A DIAGRAM-METHOD IN PROPOSITIONAL LOGIC

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1.1 The diagram-method in logic has been propagated by various authors. The most famous attempts are those of Euler, Venn, Peirce etc. Especially the so called Venn-Euler diagrams are well known. They may be used in propositional logic or quantificational logic with monadic predicative functions.

> *Vide* J. VENN, *Symbolic Logic,* 2nd. edition, London, 1894, p. 110ff; Ch. Sanders PIERCE, *Collected Papers,* vol. IV (ed. by Ch. Hartshorne and Paul Weiss), 2nd printing, Cambridge, Massachusetts, 1960, p. 293ff. For a survey of various diagram-methods in the past *vide* M. GARDNER, *Logic Machines and Diagrams,* New York, 1958, p. 28ff. For recent discussions of some aspects of the Venn diagrams *vide* Trenchard MORE, *On the Construction of Venn Diagrams* (*The Journal of Symbolic Logic,* 1959, p. 303f) and Daniel E. ANDERSON and Frank L. CLEAVER, *Venn-type diagrams for arguments of N terms, (The Journal of Symbolic Logic,* 1965, p. 113ff).

1.2. The diagrams used in this article resemble the Venn-Euler diagrams in many respects. They differ from them in that they are made suitable for more complicated operations.

1.3. It would be possible to give a completely formalized calculus of the diagram-method expounded in this article. However, we will restrict ourselves to a more informal discussion of the diagram-method in which we will try to show its usefulness as a technique in the decision procedure in logic. It is also useful as a new technique for drawing logical conclusions from complicated propositional functions, provided that the latter have not too many propositional variables.

1.4. The possibilities of application of the diagram-method are limited. Just as in the Venn-Euler diagrams it becomes difficult to survey, when the number of propositional variables is more than 5 or 6.

1.5. The logical signs used in this article are the usual ones:

1.51. p, q, r are propositional variables,
1.52. —p is the negation of p (not p),
1.53. p v q is the disjunction of p and q (p or q or both),

1.54. p.q or pq is the conjunction of p and q (p and q),
1.55. p → q is the conditional or material implication of p and q (if p, then q)
1.56. p ≡ q is the biconditional or material equivalence of p and q (if and only if p, then q)
1.57. Parentheses are used in the usual way.

2.1. The following diagram will be used in proportional functions with two variables:

1	2
3	4

This diagram, consisting of the squares 1, 2, 3, 4 represents the universe of discourse for the two propositional variables p and q. If p is true we will inscribe the sign '+' in the squares 1 and 2 and the sign '—' in the squares 3 and 4. If q is true we will inscribe the sign '+' in the squares 1 and 3 and the sign '—' in the squares 2 and 4. In the «language» of the Venn-Euler diagrams for propositional functions the diagrams 1 and 2 are shaded if p is true, and the diagrams 1 and 3 if q is true.

p q — p — q

Or inserting the variable signs into the squares, which receive the sign '+', if the variable concerned is true, we get

p q	p (– q)
(– p) q	(– p) (– q)

2.2. For the sake of clarity we will reformulate the assignments given in § 2.1. in a more precise way:
2.21. If we assign the truth value 1 (or another semantical value 1)

to p, leaving undecided what value must be given to q we will inscribe the sign '+' in the squares 1 and 2 (indicating that p is true), and the sign '—' in the squares 3 and 4 (indicating that —p is false)

+	+	p
—	—	

2.22. If we assign the truth value 0 to p, leaving undecided what value must be given to q, we will inscribe the sign '+' in the squares 3 and 4 (indicating that —p is true) and the sign '—' in the squares 1 and 2 (indicating that p is false).

—	—	— p
+	+	

2.23. If we want to assign the truth value 1 to q, leaving undecided what value must be given to p, we will inscribe the sign '+' in the squares 1 and 3 (indicating that q is true), and the sign '—' in the squares 2 and 4 (indicating that —q is false).

