ON HOW NOT TO INVALIDATE DISJUNCTIVE SYLLOGISM

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1. *Preliminaries.* I shall say of any theory of entailment, T, that it is *paradox-free* just in case,

first, none of Lewis' paradoxes (1) is a theorem of T.

second, all proofs or arguments sustaining Lewis' paradoxes must employ or presuppose at least one inference-rule or statement or principle incompatible with at least one theorem or rule or principle of T; and

third, every occurrence of '⥽' in formulae expressing the paradoxes, and of 'entails' in T, is interpreted to mean 'entails' in Moore's sense (2).

T will be said to *exclude* the paradoxes if and only if it is paradox-free.

It is evident, I trust, that on the assumption that the paradoxes are false, being paradox-free is only a necessary and not a sufficient condition of the tenability of a theory of entailment.

To claim for entailment a paradox-free analysis is to claim satisfaction (implicitly) of what I call the *invalidation-condition.* The invalidation-condition imposes upon any T, for which the paradoxes do not hold, the necessity of refuting Lewis' independent proofs of them (3). Thus, if T is paradox-free, it is incompatible with at least

(1) The paradoxes: an impossible proposition entails any proposition; a necessary proposition is entailed by any proposition; any two necessary propositions entail each other; and any two impossible propositions entail each other.

(2) For Moore, «*p* entails *q*» is the converse of «*q* follows (or is deducible) from *p*» where *following from* and *being deducible from* are exemplified by the relation in which the conclusion of an argument in *Barbara* stands to the conjunction of its premisses and by the relation in which «*x* is coloured» stands to «*x* is red». It is clear that for Lewis the only difference between strict implication in his sense and entailment in Moore's sense is *terminological.* Indeed Lewis insists that his system of strict implication, S2, gives the ordinary meaning of proof, of inference, and of deducibility. If, then, we are to read S2 as Lewis intended it to be read we should give to '⥽' the 'Moorean' interpretation.

(3) The proofs may be found in C. H. Langford and C. I. Lewis, *Symbolic Logic* (New York, Dover Publications, Inc., 1932), pp. 250-2.

one statement or rule of *any* argument offered as proving the paradoxes true; and among such will be at least one proposition or rule invoked by *Lewis'* arguments.

To claim that a condition is met is not to meet it. The creation of a paradox-free theory does not, of itself, constitute fulfilment of the invalidation-condition. At most it reveals an inconsistency between theorems of the theory and the inference-rules of Lewis' proofs. To meet the invalidation-condition one must establish that the statements of the theory, with which the rules of the proofs are inconsistent, are *true*. This is not always easy. The question is: is it always possible? As anyone knows, who has had a *reductio* argument go sour on him, a statement of the theory initially might be accepted as true only to be rejected upon the discovery that it is inconsistent with some rule of the proofs. The symmetry of *being inconsistent with* allows for, and even encourages, the appeal to one and the same fact as evidence for each of mutually incompatible views. Thus it could happen that one philosopher takes the inconsistency between a paradox-free theory and Lewis' proofs as evidence of the inadequacy of the theory, and another philosopher takes this as showing the invalidity of the proofs. Clearly, both would beg the other's question; and this should provoke us to ask whether and how the risk of *petitio* in such matters can be eliminated. More of this shortly. Let us say for now that an unarguably essential adjunct of any theory claiming to be paradox-free, is an accompanying independent defense of the implied claim that the theory does in fact satisfy the invalidation-condition. Any theory, paradox-free in this way, and whose independent defense is sound, is, as we shall say, *properly paradox-free*. Alternatively, such a theory *properly excludes* the paradoxes.

2. *The Anderson-Belnap Theory.* It is the main purpose of this paper to inquire into a theory of entailment — a joint contribution by Alan Ross Anderson and Nuel D. Belnap, Jr — which purports to be properly paradox-free. While I concede, at least for the moment, that their theory excludes the paradoxes, I have grave doubts about whether their exclusion is proper. I hope to be able to explain my doubts and, perhaps even to persuade the reader to share them.

