IN DEFENCE OF A RELEVANCE CONDITION

D. HOCKNEY & K. WILSON

In a recent paper, John Woods [5] addresses himself to the problem of the relation of irrelevance to the paradoxes of strict implication. His main contention is that there appears to be no satisfactory relevance condition which rules out Lewis' paradoxes of strict implication. After dealing with a preliminary matter, we shall contend that there is a relevance condition which Lewis' paradoxes fail to satisfy and that the relevance condition is not open to any criticism advanced by Woods.

I

Woods first asks three questions about, and lays down three conditions for, defining relevance. They are as follows.

«(i) What exactly does it mean to say that two or more propositions are irrelevant the one to the other ?» (Woods [5], p. 130).

«(ii) Is it, after all, true that the irrelevance of any two propositions (for some sense of «irrelevance» appropriate to the present context) implies that neither can stand to the other in the relation of entailment ?» (Woods [5], p. 130).

«(iii) Is it possible adequately to precise the concept of irrelevance, and so to answer question (i), without begging or *illicitly prejudging* question (ii), while at the same time *providing information* in terms of which (ii) is susceptible of a clear answer ?» (Woods [5], p. 131, our italics). «That is,» Woods continues, «is it possible to define irrelevance under the following conditions:

First: The paradoxes are clearly shown to exemplify irrelevance so defined;

Second: The decision whether A is irrelevant to B is possible without having first to determine whether A entails B; and

Third: The information yielded by the definition *is sufficient* to show that from the fact that A is irrelevant to B, it *follows that* A does not entail B» (our italics).

Woods' third question and the corresponding second and third conditions were motivated by his consideration of an interpretation of irrelevance which, as he correctly indicates, begs the question as to whether or not A entails B.

Woods' third question and third condition require some comment. First, it is not clear what distinction Woods intends between begging a question and illicitly prejudging a question. The only example that Woods gives is of begging a question and it is unclear how this would differ from illicitly prejudging a question. Perhaps Woods intends to draw no distinction here. We shall treat the two as the same fallacy. Secondly, we believe that the third condition is too strong; why should it be required that the *definiens* of 'A is irrelevant to B' entail 'A does not entail B' ? (Note the words 'sufficient' and 'follows' in the third condition.) What should be required is that the information given by the definition of 'A is irrelevant to B' be such that in terms of this information (though not necessarily exclusively in terms of this information) it would reasonable to think that A does not entail B.

II

We shall now consider and reject a condition of relevance given by Anderson and Belnap. Anderson's and Belnap's *necessary and sufficient* condition for 'A is relevant to B' is given in a subscripting technique, but is claimed to capture «the intuitive idea that for A to be relevant to B it must be possible to *use* A in a deduction of B from A» (Anderson and Belnap [2], p. 46). We shall concentrate on this «intuitive idea» taken as a necessary and sufficient condition (but we do not claim that Anderson and Belnap identify the intuitive idea and the condition).

Lewis' «independent proof» for the first paradox may be construed as follows (cf. Lewis & Langford [4], p. 250):

(a)	$A \,\&\, {\sim}A$	premise
(b)	A	from (a)
(c)	$A \vee B$	from (b)
(d)	${\sim}A$	from (a)
(e)	B	from (c) and (d)

Clearly this deduction of B from $A \,\&\, {\sim}A$ satisfies the intuitive necessary and sufficient condition for relevance, for $A \,\&\, {\sim}A$ is used in the proof to deduce B. Hence, $A \,\&\, {\sim}A$ is relevant to B. But this is exactly what Anderson and Belnap would wish to deny. They could reply, correctly, that this proof could not be given in their Pure Calculus of Entailment. However, this condition of relevance, if *restricted* to the Pure Calculus of Entailment, would then be irrelevant to Lewis' «independent proofs» of the paradoxes of strict implication. What is required is a relevant condition for relevance *.

