INFINITE CONJUNCTIONS AND DISJUNCTIONS AND THE ELIMINATION OF QUANTIFIERS

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It is a well-known fact that when we have a finite universe of discourse universal quantifiers can be replaced with conjunctions and existential quantifiers can be replaced with disjunctions. Further, quantificational rules, such as the rules of natural deduction, can be replaced with such rules as addition ($p \vdash p \vee q$) and simplification ($p \cdot q \vdash p$) from the propositional calculus (1). This fact leads one to ask whether quantifiers can always be replaced by conjunctions and disjunctions even when the universe of discourse is infinite and infinitely long conjunctions and disjunctions would be required. We will not attempt to give a final answer to this question here, but we will look at some of the problems involved and venture some suggestions in the matter.

Before one can make progress with this question it must be decided whether one is to be concerned with the elimination of quantifiers from the statements of mathematics and logic or with their elimination from empirical statements about the world if, in fact, it turns out that the latter require an infinite universe of discourse. We will here be concerned almost exclusively with mathematics and logic. Only our conclusion that there is no general and categorical objection to the use of infinite truth functions in the elimination of quantifiers will have relevance to the other question. Even when we concern ourselves solely with mathematics and logic, it turns out that there are many different conditions and circumstances which can arise. Much also depends on the purpose one has in setting out to eliminate quantifiers from a given system. This is why we can give no general answer as to the legitimacy or illegitimacy of the process of quantifier elimination, and we will instead examine the various possible contexts one by one.

It is actually quite common for mathematicians and logicians to take conjunctions and disjunctions, both finite and infinite, as basic and to define the quantifiers as abbreviations for these truth func-

(1) A discussion of these points may be found in I. Copi's *Symbolic Logic* or in any similar text.

tions. This is done by writers as various as Frank Ramsey and Abraham Robinson [2]; some do not attach great importance to these conjunctions, but others found all sorts of philosophical and logical positions on the supposed theoretical possibility of eliminating quantifiers in favor of infinite conjunctions and disjunctions. We will be mainly concerned with infinite conjunctions; almost all the same problems arise in the case of infinite disjunctions and they will generally have the same status as the corresponding conjunctions.

In the case of mathematics it is obvious from the outset that there are universally quantified statements which have an infinite range. A simple example would be the statement that every natural number is greater than —1. This would be translated as a conjunction of statements each asserting that a particular natural number has this property. Here the number of conjuncts is denumerable, but even this would not be true if we substitute the corresponding statement about the positive real numbers. Further, it is clear that variables with an infinite range are essential to mathematics and cannot be eliminated. Since we want a logic that can be applied to mathematics there is no hope of getting along with finite conjunctions if we want to eliminate the quantifiers.

The first point to be made is that infinite conjunctions are already present in most logistic systems which contain conjunction at all. The usual sort of recursive definition of a well-formed formula simply tells us how to go about constructing wffs and imposes no limitations on the length of the formulas so constructed. An example of such a definition would be:

All propositional symbols are well-formed.

If A is well-formed then ~ (A) is well-formed.

If A and B are well-formed then (A)·(B) is well-formed.

If we go through the definition an infinite number of times, we will get a formula of infinite length, and there is nothing which prevents us from constructing well-formed infinite conjunctions except our finite life spans and our inability to carry out an infinite number of operations in a finite time. However, it does not seem logically impossible that there might be some person (or angel or superman) who could conjoin two propositions in 1/2 second, a third in 1/4 second, a fourth in 1/8 second, and so on in a convergent series so

(2) Frank RAMSEY, «The Foundations of Mathematics» reprinted in a book of the same title by the same author; Abraham ROBINSON, *Introduction to Model Theory and to the Metamathematics of Algebra,* published by the North Holland Publ. Co., Amsterdam.

that he could construct an infinite conjunction in one second. Similarly, it seems logically possible that someone might be immortal and construct such a conjunction over an infinite period of time. This being the case, the fact that we do not ordinarily meet with infinite conjunctions is a fact about the world rather than a fact of logic.

Even though infinite conjunctions may be contained implicitly in the ordinary sorts of logistic systems, this is not really the point at issue. We can create a system which will contain almost any sort of formulas, but this does not necessarily imply that these formulas can be interpreted in such a way that they will have the desired meaning, even for the rest of logic and mathematics. In particular, those who want to use infinite conjunctions instead of universal quantifiers will have to insist that, on the standard interpretation of a propositional calculus, an infinite conjunction follows the same rules and acts in the same general way that finite conjunctions do. We know the meaning of a finite conjunction of meaningful statements and we must also know the meaning of an infinite conjunction of meaningful statements. Infinite conjunctions must not just be well-formed formulas, but must be mathematically meaningful in a stronger sense.