Our inquiry will be limited to an examination of Anderson's and Belnap's putative refutation of Lewis' independent argument in behalf of the first paradox, the paradox, namely, which asserts that an impossible proposition entails any proposition whatever. For convenience, the argument is here reproduced:

Assume		$p \, . \sim p$	(1)
	(1) entails	p	(2)
	(2) entails	$p \vee q$	(3)
	(1) entails	$\sim p$	(4)
	(3) and (4) entails	q	(5)

Anderson and Belnap are agreed that if, with respect to the relevant truth-functional connectives, the principles of simplification, addition and disjunctive syllogism are valid modes of inference, the argument stands and the first paradox is proved. Since, however, they do not believe the paradox to be true, they are committed (and acknowledge that they are committed) to invalidating at least one of these rules of inference. That is, Anderson and Belnap openly accept the invalidation-condition; the question for us to settle is whether they manage to satisfy it.

Ominously, their decision is to discredit disjunctive-syllogism. Writes Belnap, «We do hold that the inference from $\sim p$ and $p \vee q$ to q is an error: it is a simple inferential mistake. Such an inference commits nothing less than a Fallacy of Relevance» (4). Yet «there is a sense in which the real flaw in Lewis' argument is not a Fallacy of Relevance but rather a Fallacy of Ambiguity: the passage from (2) to (3) is valid only if the 'v' is read truth-functionally, while the passage from (3) and (4) to (5) is valid only if the 'v' is taken intensionally» (5).

Now, at first blush certainly, the repudiation of disjunctive syllogism is at least as self-evidently preposterous as Lewis' first paradox is thought to be. What could be logically more solid than that if at least one of p and q is true and p is not true, then q is true ? Is it possible, one wonders, fully to appreciate how Anderson and Belnap could possibly justify so heroic a stand ?

Central to their position is the claim that the inference from $\sim p$ and (p or q) is valid only for some *intensional* interpretation of 'or' and, not, in general, for its extensional or truth-functional interpretation. Let us understand 'OR' as expressing that intensional sense of 'or' for which disjunctive syllogism (hereafter 'DS') holds, and let us reserve for 'v' its customary truth-functional interpretation. Given

(4) Nuel D. Belnap, Jr.: *A Formal Analysis of Entailment*, Technical report no. 7, Office of Naval Research Contract no. SAR/Nonr - 609 (16) (New Haven: Interaction Laboratory, Yale University, 1960), p. 33.

(5) *Ibid.*, p. 34.

this contrast, we may now add that, in the opinion of Anderson and Belnap [6], $p\,\mathrm{N}\,(p \vee q)$ is true but $p\,\mathrm{N}\,(q \text{ OR } q)$ is not; that is, the principle of addition holds for extensional or material disjunction, but not for its intensional analogue. ('N' is to be read as 'entails'). So, then, their attack upon Lewis' argument exposes it to three possible and allegedly fatal objections:

first, if the proof asserts or implies the truth of both $p\ \mathrm{N}\ (p \vee q)$ and $(\sim p \cdot (p \vee q))\ \mathrm{N}\ q$, then the former is true but the latter is not;

second, if it asserts or implies $p\,\mathrm{N}\,(p \text{ OR } q)$ and $(\sim p \cdot (p \text{ OR } q))$ $\mathrm{N}\ q$, then the latter is true but the former is not;
and

third, if it asserts or implies $p\,\mathrm{N}\,(p \vee q)$ and $(\sim p \cdot (p \text{ OR } q))\ \mathrm{N}\ q$, both are true, but there is a Fallacy of Ambiguity on step (3) of the argument; as a valid consequence of (2) it must be read truth-functionally, but the only legitimate rôle it can play in DS requires it to be read intensionally.