We shall now state a sufficient condition for 'A is irrelevant to B' and defend it against Woods' attacks.

IC: If A and B fail to share a variable then A is *irrelevant* to B. (Notice that IC is the condition of relevance laid down by Belnap and quoted by Woods ([5], p. 132)).

We think that it is odd that Woods calls IC a «definition», for both CIII (if A and B have no variables in common, then 'A entails B' is rejected as a theorem of the system) and IC are surely to be construed as adequacy conditions. If so, it is far from clear that Belnap is committed to IC as a *definition,* any more than Tarski is committed to the view that his well-known adequacy condition for a definition of truth *is* a definition of 'true'.

Woods complains ([5], p. 131) that «it does not follow from Belnap's definition alone, that if A is irrelevant to B, then A does not entail B», and so his third condition is not satisfied. But nothing follows from a definition *alone,* without the use of logical laws. But this is, perhaps, quibbling. IC is properly construed as giving a sufficient condition for irrelevance (i.e., a necessary condition of relevance). By CIII, which is itself an adequacy condition for any systematic treatment of entailment, a *necessary* condition for entailment is given which relates entailment with the sharing of variables. Woods' error consists in confusing definitions with necessary conditions. Since CIII is an adequacy criterion for entailment, CIII should be provable for any such system of entailment. That CIII begs all the important questions is far from being demonstrated. Would Woods claim that Tarski's adequacy condition for an analysis of 'true' begs all the questions ? To do so would be rash ! To show that an adequacy condition is inappropriate or misguided, considerations must be advanced in the

* We note that Anderson and Belnap consider Lewis' «independant proof» of this paradox in [3] where a falacy of relevance is claimed to be committed by Lewis (and/or a fallacy of ambiguity). Our complaint is that their necessary and sufficient condition for relevance needs to be more general, so as to apply to general propositional systems, and not just to implicational systems.

way of arguments or counter-examples directed to those considerations on the basis of which that condition was advanced. Woods has failed to do this up to this point in his paper. IC does not *entail* CIII, but this does *not* establish the inadequacy, inappropriateness, or falsehood of either IC or CIII.

But let us be charitable, and take IC as a definition as Woods does. He claims, then, that to accept Belnap's CIII is to beg all the important questions. We view the charge of begging all the important questions as sheer dogmatism. It is true that IC does not entail CIII, but it hardly follows from this that to accept CIII is to commit a *petitio*. (In fact, given IC *as a definition*, then CIII together with IC entails that 'A entails B' entails 'A is relevant to B'. But it does not follow from this that CIII begs the question.) We shall show that given IC, it is reasonable to accept CIII.

It is notorious that logicians of the «Official School» have managed to anesthetize their pre-analytic intuitions regarding entailment by telling one another rather odd stories about the paradoxes of material and strict implication. We find it extremely paradoxical that they should manage to convince themselves that there is nothing paradoxical about the paradoxes of material and strict implication. For example, Lewis has said that the paradoxes of strict implication simply state facts about deducibility ([4], p. 251). But why then should they clash so violently with our pre-analytic intuitions ? Of course we know that these paradoxes are not paradoxical *in the sense* that they are simply deductive consequences of their respective axiom systems. But it does not follow that they are not paradoxical in the sense that they are said to state facts about *deducibility*. It is puzzling to us that anyone should hold that propositions can stand in a relation of *entailment* simply because of their truth values or their modal status.

When students are told, for example, that $A \& \sim A$ entails B, they are usually completely baffled. Often they will say, «How can that be ? There is nothing in common, no connection at all, between them.» In short, they recognize the irrelevance of $A \& \sim A$ to B. And they are correct. It is a fact about our concept of entailment that there must be some meaning content common to A and B if one entails the other. Anderson and Belnap are correct when they say:

> Informal discussions of implication or entailment have frequently demanded «relevance» of A to B as a necessary condition for the truth $A \rightarrow B$, where relevance is construed as involving some

«meaning content» common to both A and B. A formal condition for «common meaning content» becomes almost obvious once we note that commonality of meaning in propositional logic is carried by commonality of propositional variables. ([2], p. 48).