Since an infinite conjunction is a function which maps the truth-values of the arguments onto a set which again contains only the numbers 1 and 0, this kind of function satisfies the usual definition in terms of ordered n-tuples. However, there is nothing in the standard definition of a function to prevent the n-tuples being infinite and there being an infinite number of such n-tuples, or, in effect, to disallow functions with an infinite number of arguments. In fact, the norm function on Hilbert space is just such function. The introduction of these conjunctions will not generate any new paradoxes in logic or mathematics.

While the crucial question is not the possibility of paradox, it is important to see whether there is a general method of specifying infinite conjunctions. Any finite conjunction makes sense in mathematics and logic because, among other things, there is always a way of specifying the function — making clear the mapping in question. We must now ask whether this is always so in the case of infinite conjunctions.

There are in general two different ways of specifying a function. Since a function is a set of ordered n-tuples the first element of which represents the value of the function while the other elements represent the values of the arguments, the most direct way of specifying a function is via a matrix from which these n-tuples can be read off. In fact, the standard way of specifying a truth function is to use a truth-table which is just such a matrix. But one can also give a rule by which one can determine the value for the function for any given set of values for the variables. When we do this, we are usually specifying one function not absolutely but in terms of some other already known function. Thus, using 1 and 0 as truth values, the truth function, $\sim p$, can be given as 1—p. In the case of most mathematical functions it is usual to give a rule, which might be represented by an expression such as $x^2 + y$. Here the required matrix would be infinite and would not even be denumerable since the function is on the real numbers; hence the matrix cannot in fact be constructed. Nevertheless, we can be assured that such a matrix exists because the components of it are real numbers and we know that it is possible to form n-tuples (in this case ordered triads) of such numbers. It is characteristic of such functions as $x^2 + y$ that there is a regular relationship that holds between the values of the variables and the value of the function, or between the first member of an n-tuple and the other members. Hence we do not really need the matrix to look up the value of the function but can determine it in accordance with the rule.

Let us now consider a set of infinite n-tuples where the non-first elements represent all values which might be given to a set of variables, but where the first elements bear no more than a random relationship to the others. Such a set of n-tuples satisfies the definition of a function just as much as any other set, but there is now no special relationship which holds between the values of the variables and the values which the function takes. Consequently, there is no rule which we can state in a finite way. It then follows that there are functions which we cannot in fact distinguish from one another or specify in any way; in fact, we cannot say anything about them at all except that they exist and that they are not identical with any of the functions which we can specify. Of course, not all functions with an infinite number of arguments are unspecifiable in this way and infinite conjunctions seem much less strange when compared with these «inaccessible» functions.

The important question now is whether one wants to eliminate quantifiers from a particular system of logic or mathematics, or whether one wants to eliminate them everywhere they occur, at least in theory. The difference becomes apparent when we attempt to give a rule for determining the truth value of an infinite conjunction. We might state it as $C(x_1, x_2 \ldots, x_n) = (x_1 . x_2 . \ldots . x_n)$ where «C» represents the conjunction function and the right-hand dots represent multiplication. However, there is implicit quantification over the variables. Similarly, we might say in English that the conjunction has the value «true» if and only if all the conjuncts are true. But here the «all» again involves implicit universal quantification. Thus we run into difficulties if we seek to give a rule and eliminate quantifiers in all languages since our procedure would ultimately be circular. However, this problem disappears if we are concerned only with a particular system. In the language of the system we can define the quantifiers in terms of infinite conjunctions and disjunctions and, when we come to interpret the language, we have a metalinguistic rule for the evaluation of these truth functions. There is then no harm in having irreducible quantifiers in the metalanguage. This situation may explain the fact that philosophers often take a dimmer view of infinite conjunctions and quantifier reduction than mathematicians; the philosophers tend to think of the more ambitious project. As stated in the beginning, the most ambitious project which we will consider is the elimination of quantifiers from systems of mathematics and logic. There will always be an English metalanguage (if not an England) in which quantifiers will remain.