3. *Intensional disjunction.* The whole force of the refutation of Lewis' argument turns on a conception of intensional disjunction for which DS is guaranteed to hold and for which addition is guaranteed to fall. What is there, then, in the meaning of «p OR q» over and above its truth-functional *vis-à-vis*; and what is there about the meaning of 'OR' that resists addition and tolerates DS, and about the meaning of 'v' which simply switches this order of resistance and toleration ?

Notwithstandig their relative inattention to such questions, Anderson and Belnap do make two suggestive comments [7]:

I. If p OR q, then if p were not the case, q would be the case. That is, the truth of any intensional disjunction guarantees the truth of a corresponding subjunctive conditional to the effect that if all but one of the disjuncts were false, the remaining one would be true.

II. Someone asserts that p OR q when he asserts that either p is the case or q is the case but that he does not know which. He cannot mean to assert simply that $p \vee q$ because there are cases in which having asserted that $p \vee q$ solely on the strength of p, the subsequent discovery that p is false would require him to withdraw his assertion of $p \vee q$; in which case he would not be prepared to assert the cor-

(6) *Idem.*

(7) Allan Ross ANDERSON and Nuel D. BELNAP, Jr.: «Tautological entailments», *Philosophical Studies,* XIII (1962), 9-24, p. 22.

responding subjunctive conditional to the effect that *had* q been false p would have been true, in violation of I. Presumably, then, one correctly asserts that p OR q only when knowledge of the truth of $p \vee q$ is claimed independently of the knowledge of the truth-values of p and q.

I interpret I and II as constituting the heart of the attack upon DS: $p \vee q$ does not generally license the corresponding subjunctive conditional that if p where false q would be true. It does not license its corresponding subjunctive conditional because there are situations where the only available grounds upon which to assert that $p \vee q$ is the truth of p. If then, p where to be false, one would not any longer, in those circumstances, be prepared to assert the subjunctive conditional, and this simply because one would not have the warrant to assert that $p \vee q$.

Just such a situation may be thought to obtain within the context of Lewis' argument for the first paradox. Given that $p \cdot \sim p$, we are entitled to move to p; and from p we may move to $p \vee q$. Now where q is an indeterminate, arbitrarily selected proposition, the only grounds we have for materially disjoining it to p is the truth of p. But given $p \cdot \sim p$, we must also move to $\sim p$, in which case we deprive ourselves of the only available grounds upon which to assert the disjunction of p with any arbitrary q. This is just to say that we thereby deprive ourselves of the only available grounds for asserting q — disjunctive syllogism therefore fails.

What, then, is the difference in meaning between 'OR' and 'v'? Simply that p OR q does, but $p \vee q$ does not, license the corresponding subjunctive conditional.

Why is it that 'OR' does not, but 'v' does, tolerate addition? Simply because it is not the case that the truth of the result of disjoining to a given true p any arbitrary q is always knowable independently of knowledge of the truth-values of its disjuncts.

And why is it that 'OR' does, but 'v' does not, tolerate DS? Simply because 'OR' alone generates the corresponding subjunctive conditional (which encapsulates, after all, the whole logical force of DS); 'v', on the other hand, cannot generate the corresponding conditional because in those cases where $p \vee q$ is affirmed solely on the strength of p, one could not say: «$p \vee q$ is true, and p is false, so q is true», just because one could no longer say that $p \vee q$.

4. *Properly paradox-free?* If I am right in thinking that this is what it is for DS to fail, it is an elementary, but gross, error to sup-