Now what IC says is that if A and B fail to share a variable in common then A is irrelevant to B. Hence, it is perfectly in keeping with our pre-analytic intuitions to accept CIII. For CIII connects the formal counterpart of our intuitions about relevance with entailment, parallel to the way in which relevance and entailment are connected in our informal thought. So to accept CIII is not to beg all the important questions. Rather, it is to state formally what we have thought informally all along. CIII is thus a formal statement about the concept of entailment and in our opinion reveals why the paradoxes were thought to be paradoxical in the first place.

We turn now to Woods' counter-examples ([5], p. 132), which, in our opinion, are singularly misguided. The claim is that we must deny the following statements, given Belnap's CIII:

(1) 'The book is blue' entails 'The book is coloured'.

(2) 'Proposition A is true' entails 'Proposition A is truth-valued'.

(3) 'The figure is square' entails 'The figure is rectangular'.

Woods then goes on to claim that CIII «rules out all entailment statements which hold either in virtue of a determinate-determinable relation, or in virtue of a so-called genus-species relation.» And this alleged fact, claims Woods, violates another of Belnap's conditions for a viable analysis of entailment, namely

(CII): A theory of entailment ought to be only as strong as is consistent with its fundamental aims. E.G., it ought not rule out any *non-controversial* entailment statement.

Should the sentences above actually be counter-examples, then Woods' claim would be correct. But we shall show that these sentences do not constitute genuine counter-examples. Statements (1) and (3) are simply not adequately analyzed in *any* propositional system, and so we have to turn to a predicate calculus of entailment to show that these statements are not ruled out by CIII. Now the propositional function 'x is colored' we analyze naturally, as 'x is blue or x is red or ...', in which case statement (1) is not ruled out by CIII. Now Woods cannot reply that 'x is colored' cannot be analyzed, for from this it would follow that 'x is colored' is an atomic statement (in Carnap's sense), and then 'x is colored' could not be entailed by any other atomic statement, and 'x is blue' is a paradigm of such a statement. In that case, of course, CIII would rule out Woods' first alleged counter-example, but it would do so quite properly, for (1) would be false in that case. A similar analysis can be given for (3). If he should reply that what we have done here is illegitimate (i.e., for some reason we should be content to analyze (1) and (3) in propositional logic), then we would reply that he should also reject all propositional logics (whether of material, intuitionistic, or strict implication): for the Barbara syllogism, whose logical form on that (superficial) analysis is $(A \& B) \rightarrow C$, is demonstrably invalid on any of these analyses; but Barbara is an obviously valid argument.

Counter-example (2) meets with a similar fate on inspection. Here we may analyze 'A is truth-valued' as '(A is a wff & (A is true or A is false))'. In which case (2) is not ruled out by CIII. On the other hand, if 'A is truth-valued' cannot be analyzed in this or in a similar way, then shall we say it is an atomic statement? If so, then (2) should be ruled out, for it would be false. If not, then presumably, it is analyzable in some other way, and the onus is on Woods to give such an analysis. Thus, we conclude that Woods' counter-examples are unsatisfactory as they stand.

We also note, however, that should our attempts to rescue CIII from such counter-examples fail, so that Woods' counter-examples are successful against Anderson's and Belnap's analysis of entailment, they will be equally successful (for approximately the same reasons) against *all* systems of implication formulated to date, even those comparatively weak systems of Lewis.

Woods next claims ([5], p. 133) that «each step of Lewis' proofs of the paradoxes satisfies the relevance-condition and that each step would appear to be sanctioned by a valid rule of inference.» Thus he charges, «by CI, we must agree that the paradoxes are true (albeit truths easily overlooked), which by CIII we must deny.»