At this point questions as to the legitimacy of infinite conjunctions seem to be closely related to the question of whether infinite conjunctions are meaningful. However, philosophical criteria of meaningfulness will not help us much here. Given a statement in hand one can apply one or the other of the many possible theories and decide whether the statement has meaning, or decide what kind of meaning it has. But logic and mathematics constitute a very special context since the primitives of the systems are not even supposed to have meaning in themselves but are only supposed to be interpretable. Further, one of the main problems about infinite conjunctions, as we will soon see, is that it is not obvious how to construct them in the first place. This is a problem which never occurs to philosophers who promulgate theories of meaning. Thus it does not seem very rewarding to ask whether an infinite conjunction can be meaningful. We have to ask instead whether it will serve its intended purpose.

We can now state four different demands which someone might make of infinite conjunctions before admitting their legitimacy. We will see how far these demands can be satisfied first and later see how important each demand is to persons of various points of view. These demands are as follow:

1) That there be a rule for determining the value of the function given the values of the arguments.
2) That there be a rule for the construction of the conjunction.
3) That there be a rule for constructing a matrix for such a function.
4) That there be rules for proving conjunctions of this sort as theorems in a system and also that there be rules for deriving non-theorems.

We have already discussed the first demand in some detail and have seen that it can be satisfied. We now pass to the second which might be thought to be the most basic of all.

The universally quantified variables of a system have some range which is specified in the interpretation of the system, but this does not necessarily tell us how to construct the conjunction. If the quantifiers of the language range over the natural numbers, for instance, there is no problem in reconstructing such a statement as the one which asserts that all the natural numbers are greater than or equal to 0. We can take the natural numbers in their natural order, let the first conjunct say that 0 is greater than or equal to 0, let the second say the same thing about 1, and so on. If, however, the original variables ranged over the real numbers, there is no such obvious principle to follow.

In constructing a conjunction, or a sequence of any kind, the important thing is to know where to start and, at any point, where to proceed next. I think that this is what we mean by knowing how to do something. Thus it seems reasonable to say that we know how to construct an infinite conjunction if we are told what the first conjunct is and have a rule for the selection of the next conjunct at any given point. Thus the importance of well-ordering is obvious here. If the set of conjuncts can be well-ordered then every subset will have a first element. By taking the set itself as a subset we are assured that it will have a first element and this gives us our first conjunct. We then consider the remaining conjuncts and, since they constitute a subset, we take the first element of that subset as our next conjunct, and so on. We are then in a very good position as far as eliminating quantifiers when the variables range over the natural numbers, the integers, or the rational numbers. Since these sets can all be well-ordered and there will be a conjunct for each number, we can claim that there is a method for constructing the required infinite conjunction. We can also, of course, decide immediately whether any given statement is included in the list of conjuncts.

If we are quantifying over the real numbers the situation is more difficult. We know of no way of well-ordering them and it can be proven that they can be well-ordered only if we assume the choice axiom or an equivalent principle. As long as we do not know how to well-order the real numbers we do not know how to order the conjuncts in an infinite conjunction corresponding to a statement about the real numbers. Hence we do not know how to construct such a conjunction and still would not know how to even if we could perform an infinite number of operations in a finite time. On the other hand, it is not as if we were in ignorance about the contents of such a conjunction. A statement will be one of the conjuncts if and only if it predicates some specified property, F, of a real number. Hence we can give a criterion for membership in the conjunction but not a criterion for constructing the conjunction.

The next demand concerns the construction of matrices. This demand is likely to be made by someone who considers that giving a rule only specifies one function in terms of some other function. It then appears that ultimately some functions must be specified by matrix. One must decide whether conjunction is to be one of these when one decides whether or not to make the demand. A crucial fact here is that a truth table has 2^n rows where n is the number of arguments. Hence no infinite conjunction will ever have a denumerable truth table, and, according to our previous argument, we will know how to construct such a matrix only if we know how to well-order the set. This is not guaranteed even if the choice axiom is assumed. Thus the third demand is largely unsatisfiable.

