WHAT NUMBERS ARE

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Cardinal numbers are non-distributive properties of certain manifolds. This thesis, the *P-C thesis,* though often (in effect) proposed, does not seem to have been fully worked out, and has in fact been subject to repeated attacks. Recently the thesis that numbers are properties of classes has been criticised by Professor Benacerraf (1). My main aim here is to elaborate the initially plausible P-C thesis and to defend it against criticisms.

I

Formal details of the definition of *cardinal number 2* and of *natural number,* as spelled out under the P-C thesis, are sketched. To facilitate later discussion, a 3-valued significance logic is adopted as a basic logic. An equivalence relation '$\equiv$' between properties of manifolds is first defined:

(D1): $f \equiv g =_{Df} (\Pi w)\,(f(w \simeq g\,(w))$,

i.e. f is weakly identical with g if and only if, for every possible (2) w, f (w) is extensionally synonymous with g (w), where '$\simeq$', reads 'is extensionally synonymous with', is a 3-valued connective with matrix:

$\simeq$	1	—i	—1
1	1	—1	—1
—i	—1	1	—1
—1	—1	—1	1

In this and succeeding matrices, '1', '—1' and '—i' symbolise respectively 'true', 'false' and 'non-significant'. With (D1), predicate

(1) In his stimulating article 'What numbers could not be', *Philosophical Review* LXXIV (1945), 47-73.

(2) On the quantifiers 'Π' and 'Σ' which stretch out over possible items, see R. Routley 'Some things do not exist', *Notre Dame Journal of Formal Logic* to appear in vol. 7, 2, April 1965. Quantifiers with even more extensive range could be used.

descriptions can be defined as binary quantifiers:

(D2): $(\imath f)\ (g(f),\ h(f)) =_{Df} (\Sigma f)\ ((\Pi f')\ (g(f') \simeq .f' \equiv f)\ \&\ h(f))$.

Here and below 3-valued significance analogues of classical 2-valued connectives are used: these have matrices which coincide with usual 2-valued matrices for significant values and which have value —i otherwise; e.g. '&' and '~' have matrices:

&	1	—i	—1
1	1	—i	—1
—i	—i	—i	—i
—1	—1	—i	—1

~	
1	—1
—i	—i
—1	1

Conventions can be introduced for replacing '$(\imath f)\ (g(f), h(f))$' by the more familiar '$h((\imath f)\ g\ (f))$' when 'h' is not so compounded as to produce scope ambiguities. Similarity of manifolds, symbolised 'sm', is defined in the customary set-theoretic way except that 'Π' and 'Σ' replace usual existential-loaded quantifiers (3). Furthermore the manifold and attribute theories presupposed are such that suitable paradigmatic manifolds, having any required numbers as properties, can be selected: the paradigm manifold which instantiates number n is represented 'pd(n)'. For example, given a suitable manifold logic, the recursion scheme:

(P): $pd(0) = \wedge$
$pd(n) = [0, 1, \ldots, n\text{-}1],\ n \geqslant 1$

could serve to provide paradigm manifolds. Then the cardinal '2' is defined:

(D3): $2 =_{Df} (\imath f)\ (\Pi w)\ (w \text{ sm pd } (2) \simeq f(w))$,

(3) In this way objections levelled by W. Kneale, F. Waismann & L. Wittgenstein against definitions of *natural number* built on a similarity relation sets or manifolds, because the existence of 1-1 relations required by the definitions cannot be established in a suitably independent way, are avoided: existence of the relation is *not required*. For the objections see W. & M. Kneale, *The Development of Logic*, Oxford (1962), 461-464, & F. Waismann, *Introduction to Mathematical Thinking*, Ungar, New York (1951). It is important here to distinguish 'for some possible R, R is 1-1 and...', i.e. '$(\Sigma R)(R \varepsilon$ 1-1 &...)' from 'it is possible there exists a relation R which is 1-1 and...' i.e. '$\Diamond(\exists R)$ $(R \varepsilon$ 1-1 &...)'. The first implies the second, but the converse implication does *not* hold.

i.e. 2 is the property of *all and only* those possible manifolds which are similar to its paradigm manifold. In general,

(D4): $n =_{Df} (\iota f)\ (\Pi w)\ (w \text{ sm pd } (n) \simeq f(w))$,

'pd (2)' can be eliminated from (D3), e.g. thus:

(D3′): $2 = (\iota f)\ (\Pi w)\ (\Sigma v)\ (w \text{ sm } v\ \&\ (\Sigma x)\ (\Sigma y)\ (x \neq y\ \&\ v = \iota^{\langle} x \cup \iota^{\langle} y))\ \simeq f\ (w))$.