Now the fact that each *step* of Lewis' proofs of the paradoxes satisfies the relevance-condition is nothing in favour of the paradoxes. Clearly it does not follow from the fact that each step of a proof satisfies the relevance-condition that the *premise(s)* of the proof are relevant to the conclusion, for relevance is not a transitive relation. (See Anderson and Belnap, [3], p. 11) In the *first proof* '$p \& \sim p$' does not share a variable with 'q'; in the *second proof* 'p' does not share a variable with '$q \vee \sim q$'; in the *third proof* '$p \vee \sim p$' does not share a variable with '$q \vee \sim q$', and in the *fourth proof* (not actually given by Lewis) '$p \& \sim p$' does not share a variable with '$q \& \sim q$'. So far so good. However, Woods may still claim that by CI we must admit that the paradoxes are true and consequently that CI and CIII are in conflict. We believe that this charge can be met.

Let us reconstruct the first independent proof of the paradoxes which Woods believes to be «impeccable». Here we shall follow Anderson and Belnap. In the proof, the following modes of inference are used (See Anderson and Belnap [3], p. 18):

1. from A & B to infer A,
2. from A & B to infer B,
3. from A to infer A v B, and
4. from A v B and ~A to infer B.

The independent proof is as follows:

(a)	A & ~A	premise
(b)	A	from (a) by 1
(c)	A v B	from (b) by 3
(d)	~A	from (a) by 2
(e)	B	from (c) and (d) by 4.

What are we to make of this independent proof of Lewis' first paradox? Lewis says:

> For the validity of the trains of reasoning in these proofs we must, of course, appeal to intuition. A point of logic being involved, no other course is possible. But let the reader ask himself whether they involve any mode of inference which he is willing to be deprived of — for instance, in making deductions in geometry ([4], p. 252).

Now upon appealing to *our* intuitions we find the proof suspect. That any proposition whatsoever may be deduced from A & ~A strikes us as being preposterous and is accordingly ruled out by CIII. Thus we conclude that there must be a mode of inference which is invalid. It seems obvious to us that Anderson and Belnap are perfectly justified in objecting to the unrestricted rule 'from A v B and ~A to infer B'. We are well aware that the rejection of rule 4 appears radical. However, we believe it to be well founded both intuitively and formally (See Anderson and Belnap [3], pp. 18-20). It is true that either Churchill was Prime Minister of England or that Fords are built in London (with the truth functional 'or'); but it does not *follow* that if Churchill should not have been Prime Minister of England that Fords *would* be built in London. The truth-functional 'or' is simply not strong enough to warrant the plausibility of rule 4. We do admit that 4 is a valid mode of inference when, for example, ~A entails B, viz. when the 'or' is taken intensionally. But there is nothing in rule 4 to guarantee that this is the case. Indeed A and B do not share a variable. Also, the inference from $\sim A$ and A-or-B is acceptable when 'or' is construed truth functionally under the following restriction: «If $A_1 \vee \ldots \vee A_n$ is a tautology in Boolean disjunctive normal form, then $\sim A_1 \,\&\, (A_1 \vee \ldots \vee A_n) \rightarrow A_2 \vee \ldots \vee A_n$ is provable» (Anderson and Belnap [2], p. 21). However, in the general case, $\sim A \,\&\, (A \vee B) \rightarrow B$ is not acceptable. (For the formal proof of this see Anderson and Belnap [3], p. 21.) Finally, in accordance with Lewis' demands we would be prepared (and we believe without loss) to give up rule 4 in its unrestricted form for making deductions in geometry.

We also find difficulties in Lewis' independent proof of the second paradox ([4], p. 251). It is constructed to show that a necessary proposition of the form $B \vee \sim B$ is implied by any proposition. Lewis' proof is essentially as follows:

(1) Assume A
(2) (1).→: $A \,\&\, \sim B \,.\!\vee\!.\, A \,\&\, B$
(3) (2) .=: $A \,\&.\, B \vee \sim B$
(4) (3).→. $B \vee \sim B$.