pose that susceptibility to such failure, to failure of this ***kind***, establishes the *invalidity* of DS. Nothing we have interpreted Anderson and Belnap as having meant in any way defeats the claim that *if* $p \vee q$ were true and if p were false, then q would be true. At most, it has been shown that where p is the only available grounds for asserting $p \vee q$, then DS cannot establish the *truth* of q simply because in the very application of it I deprive myself of the right to say that my deduction is *sound* (i.e. valid *and* possessed exclusively of true premisses). The compound categorical inference (p, so ($p \vee q$); and since $\sim p$, therefore q) is self-defeating; it cannot be other than unsound. It is unsound because it embodies the categorical assertion of both p and $\sim p$; the premisses of the inference are internally inconsistent. But unsoundness, as Anderson and Belnap should well know, is not invalidity. A valid argument can live with false or inconsistent premisses, a sound argument cannot. Suppose, then, (*per impossible*, of course) that p is true and $\sim p$ is true. The truth of p would provide full authority for asserting $p \vee q$, and the truth of $\sim p$, together with $p \vee q$, would provide full authority for asserting q. So if it were the case that $p \,.\, \sim p$, we should have full authority for asserting the truth of any q we please: This just is Lewis' first paradox — a paradox whose truth is entirely compatible with the unsoundness of Lewis' argument in its behalf.

Anderson and Belnap have, then, confused invalidity with unsoundness — an egregious howler if ever there was one; and the very enormity of the error, in conjunction with the fact that those who make it are logician's of high rank, is liable to produce second thoughts about the correctness of our interpretation of their dismissal of DS. Indeed one might be tempted to say that, strictly speaking, Anderson and Belnap are not alleging the *invalidity* of DS, but rather, if you like, the «systemmatic unsoundness» of any application of it in which the only basis for asserting $p \vee q$ is negated. Such temptations are better resisted. If it is the systemmatic unsoundness of Lewis' argument that bothers Anderson and Belnap, they need not have troubled themselves with DS, for the argument is rendered unsound at the very outset, at its first step, $p \,.\, \sim p$. What is more, strictly speaking that argument is ***not unsound***; it takes the form of a conditional proof, in which the «premise» is not asserted, in which, that is to say, $p \,.\, \sim p$ does not occur as a premiss, but only as an assumption. Any assessment, therefore, in terms of soundness and unsoundness is irrelevant; it misconstrues the very nature of the argument thus assessed.

It is arguable, of course, that if it were true both that p and $\sim p$, it would be true that $p \vee q$. But the joint truth of $p \vee q$ and $\sim p$ would not yield q unless $\sim p$ were to *negate* p, unless, that is, it were not the case both that p and $\sim p$. Since, by hypothesis, it *is* the case that p and $\sim p$, $\sim p$ does not negate p, and q cannot follow from the joint truth of $(p \vee q)$ and $\sim p$.

How is this to be answered ? ([8]) The very application of DS to the conjunction of $p \vee q$ and $\sim p$ presupposes the truth of $\sim (p \,.\, \sim p)$. Yet the first step of the argument gives the contradictory of this. Surely, then, the application of DS is inconsistent with the first step of the proof. I think not; and this because the first step of Lewis' proof is not asserted, even though the truth of the *denial* of the first step, by the application of DS, *is* asserted, by implication at least. So nothing which Lewis' proof commits us to asserting is incompatible with anything else the proof asserts. In applying DS I implicitly assert $\sim (p \,.\, \sim p)$; in supposing or assuming the first step, I suppose or assume, but do not assert, the contradictory of this. My supposing what I suppose is compatible with my asserting what I assert, although *what* I suppose is not compatible with *what* I assert. I do not contradict myself when I say: «Suppose, although it is false, that p»; neither do I contradict myself when I say «Suppose, *per impossible*, that p and $\sim p$.»

In the absence, then, of a viable defense of it, the most to be said of the case of the invalidity of DS is that it merits a Scotch verdict of «Not proven». Because they have not adequately justified their repudiation of DS, Anderson and Belnap cannot maintain satisfaction of the invalidation-condition. As it stands, therefore their theory is not properly paradox-free.