Identity relations are discussed below.

Once a suitable recursive progression of paradigms such as (P), on which to fix (quasi-ostensively) the similarity relations, is selected, the *successor* relation 's' can be defined in such a way that property f succeeds property g iff, if g is the property of all and only those manifolds similar with some member of the progression, then f is the property of all and only those possible manifolds similar to its successor in the progression. Finally the *positive integers* can be defined as the set of property ancestors of 1 under the successor relation, and the positive integer property 'NN' can be defined. Thus positive integers are finite cardinals ordered under the successor relation.

II

Features of these definitions of cardinal numbers, which enable objections to the P-C thesis to be overcome, should be stressed.

(1) The property used to define a given cardinal is *unique* up to a suitable identity relation on properties. For suppose on the contrary the property used to define '2' were not unique. Let f and g be properties which satisfy the requisite conditions. Then it follows from (D3)

$$\vdash (\Pi w)\ (w \text{ sm pd } (2) \simeq (f(w))\ \&\ (\Pi w)\ (w \text{ sm pd } (2) \simeq g(w)).$$

Hence $\vdash (\Pi w)\ (f(w) \simeq g(w))$,

i.e. by (D1) $\vdash f \equiv g$.

Thus the uniqueness condition required for (D2) is met. That this uniqueness condition can be met is a consequence of the expansion of the range of manifold-variables to include possible sets [4]. In this

[4] The identity relation of (D1) could be apparently strengthened by putting '$\Box$' across the definiens; then the argument would run as before once (D3) were supplanted by

(D3″): $2 =_{Df} (\iota f)\ \Box\ (\Pi w)\ (w \text{ sm pd}(2) \simeq f(w))$.

Compare CARNAP's weak conditions for the identity of properties: *Meaning & Necessity*, Enlarged edition, Chicago (1956), 18.

way the «absolutely fatal formal defect» ([5]) of abstraction definitions used in elaborating P-C related thesis, that uniqueness of the property cannot be established, is repaired.

(2) The definitions given can be combined with further definitions, for example definitions of numerical quantifiers. Jointly these definitions suffice for an explanation of most non-metaphorical occurrences of cardinal number expressions in English. This is as well. Conditions of adequacy for any definition of 'finite cardinal number' include:

(I) The definition should provide a basis on which non-metaphorical occurrences of number words, both as nouns and as predicative adjectives and as attributive adjectives (or as quantifiers), can be defined and explained.

(II) Significance restrictions on the occurrence of number-words should be reflected in the definitions.

To enlarge on (I). Number words do occur predicatively ([6]) as in

(a): The moons of Jupiter are four.

Sometimes admittedly 'in number' or 'strong' are added to predicative uses to avoid ambiguity but this does not detract from the point. Predicative uses of number adjectives are perhaps not so common as they were, but this can very likely be ascribed just to current linguistic fashion. Certainly (a) has to be distinguished from

(b): The moons of Jupiter are red,

as the property red is a distributive property of manifolds, that is a property which belongs (distributively) to each component of the manifold. Properties of manifolds which do not distribute onto components of the manifold may be called *non-distributive* properties of manifolds. Predicates which specify such properties are plentiful, e.g. '(are) six', 'seventeen', 'numerous', 'few', 'many', 'vanishing', 'represented'. Not all predicative adjectives function in the same way. A classification of adjectives which qualify manifold expressions like

([5]) Explained at length by B. Russell: *The Principles of Mathematics*, Allen & Unwin, London (1903), 114-116.