+	—	q
+	—	

2.24. If we want to assign the truth value 0 to q, leaving undecided what value must be given to p, we will inscribe the sign '+' in the squares 2 and 4 (indicating that —q is true) and the sign '—' in the squares 1 and 3 (indicating that q is false).

—	+	— q
—	+	

2.3. From the preceding indications it may be clear that the diagrams for p v q, p → q, pq, etc are

+	+
+	—

p v q

+	—
+	+

p → q

+	—
—	—

p . q

+	—
—	+

p ≡ q

We have 16 diagrams according to the 16 different possibilities of assigning the truth values 1 or 0 to p and q as is well known in the truth table method to which this diagram-method is, of course, related. The diagram

+	+
+	+

may count as indicating a tautology and the diagram

—	—
—	—

may count as indicating a contradiction.

3.1. So far this diagram-method does not differ essentially from the Venn-Euler or other related diagrams. This method becomes useful by the following assignments for operations with two diagrams: Let P and Q stand for two diagrams, then

3.11. P v Q means: construct a new diagram D in which the sign '+' must be inscribed in the squares where P *or* Q have a sign '+' in the corresponding squares and in which the sign '—' must be inscribed in the squares where P *and* Q have a sign '—' in the corresponding square.

Or, in another formulation: Let /P followed by the sign '+' or '—' mean: the square or squares where the diagram P has the sign '+' resp. '—' and let /Q followed by the sign '+' or '—' mean: the corresponding square (or squares) in the diagram Q has (or have) the sign '+' resp. '—' and let //D followed by the sign '+' of '—' mean: inscribe in the corresponding square in D the sign '+' resp. '—'. We may then reformulate our assignment in § 3.11 in the following way:

/P+ /Q+ //D+
/P+ /Q— //D+
/P— /Q+ //D+
/P—/Q—//D—

3.12. P.Q means:

/P+ /Q+ //D+
/P+ /Q— //D—
/P— /Q+ //D—
/P— /Q— //D—

3.13. P → Q means:

/P+ /Q+ //D+
/P+ /Q— //D—
/P— /Q+ //D+
/P— /Q— //D+

3.14. P ≡ Q means:

/P+ /Q+ //D+
/P+ /Q— //D—
/P— /Q+ //D—
/P— /Q— //D+

3.15. Of course it would be possible to give more operation assignments for operations between two diagrams, but those mentioned in §§ 3.11-3.14. are sufficient for this article.

3.2. With the help of the operation assignments of §§ 3.11-3.14. the diagrams of § 2.3. can easily be derived from the diagrams of §§ 2.21-2.24 as the reader may check. We will give one example:

p → q in diagrams

+ 1	+ 2
— 3	— 4

p

→

+ 1	— 2
+ 3	— 4

q

square 1: /+ /+ //+   3: /— /+ //+
2: /+ /— //—   4: /— /— //+

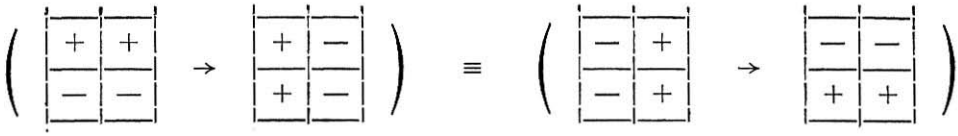
This yields the following diagram:

+	—
+	+

p → q

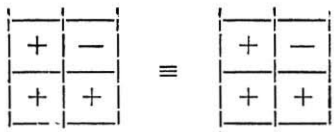
4.1. This diagram-method is very suitable for checking as to whether a given propositional function is a tautology oı not. We will give some examples:

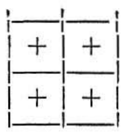
4.11 (p → q) ≡ (—q → —p). In diagrams:



p q —q —p

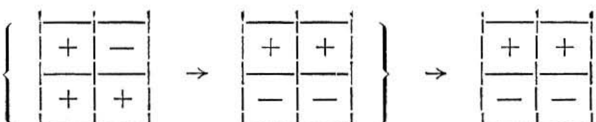
We successively carry out the operation assignments of §§ 3.11-3.14:



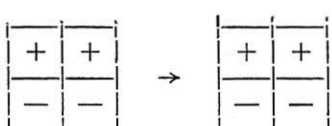


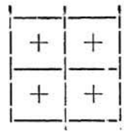
i.e. (p → q) ≡ (—q → —p) is a tautology.