5. *Consistency and Petitio*. I have, in effect, argued that Anderson and Belnap have not *shown* DS to be invalid; but this alone does not allow me to say that I have shown it to be valid. Surely I can do better. I wonder, frankly, whether I can. It can be demonstrated, although I shall not trouble you with such matters here, that, on the basis of principles either explicitly accepted by Anderson and Belnap or derivable from those they accept or so obviously tenable that there could be no good reason why they would not be accepted, the denial of DS begets an explicit self-contradiction. It is here that problems

([8]) For a fuller discussion see John Woods, «The Contradiction-Exterminator», *Analysis*, XXV (1965), 49-53.

arise. Two of these principles, both «so obviously tenable that there could be no good reason why they would not be accepted» are these:

1. If $\sim(p \text{ N } q)$ then $(p \circ \sim q)$. That is, if p does not entail q, then, whatever the correct analysis of consistency might be, p is consistent with the denial of q.
2. If $(p \circ q)$ then, if $\text{M}(p \vee q)$ then $\text{M}(p \,.\, q)$. That is, it is a necessary, but not perhaps a sufficient condition of «p is consistent with q», that if either p or q is logically possible, both are logically possible. Notice that this conception of consistency departs markedly from Lewis'. For Lewis, two propositions are mutually consistent if and only if their conjunction is logically possible. On this view an impossible proposition is inconsistent with every proposition, in violation of our strong intuition that, notwithstanding the impossibility of «7 bachelors are married», it, manifestly, *is* consistent with «At least 5 bachelors are married». If only because it avoids this consequence, the interpretation of consistency which I have offered seems difficult to quarrel with.

Here, then, is my problem. If you give me these two principles, the rejection of DS is open to a conclusive *reductio*. And it seems that you should give me these principles for what, pray, could be more obvious than they ? But if you do allow me their use, you aid and abet me in begging a key question against Anderson and Belnap. For our two principles, straightaway establish the truth of Lewis' paradox, the very paradox that the denial of DS was to rid us of. This is obvious. If $\sim((p \,.\, \sim p) \text{ N } q)$ then, by our first principle, $(p \,.\, \sim p) \; o \sim q$. Then, by our second principle, if $\text{M}(p \,.\, \sim q) \; v \sim q)$ then $\text{M}((p \,.\, \sim p) \,.\, \sim q)$, which, latter is false on inspection; from which follows the truth of $(p \,.\, \sim p) \text{ N } q$.

It is possible, of course, that we might attempt to soften our conception of consistency in an effort to, as it were, «unbeg» the question. Such a sense of consistency would have to capture at least one essential ingredient: if p is consistent with q, then there is nothing in the meaning of 'p' and of 'q' which makes it impossible for p and q to have the same truth-values.

Thus,

2*. If $(p \circ q)$ then $\text{M}(p \equiv q)$. It is a necessary, but not sufficient condition of «p is consistent with q» that the material equivalence of p and q be logically possible.

Surely, then, 2* squares very nicely with our intuitions about consistency, and it certainly does not appear to give us the paradox — $\text{M}((p \,.\, \sim p) \equiv \sim q)$ is obviously true.

Could we not, then, substitute into our *reductio* defense of DS this weakened conception of consistency and thus vindicate DS without begging the issue against Anderson and Belnap ? We could not. In the first place, such a substitution invalidates the *reductio*; the inferences which the stronger sense of consistency generated are flatly invalid for its weaker counterpart. And secondly, whereas the weaker sense of consistency does not immediately generate the first paradox, it does immediately generate the fourth from which the first directly follows. It is obviously not possible that $(p \,.\, \sim p)$ have the same truth-value as $(\sim q \vee q)$, from which it follows that $(p \,.\, \sim p)$ N $(q \,.\, \sim q)$, and hence also that $(p \,.\, \sim p)$ N q.

Both the strong and weak conceptions of consistency generate the paradoxes and render otiose the need for an independent defense of DS. What, then, are Anderson and Belnap to make of this? If they deny the truth of either (2) or (2*) then I want to say that they are not talking about consistency save in some non-standard sense; and if they deny (1), then I want to say that they have ceased talking about entailment in Moore's sense, and hence that their theory of entailment violates condition-three of a paradox-free analysis of entailment. This is what I want to say, but I confess that I do not know how to *prove* what I want to say without begging the question. So perhaps, in the end, we shall have to content ourselves with the earlier verdict of «Not proven».

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