([6]) Contra Benacerraf, *op.cit.*, 59-60. Many predicative occurrences of English number words do not strike questioned native informants as peculiar or implausible (contra 60). Some of Benacerraf's grammatical evidence is decidedly dubious, e.g. the suggestion that examples like (2) probably came into English by the deletion of 'in number'. Even if the evidence were correct it remains far from clear what it would show.

'the moons of Jupiter', in the style of Goodman's classification [7] of predicates of individual expressions, would prove valuable.

Numerals also occur as nouns and as attributive adjectives. These uses have to be accounted for if the initial definitions are to prove adequate. Noun occurrences of numerals are already accounted for: these demarcate properties of manifolds as opposed to the property-instances specified by number predicates. For instance

(c): The number of Jupiter's moons is (=) four [8]

can be derived from (a) as follows. Let 'w_1' symbolise the manifold expression 'the moons of Jupiter'. By (a), $4(w_1)$. Also, NN(4); hence $4(w_1)$ & NN(4). It follows,

$$(\Pi f')\ (f'(w_1)\ \&\ NN(f') \simeq .f' \equiv 4)$$
$$(\Pi f')\ (f'(w_1)\ \&\ NN(f') \simeq .f' \equiv 4)\ \&\ 4 = 4$$
$$(\Sigma f)\ (\Pi f')\ (f'(w_1)\ \&\ NN(f') \simeq .f' \equiv 4)\ \&\ 4 = f)$$
$$(\iota f)\ (f(w_1)\ \&\ NN(f)) = 4.$$

Moreover the usual arithmetic properties of numbers follow once addition, multiplication, etc. of number-properties are defined; thus in '2 + 2 = 4', '2' demarcates the (number-)property two, or if you prefer, since numbers are properties, the number two. With numerals there is no further problem, like that which arises with common nouns such as 'triangle' and 'cat', of distinguishing the properties, triangularity and felinity, from the individual universals, the triangle and the cat, yet explaining their interrelations. For 'the two', if it means anything when it does not refer to a paricular association or pair as it does in 'the (big) Two' or 'That's the two', means 'the number two'. Compare, on these points, number properties with colour properties.

Numerals occur attributively in the expressions

(d): Jupiter has four moons, and

(e): Mars has two moons circling it

and as quantifiers in the slightly unidiomatic

(f): There are four moons of Jupiter, and in

(g): There exists two moons circling Mars.

(f) and (g) may be symbolised, assuming 'there are' carries existential

[7] See N. Goodman, *The Structure of Appearance*, Cambridge Mass. (1951), 48-51.

[8] Frege's stock example, *The Foundations of Arithmetic*, translated J. L. Austin, Blackwell, Oxford (1950), 69.

import, respectively, '$(\exists 4x)$ mj (x)' and '$(\exists 2x)$ mm (x)'. These expressions directly connect with earlier ones through the definition:

(D5): $(Enx)\ A(x) =_{Df} (\Sigma w)\ (n(w)\ \&\ w = [x{:}\ A(x)])$.

Thus for (f) it follows:

$$(\exists 4x)\ mj\ (x) \simeq (\Sigma w)\ (4(w)\ \&\ w = [x{:}mj\ (x)]\ \&\ E(w))$$
$$\simeq (\exists w)\ (4(w)\ \&\ w = [x{:}mj\ (x)]).$$

Now symbolise (a) in more detail and make the usually presupposed existential import explicit. Then $4(w_1)$, where $w_1 = [x{:}mj\ (x)]$ and $E(w_1)$.

Thus (a) $\simeq 4(w_1)\ \&\ w_1 = [x{:}mj\ (x)]\ \&\ E(w_1)$
$\simeq (\exists w)\ (4(w)\ \&\ w = [x{:}mj\ (x)])$
$\simeq$ (f).

Furthermore (f), and therefore (a), is at least extensionally synonymous with a first-order (weak) translation of (f). For simplicity consider a first-order rendering of (g), viz.