The first question that occurs to us is how does Lewis get from (1) to (2) ? Lewis' proof of (2) as a theorem of his system (Theorem 18.9 in [4]) involves the use of a rather paradoxical theorem, viz. 18.61:

$$[\sim \Diamond \sim p \,\&\, (p \,\&\, q . \rightarrow r)] \rightarrow (q \rightarrow r).$$

That is to say, if a necessary proposition and a second proposition together entail a third proposition, then the second proposition (alone) entails the third. (The corresponding paradox for material implication is that if A and B imply C, and A is true, then B implies C.) What is paradoxical about 18.61 (and the corresponding paradox of material implication) is that strict (or material) enthymemes cannot be distinguished from valid arguments. But we do not wish to belabour this point. (For more on this, see Anderson and Belnap, [1].) The crucial point to be made here is the following one. If $A \,\&\, \sim B \,.\!\vee\!.\, A \,\&\, B$ is put into conjunctive normal form, we get:

$$A \,\&\, (A \vee B) \,\&\, (A \vee \sim B) \,\&\, (B \vee \sim B).$$

But $A \rightarrow [A \,\&\, (A \vee B) \,\&\, (A \vee \sim B) \,\&\, (B \vee \sim B)]$ is not a tautological entailment, for A does not entail $B \vee \sim B$. (See Anderson and Belnap [3].) So this inference must be rejected, since a fallacy of relevance is committed. If we are correct in our estimate of Anderson's and Belnap's discovery, then clearly Woods is in error in saying that CIII is inconsistent with CI.

Finally, the problem that Woods raises about simultaneous and piecemeal (uniform) substitution for CIII and CI is easily disposed of. One only need employ the method of using metalinguistic variables ranging over wffs which are specified in the usual way after having specified propositional variables. Then by using axiom schemata, there is no need for any substitution rules. And this is exactly the procedure employed by Anderson and Belnap, ([3], p. 14) the only two rules of their system E being modus ponens and adjunction. Since substitution rules are not used in this axiom system, one can hardly claim that CI and CIII conflict on the basis of considerations arising from substitution rules.

Particular rules of inference are, of course, system relative, not absolute for all systems. And given the difficulties logicians historically met with in giving satisfactory substitution rules, it would hardly be appropriate to complain that a system does not have such rules.

By way of summary, we would remark that we believe we have shown that there is a sufficient condition of irrelevance such that (a) it fits our intuitive ideas about irrelevance; (b) the paradoxes clearly exhibit irrelevance; (c) one can determine whether A is irrelevant to B without first having to determine whether A entails B; (d) the information yielded by the relevance condition is such that it is plausible to maintain that if A is irrelevant to B, then A does not entail B; and (e) no criticism advanced by Woods against this relevance condition holds.

We do not claim to have given anything like a full analysis of irrelevance, but we do claim to have defended a promising beginning of such analysis suggested by Belnap (and further developed by Anderson and Belnap) against a *prima facie* plausible attack.

The University of Western Ontario. D. Hockney & K. Wilson

BIBLIOGRAPHY

1. Anderson, Alan Ross, and Nuel D. Belnap, Jr., «Enthymemes», *Journal of Philosophy*, LVIII (1961), 713-23.

2. ———, «The Pure Calculus of Entailment», *Journal of Symbolic Logic*, 27 (1962), 19-52.
3. ———, «Tautological Entailments», *Philosophical Studies*, XIII (1962), 9-42.
4. LEWIS, Clarence Irving, and Cooper Harold LANGFORD, *Symbolic Logic*, New York and London, Appleton-Century, 1932.
5. WOODS, John., «Relevance», *Logique et Analyse*, Nouvelle Série, 7e Année, N° 27, Oct. 1964, pp. 130-137.