4.12. Another example
{(p → q) → p} → p (Peirce's law).



p → q p p



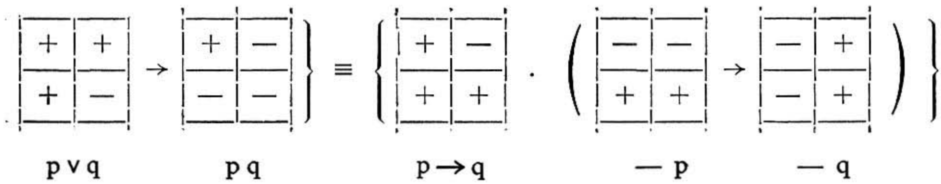


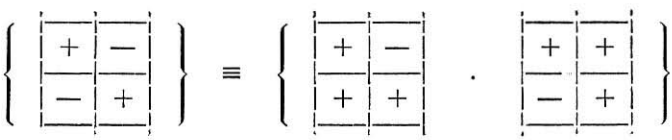
i.e. $\{(p \rightarrow q) \rightarrow p\} \rightarrow p$ is a tautology.

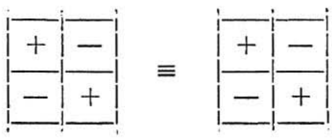
4.13. A great advantage of this diagram-method is that it can be carried out mechanically.

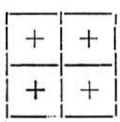
4.2. Furthermore this method may be applied to more complicated forms where other decision methods are more difficult to handle. E.g.:

4.21. $\{(p \vee q) \rightarrow (pq)\} \equiv \{(p \rightarrow q) \,.\, (-p \rightarrow -q)\}$



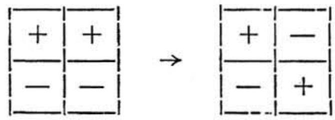


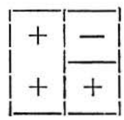




i.e. $\{(p \vee q) \rightarrow (pq)\} \equiv \{(p \rightarrow q) \,.\, (-p \rightarrow -q)\}$ is a tautology.

4.22. $\{(p \rightarrow q) \rightarrow (pq)\} \rightarrow \{(p \vee q) \equiv (pq)\}$

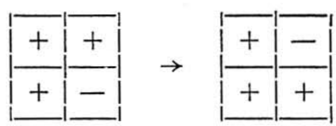


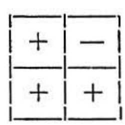


i.e. $\{(p \rightarrow q) \rightarrow (pq)\} \rightarrow \{(p \vee q) \equiv (pq)\}$ is not a tautology.

4.3 A second advantage of this method is its possible application to questions as to what conclusions can be derived from certain given propositional functions. E.g. what conclusion may we derive from $(p \vee q) \rightarrow (p \rightarrow q)$?

In diagrams:





Looking in our index of diagrams for two propositional variables a part of which has been listed in § 2.3, we easily find that this is the diagram for $p \rightarrow q$.

So: $\{(p \vee q) \rightarrow (p \rightarrow q)\} \equiv (p \rightarrow q)$.

5.1. This diagram-method can easily be extended to propositional functions with 3 or more variables. We will give two examples using the following diagram:

1 p q —r	2 p —q —r	5 p q r	6 p —q r
3 —p q —r	4 —p —q —r	7 —p q r	8 —p —q r

For p and q the squares 1 and 5, 2 and 6, 3 and 7, 4 and 8 run parallel. They differ in that in the squares 1, 2, 3, 4 (— r) is true and in the squares 1, 2, 3, 4, r is true.