(h): $(\exists x_1)\ (\exists x_2)\ (mm\ (x_1)\ \&\ mm(x_2)\ \&\ x_1 \neq x_2\ \&\ (\Pi y)\ (mm(y) \supset . y = x_1 \vee y = x_2))$.

Using the relations:

1. $w\ sm\ [0,1] \simeq (\Sigma x_1)\ (\Sigma x_2)\ (x_1\ \varepsilon\ w\ \&\ x_2\ \varepsilon\ w\ \&\ x_1 \neq x_2\ \&\ (\Pi y)\ (y\ \varepsilon\ w \supset . y = x_1 \vee y = x_2))$,
2. $E(w) \simeq (\Pi x)\ (x\ \varepsilon\ w \supset E(x))\ \&\ (\Sigma x)\ (x\ \varepsilon\ w)$,
3. $((x\ \varepsilon\ w)\ \&\ w = [x{:}mm(x)]) \simeq mm(x)$,

it soon follows that $\Box\ ((g) \simeq (h))$.

In fact 'x_1' and 'x_2' designate Phobos and Deimos. But because there is a logically equivalent analysis of a statement which eliminates the appearance of a property, an analysis which *shows* the property instance, for example by ostensive (subscripting) devices, instead of explicitly stating its occurrence, it does not follow that the relevant property is eliminated or does not occur instantiated. Compare certain vogue eliminations of properties and universals (9).

Attributive uses of numerals can be treated as grammatical transformations of the predicative uses considered. Thus (e) can be symbolised '2 ms (m)', and this transformed to '2 (ms «m)', i.e. '$2\ (w_2)$' where the manifold w_2 is that designated by 'ms « m', i.e. by 'the moons of Mars'. Alternatively, non-metaphorical attributive uses of

(9) Compare, too, RUSSELL, *op.cit.*, 112-113: «Numbers are, it will be admitted, applicable essentially to classes. It is true that, where the number is finite, individuals may be enumerated to make up the given number, and may be counted one by one without any mention of the class-concept. But all finite collections of individuals form classes, so that what results is after all the number of a class.»

numerals could be introduced through talk of *quantitative* predicates and noun phrases like '(has) four moons' and '(eats) four red apples'. For it can be argued that quantitative predicates can be decomposed into a pair consisting of a purely quantitative or numerical predicate and of a manifold determining non-quantitative predicate (or referring expression), and that the numerical predicates specify number property-instances of the manifolds determined.

(3). Numbers are properties of manifolds. They are not properties of sets, because sets are units compressed or condensed from manifolds. Thus predicates of set-expressions are singular in number. P. Bernays' theory of classes (10) can be taken as a 2-valued approximation to the logic of manifolds, once '∀' and '∃' are replaced. For manifolds, such as those designated by 'the moons of Jupiter', 'the trees of Sherwood forest', 'living men', 'the Apostles', are, when they have more than one component, pluralities. It is not significant to say that they belong as units or elements to some set or manifold. Manifolds are not, however, mere aggregates. They can be determined by an ordered couple consisting of a referring expression and a dividing predicate, e.g. ⟨Sherwood forest, tree-wise⟩, ⟨the Red army, men-wise⟩. Since the components of a manifold are determined by a property, Frege's objections (11) to describing numbers as properties of certain classes or groups can be met.

With almost every manifold is associated a set into which the manifold can be condensed. When the relation obtains, the set *represents* the manifold with which it is associated. When 'α represents w', symbolised 'α rep w', is defined, the representing set $\hat{w}$ of w can be defined: $\hat{w} =_{Df}$ rep 'w. Every set represents some manifold. A set is *n-membered* if the manifold it represents has number n. These relations serve to explain the connexion between (a) and

(j): The set of moons of Jupiter is four-membered (12).

In contrast the sentence 'The set of moons of Jupiter is four' is not

(10) P. Bernays & A. A. Fraenkel, *Axiomatic Set Theory*, North Holland, Amsterdam (1958). For the picturesque expressions 'compressed' and 'condensated', used in describing the relation of classes to sets, see E. W. Beth, *The Foundations of Mathematics*, North Holland, Amsterdam (1959), 392.