Let us now look to the last demand. Since a conjunction is a non-thesis if any of its conjuncts are non-theses, it follows that it will be practically possible to find counter-instances to a proposed thesis of this form; one needs only to find one conjunct which is not provable. However, it would be more difficult to prove a thesis of this form as a theorem if we do not just treat an infinite conjunction as an alternative notation for universal quantification and use the usual quantification rules. If one takes the view that infinite conjunctions are really basic and that universally quantified expressions are just abbreviations for them, one has to ask whether, on the one hand, it is sufficient to prove that one arbitrarily chosen conjunct is true and infer that the rest are, or, on the other hand, to prove that each conjunct is true. Much depends here on whether we are thinking of logic or mathematics. It is traditional in mathematics to select an arbitrary member of a set and then generalize the result, as in Euclidean geometry. Even in modern mathematics it is customary to use induction to prove something about an infinite number of numbers in a finite way by generalizing a property of an arbitrarily chosen number in the assumption of the general case. In a logistic system, however, this use of infinite conjunctions has the effect of reducing the predicate calculus to the propositional calculus. This means that the only rule of inference we would have would be modus ponens or something similar, and that only the traditional rules of the propositional calculus would be derivable from it. Hence there would be no way of generalizing from an arbitrarily chosen conjunct, and to prove such a thesis we would have to prove each conjunct. Since we feel that theses of this form are provable in logic, we would then be committed to proofs of infinite length.

The idea of proofs of infinite length may be upsetting if one thinks that there ought to be an effective (finite) method of deciding whether any given wff is a theorem or not. However, one could under some circumstances admit proofs of infinite length and still satisfy this demand. The system might be such that one could in the metalanguage characterize the form of an infinite conjunction, perhaps using metalinguistic quantifiers, without stating the conjunction. One might then have an effective decision procedure, again stated in the metalanguage, which would tell one whether any given wff is a theorem, or, in other words, whether a proof, either finite or infinite, exists which has any specified last line. However, this sort of system would be unusual, and one would in any case no longer be able to determine in a finite and mechanical way whether any given sequence of formulas constitutes a proof.

It is now time to summarize our results and draw conclusions. The first is that if one is just considering a particular mathematical system and is not trying to reduce it to logic, there is no particular problem in taking infinite conjunctions and disjunctions as primitive and introducing quantifiers as defined terms. Even though we cannot construct a matrix for these truth functions and cannot give rules for doing so, we can, given an appropriate metalanguage with quantifiers, express the function arithmetically, and in these respects infinite conjunctions and disjunctions will be like most other mathematical functions. Statements involving these expressions will be unusual in that we cannot in practice write them out in their unabbreviated form, and, if they are not denumerable we cannot give rules for so doing. However, we do have a perfectly adequate way of abbreviating them, and there are other cases where the object language is too cumbersome to deal with unless we have abbreviations of some sort. Some systems, for instance, even have an infinite number of axioms. Most important, we can deal with statements of this sort in that we can prove them to be non-theorems by counterexample and prove them to be theorems using the methods discussed above. While we are in a slightly better position when the infinite conjunctions are denumerable, even when they are not, there seems to be no decisive objection to reducing quantifiers ultimately to conjunctions and disjunctions in a typical mathematical system.

The situation is still largely unchanged even when we consider a system of logic provided that we are not trying to collapse all of logic and mathematics into one large system. The main difference here is that we are likely to have very limited rules of inference which will not allow us to prove theorems involving infinite conjunctions or disjunctions in the object language. Instead, we will have to prove metalinguistically that such infinite proofs do exist in the object language. Again, this need not be a fatal difficulty, and there may even be an effective decision procedure for theoremhood.

More difficulties appear when we come to constructing a unified system of logic and mathematics (including at least arithmetic and set theory). Historically, it has been primarily the logicists who have wanted to do this, but the program need not be limited to those who have wanted to found mathematics in logic. One might also want to bring the formalist program to its logical conclusion and model as many systems as possible in a given system, which might turn out to be a propositional calculus, a system of arithmetic, or some other kind of system. We need not ask what sort of system is to have all the others modelled in it in order to see what happens when we eliminate quantifiers in it. We are assuming that we still have the quantifiers, «all» and «some» in English so that we can still express the rule for the evaluation of infinite conjunctions in a metalanguage and in a completely non-symbolic way. There does not seem to be any great loss here: the rule is still the same whether we can state it in arithmetic, in logical symbolism, or just in English.