5.11. We inscribe the sign '+' in the squares 5, 6, 7, 8 if the variable r is assigned the truth value 1 (indicating that r is true) and the sign '—' in the squares 1, 2, 3, 4 (indicating that — r is false), leaving undecided what value must be given to p and q.

1 —	2 —	5 +	6 +	r
3 —	4 —	7 +	8 +	

5.12. We inscribe the sign '+' in the squares 1, 2, 3, 4 if the variable —r is assigned the truth value 1 (indicating that —r is true) and the sign '—' in the squares 5, 6, 7, 8 (indicating that r is false) leaving undecided what value must be given to p and q.

1 +	2 +	5 —	6 —	— r
3 +	4 +	7 —	8 —	

5.13. The other diagrams for p, — p, q, — q are drawn according to the same method of assignments as developed in § 2.21ff, only extended symmetrically to the squares 5, 6, 7, 8.

+	+	+	+
—	—	—	—

p

—	—	—	—
+	+	+	+

—p

+	—	+	—
+	—	+	—

q

—	+	—	+
—	+	—	+

—q

5.2. We will give two related examples of a decision method with the help of the diagrams for three variables. The usefulness of this diagram-method as well as the usefulness of symbolic logic in general may become clear and with that the pedagogical aspect of our method.

5.21. People without knowledge of symbolic logic would certainly be tempted to drawing the conclusion '$q \rightarrow r$' from the premisses '$pq \rightarrow r$' and '$(p.—q) \rightarrow (—r)$'. Still the following propositional function is not a tautology:

$$[\{(pq) \rightarrow r\} . \{(p.—q) \rightarrow (—r)\}] \rightarrow (q \rightarrow r)$$

[{

+	—	+	—
—	—	—	—

pq

→

—	—	+	+
—	—	+	+

r

} . {

—	+	—	+
—	—	—	—

p—q

→

+	+	—	—
+	+	—	—

—r

}] → (

+	—	+	—
+	—	+	—

q

→

—	—	+	+
—	—	+	+

r

)

[

—	+	+	+
+	+	+	+

.

+	+	+	—
+	+	+	+

] →

—	+	+	+
—	+	+	+

—	+	+	—
+	+	+	+

→

—	+	+	+
—	+	+	+

+	+	+	+
−	+	+	+

But seeing that

−	+	+	−
+	+	+	+

≡ (

+	+	+	+
−	−	−	−

→

−	+	+	−
−	+	+	−

)

we find 5.22.

5.22. The following propositional function is a tautology:

$$[\{(pq) \rightarrow r\} . \{(p.—q) \rightarrow (—r)\}] \equiv \{p \rightarrow (q \equiv r)\}$$

As we have seen in 5.21 the first part of this function $\{(pq) \rightarrow r\}$. $\{(p.—q) \rightarrow (—r\}$ yields the following diagram

−	+	+	−
+	+	+	+

Thus we get:

−	+	+	−
+	+	+	+

≡ {

+	+	+	+
−	−	−	−

→

p

(

+	−	+	−
+	−	+	−

≡

−	−	+	+
−	−	+	+

)}

q r

—	+	+	—
+	+	+	+

$\equiv$ {

+	+	+	+
—	—	—	—

$\rightarrow$

—	+	+	—
—	+	+	—

}

—	+	+	—
+	+	+	+

$\equiv$

—	+	+	—
+	+	+	+

+	+	+	+
+	+	+	+

So $[\{(pq) \rightarrow r\} . \{(p. -q) \rightarrow (-r)\}] \equiv \{p \rightarrow (q \equiv r)\}$ is a tautology.

6.1 But we repeat: the diagram-method expounded in this article does not pretend to replace any other method of the usual decision procedure techniques. It has only some restricted advantages for certain purposes. Moreover, it may have a pedagogical use in the teaching of logic.

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