (x_1, ..., x_i), f^2_1\ (x_1, x_n), x_r)$, where, if i designates the type of an indipidual fariable or constant, then S_2 is a second level predicate constant of type $a_1 = ((i, i), (i, i))$ and A^4_m is a fourth level predicate variable of type $a_2 = ((((i,i),(i,i))), (i, ..., i), i, i)$.

3. *Interpretation and syntax*

Let us take a numerical set S as the domain of interpretation for the individual variables, with the constant R_1 associated with a subset of S. The wf $S_2(x_1, x_2, R_1)$ may then be satisfied by some pairs of numbers and not by others. When S is a numerical set, a *dialectic number* relative to R_1 will be any pair (x_1, x_2) that satisfies $S_2(x_1, x_2, R^1)$. For example, if $S = \{0, 1, i\}$ and $R_1 = \{0, 1\}$, then the only dialectic numbers are $(0, i)$ and $(1\ i,)$. Obviously, R^1 must be as sociated with a non-empty proper subset of the numerical domain of interpretation for the collection of dialectic numbers to be non-empty. If wfs have occurrences of ε, S_2, ..., S_k, they cannot be logically valid with respect to arbitrary domains of interpretation (logical validity being defined with respect to individuals, not with respect to predicates: see [5, p. 129] and [2, pp. 308-309]). These wfs are, at most, satisfiable.

The synthesizing predicates S_2, ..., S_k play a role similar to that of ε. They, too, are binary predicates, and if S_2 is satisfied by pairs of individuals (once the interpretation of R_1 is fixed), then S_j is satisfied by what we may call «dialectic numbers of N(j) components», the components in odd places being members of R_1 and the components in even places being members of the complement of R_1 relative to S. Let us assume that R_1 is a numerical field — for example, the field of real numbers R. Let us also assume that the complement of R_1 relative to S is a disjoint numerical field with respect to R_1 — for example, the field Ri (the set of pure imaginary numbers) using arithmetic laws $ai+bi = (a+b)i$ and $ai \cdot bi = (ab)i$. ($R \cup Ri$ is the domain of interpretation). Then, omitting some obvious parentheses, dialectic numbers assume the forms

$$(a_1,a_2 i),\ (a_1,a_2 i,a_3,a_4 i),\ \ldots,\ (a_1,a_2 i,\ \ldots,\ a^{N(k)} i).$$

By defining the arithmetic for dialectic numbers,

$$(a_1,a_2 i)+(b_1,b_2 i) = ((a_1+b_1),(a_2+b_2)i) \text{ and}$$

$$(a_1,a_2 i) \cdot (b_1,b_2 i) = ((a_1 \cdot b_1),(a_2 . b_2)i),$$

and then generalizing, each collection of dialectic numbers of order N(j) constitutes a commutative ring with identity. As an interesting exercise, analyze the various algebraic properties that can be obtained for dialectic numbers by defining their addition and multiplication in different ways.

Given a pair of disjoint numerical fields F_1 and F_2 whose union constitutes the domain of interpretation, the collections of dialectic numbers can be converted into suitable linear algebras over F_1 or F_2.

The essential characteristic of dialectic numbers, as compared to complex and hypercomplex numbers, is that the components of odd and even places are not drawn from the same set but from complementary sets. If dialectic is meant to provide a «synthesis of contraries without erasing their differences», then it is clear that complex numbers are not dialectic in this sense, since they erase differences by drawing components from the same set (imaginary units merely marking the order of the components in notations like $a_1+a_2i_1+\ldots+a_{k+1}i_k$).

As to the syntax of the dialectic predicate calculus and that of what we might call «formal dialectic number theory», these require the establishment of several groups of postulates — axioms, axiom schemas, and rules of inference.

The first group is for the pure predicate calculus of order ω. (See, for example, [5]).

The second group should be for an applied predicate calculus that involves the predicate constants ε, S_2, ..., S_k. But here there are too many existing applications of dialectic that are far from convergent. Even Hegel keeps introducing basic changes until the dialectic process has reached its highest synthesis [3], [4], [7]. For this reason, the formation of the second group of postulates must await the completion of a study of definitions and uses of dialectic from the point of view of mathematical logic (and there is indeed an extensive and bewildering literature to review).

Finally, a third group should set down the formal arithmetic of dialectic number theory. The establishment of this final group would depend on the axioms selected for the second group — the algebraic possibilities of formal number theory being guided by the predicate calculus that forms the nucleus of such theory.

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