(11) G. Frege, *op.cit.*, 28-33. But certainly Frege's objections to construing numbers as properties of actual agglomerations only or as properties of mere aggregates not uniquely differentiated into components by some property are substantial.

(12) Thus it is a short step from 'rep w has 17 members' to 'w is 17'. Compare Benacerraf, *op.cit.*, 60.

significant, i.e. $\sim \mathbf{S}(4(\hat{w}_1))$. Granted the sentence 'The set of moons of Jupiter is (not) one' is significant, but 'one' occurs here in a different sense in which it means something like 'a unit' or 'a singular integral item'.

Theories of manifolds have their problems. First, there is the minor grammatical puzzle ([13]) that in expressions like 'a manifold is...', the singular is used even when the manifold may be several in number. Likewise manifolds have properties though they cannot (significantly) belong to sets. Second, the logic of manifolds is not sufficiently explicit. For instance it has not been explained that, or why, it is not true that every manifold has a number.

(4) Objections justifiably made to ordinal and set-of-set analyses of *number* — as analyses or definitions, *not* as partial replacements or mere explications — on the ground that they admit as true or false what is not significant of numbers, e.g. that «numbers» belong to other «numbers» and that «numbers» are identical with sets or with individuals, cannot be lodged against the P-C analysis presented. For the requisite property logic can be so developed that membership relations between properties and identity relations between properties and sets or individuals are not (well-)defined. With significance logics, however, more positive conclusions can be reached: numbers are not significantly or truly identified with concrete individuals or sets (more precisely: $\sim \mathbf{S}(\alpha = n)$ & $\sim T(\alpha = n)$, where Tp has value 1 when p has value 1 and value —1 otherwise), and numbers are not truly or significantly elements of or contained in other numbers (e.g. $\sim T(6 \subset 8)$, $\sim \mathbf{S}(6 \subset 8)$, $\sim T(4 \varepsilon 5)$, $\sim \mathbf{S}(5 \varepsilon 6)$). To arrive at these results it is necessary to develop theories of identity and membership. Only the identity issue is tackled here. Benacerraf's contention that '4 = Julius Caesar' and $\hat{\Lambda} = 1$' are not significant seems to me correct. And it follows from $\sim \mathbf{S}$ (= 4 Julius Caesar) that $\sim T$ (4 = Julius Caesar). The important question is: how to *define* identity so that such consequences ensue and yet so that other needed features of identity such as intratypical transitivity ([14]) are preserved.

([13]) Pointed out by RUSSELL, *op.cit.*, 69-70.

([14]) That is transitivity within types or significance-ranges. Intertypical transitivity, where expressions of different types are related and significance-ranges are crossed, *should not* hold generally. The reasons for this are similar to reasons why Behmann formulae do not hold generally: see S. HALLDEN, *The Logic of Nonsense,* Uppsala (1949), 3.

The definition

(α): $\quad x = y =_{Df} (\Pi \text{ ext } f).\, f(x) \equiv f(y)$

where 'ext' abbreviates 'extensional', of contingent identity is too restrictive, since it eliminates as non-significant all heterogeneous identifications [15]. On the other hand

(β): $\quad x = y =_{Df} (\Pi \text{ ext } f)\ (\mathbf{S}f(x)\ \&\ \mathbf{S}f(y) \supset .\, f(x) \simeq f(y)),$

is too liberal, since under it all identity sentences are significant.

In order to present a definition of contingent identity which appears to have the required features further notation is needed. The significance-range of a given monadic predicate 'f', i.e. the class of expressions '...' such that 'f(...)' is significant, is specified by a superpredicate '$\hat{f}$', the relation between property and superproperty being given by these superpredicate theorems:

$$\mathbf{S}f(x) \equiv \mathrm{T}\hat{f}(x)$$
$$\equiv \mathbf{S}\hat{f}(x)\ \&\ \mathbf{P}\hat{f}(x),$$

where $\mathbf{P}p$ has value —1 when p has value —1 and value +1 otherwise. For example the superpredicate of 'is prime' is 'is a natural number', of 'is red', 'is coloured (or colourless)'. A predicate stands to its superpredicate in approximately the relation that a species predicate stands to its genus predicate. A substantival expression '- - -' has an associated monadic predicate 'is (a) - - -' and a predicate of this form has an associated substantival expression '- - -'. Finally, '$\hat{x}$' is the associated substantival expression of the superpredicate '$\hat{f}$' of the associated predicate 'f' of 'x'; e.g. if 'x_0' is 'prime number' then '$\hat{x}_0$' is 'natural number'.