If one takes the logicist rather than the formalist point of view, one is not unifying systems just for purposes of elegance or to bring about greater uniformity in mathematics, but ultimately in order to insure the reliability of mathematics. The propositional calculus is

(3) J. M. Keynes, *A Treatise on Probability*; Keynes' views on the foundations of mathematics are in chapter II of this book.

the system with the most intuitive axioms and such logicists as Keynes would have been very much attracted by the thought of reducing the rest of logic to the propositional calculus via the elimination of quantifiers, and then reducing mathematics to this condensed logic ([3]). However, much of the attraction is lost if it turns out that we cannot actually produce the required proofs in the object language of the propositional calculus, but can only prove that such proofs exist, and do that only in the metalanguage. In order to produce these metalinguistic proofs we must be using all sorts of principles of inference that go far beyond the rules of the propositional calculus. Even if the metatheorems are self-eliminating in that they give us directions for carrying out the proofs in the object language, still *that* must be proven in some way. From this point of view nothing can ever replace the lack of real finite proofs in the object language that one can look at and check. If one's point of view is less logicist and one is just trying to unify systems and eliminate quantification at the same time, this is much less troublesome. However, even here one may find that more than one had initially supposed has gravitated from the object language to the metalanguage and that most of the real mathematical and logical argument remains in a completely unformalized or semi-formalized state in various metalanguages.

Of course, there will again be no rule for the construction of a matrix corresponding to an infinite conjunction unless we presuppose the axiom of choice and discover a way of well-ordering the rows of a truth table of this sort. This fact continues to constitute no difficulty for the formalist, even one who wants to unify systems to this extent and to eliminate quantifiers. He is probably willing to admit the axiom of choice in the first place, and in the second place, he would be unlikely to make a demand of truth functions in logic which he does not make of other functions in other branches of mathematics. Thirdly, and most important, if the formalist models everything else in logic, the primitives of logic will remain uninterpreted. This means that an infinite conjunction need not be a function at all unless one chooses to interpret it that way. Even if one does so interpret logic, the specification of the functions is part of the interpretation of the system and not part of the system itself. The logicist, on the other hand, regards the primitives of the propositional calculus as having a set meaning, or he at least regards the customary interpretation of the system as being of prime importance. Further, it is characteristic of Ramsay's kind of logicism to place great emphasis on the concept of a truth-table tautology for philosophical reasons most of which we cannot discuss here. One reason, however, is that if the axioms of the system can be seen to be truth-table tautologies they are certain in a very strong sense, and thus of crucial importance to the logicist. If the system has as axioms conjunctions of infinite length, we cannot be assured that they are tautologies in the same way that we should if they were finite; the situation is made worse if they are not even denumerable. Of course, we might still convince ourselves that the axioms and theorems are all tautologies, but we would have to do this in the English metalanguage in an intuitive way, and one could not really call this giving a proof. Another difficulty is that the Ramsayan logicist is likely to think that truth functions in particular must be specified by matrix rather than by rule because they come to have a very special role that other functions do not have. But we have seen that we do not even know how to go about constructing these matrices.

On the other hand, even if the system does contain infinite conjunctions and disjunctions, these need not necessarily appear in the axioms. In particular, we know that the rules of quantification can be reduced to the propositional rules of addition and simplification so it might be possible to do without any expressions in the axioms that would ordinarily involve quantification. In that case the Ramsayan logicist's situation would be considerably improved because we could then actually construct truth tables for the axioms. Of course, if there are theorems of infinite length, as there must be, then there will somewhere have to be proofs of infinite length with the difficulties that we mentioned above. Thus the axioms may now be certain in a very strong sense, but we will never be quite so certain that the theorems follow from the axioms. Much of the motivation for this kind of program lies in uneasiness about the usual methods of proof in mathematics and even in logic. Hence the attempt to find principles, such as truth-table tautologies, which are impossible to doubt (although some have been doubted), and the consequent reduction of all other principles to them. It is here that the necessity of having infinite conjunctions and disjunctions in the axioms or infinite proofs intrudes. These do not bother the ordinary working mathematician or logician, who never has to deal with them in practice, but they lack the kind of certainty which finite truth-table tautologies have and their presence vitiates this kind of program.

These difficulties apply only if one is trying to make mathematics and logic more reliable by performing this sort of reduction. If one is not worried about arithmetic tottering, as Frege was, but is asserting that the whole consists of tautologies just as an attempt to define the *nature* of mathematics and logic, we have an entirely different case. An infinite tautology with a truth-table we do not know how to construct is still a tautology, and the presence of infinite conjunctions and disjunctions does not in any way cast doubt on this thesis about the nature of mathematics whatever other difficulties it may have. In fact, it is as likely that the follower of Ramsay is saying something about the nature of mathematics as it is that he is trying to provide mathematics with a more secure foundation. Hence, our conclusion about the legitimacy of infinite conjunctions and disjunctions must be that it all depends on what one proposes to use them for.

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