Now consider

(γ): $\quad x = y =_{Df} (\Pi \text{ ext})f).\mathrm{T}f(x) \supset \mathbf{S}f(x)\ \&\ (\mathrm{T}f(y) \supset \mathbf{S}f(y))$
$$\rightharpoonup .f(x) \equiv f(y),$$

where '$\rightharpoonup$' represents a 3-valued conditional with matrix:

$\rightharpoonup$	1	—i	—1
1	1	—i	—1
—i	1	1	1
—1	1	1	1 .

(15) On definitions (α) and (β), and on the defects of (α), see V. MACRAE & R. ROUTLEY, 'On the identity of sensations and physiological occurrences', *American Philosophical Quaterly*, to appear in April 1966.

Although some vagueness has been taken up in (γ), by opting for a cut-off point after climbing just two places in species-genus trees, there is a wide measure of arbitrariness left in (γ) owing to latitude in superpredicate determination. But despite certain inevitable vagueness in application (γ) seems to capture contingent identity moderately well, and much better than either (α) or (β) does. First, intra-typical transitivity follows from (γ); e.g. if k, l and m are natural numbers, then, if (k = l) & (l = m), then (k = m), because $Tf(\overset{\frown}{m}) \supset \mathbf{Sf}(m) \equiv . Tf(\overset{\frown}{l}) \supset \mathbf{Sf}(l)$. Secondly, given acceptable assumptions (γ) yields anticipated significance results. To illustrate: it follows, taking the predicate 'has (at some times) spatial locations', that the sentence 'Julius Caesar = the first prime number' is not significant. For the superpredicate of 'is Julius Caesar' is 'is a person' and the supersuperpredicate is 'is a material thing'; and both 'a material thing has spatial locations' and 'Julius Caesar has spatial locations' have value 1. But the supersuperpredicate of 'is the first prime number' is 'is a number'; and 'a number has spatial locations' has either value —i or value —1. Thus the antecedent of the right-hand side of (γ) for the particular predicate 'has spatial locations' has value 1, but the consequent has value —i since 'the first prime number has spatial locations' has value —i. Therefore the universally quantified right-hand side has value —i; and so 'Julius Caesar = the first prime number' is not significant. The argument can be further elaborated: for non-significance claims can be established using superpredicate theorems and approximate choices of superpredicates can be vindicated. By similar arguments it can be shown (i) 'Julius Caesar = 4' is not significant and (ii) '$1 = \hat{\Lambda}$' is not significant. For (i) take, say, the predicate 'is coloured' or 'is an observable individual'; for (ii) the predicate 'belongs to a set'. The superpredicate of 'is 1' and 'is 4' (as predicates) is 'is a well-orderable manifold' and the supersuperpredicate roughly 'is a manifold'.

The P-C thesis, when developed by coupling with it such conditions as (γ), provides an analysis of number, the *P-C analysis*, which is definitely superior to all heterodox analyses such as ordinal, numeral or set-of-set analyses, because it does not let in as significant (or true) what the intuitive notion of number rules out as non-significant (or false). In contrast with these heterodox analyses, the P-C analysis classes as non-significant most sentences which are dismissed under the intuitive notion as nonsense, and the analysis of the intuitive notion is our main exercise. To endorse a heterodox analysis is to make a mistake, a bit like committing the naturalistic fallacy. On the other hand the P-C analysis does not admit as significant all sentences which are recognised as significant under the intuitive notion unless the analysis is supplemented by further definitions. For example division of certain properties (number properties) is only well-defined given appropriate definitions. But the same goes for heterodox analyses. e.g. division of certain sets has likewise to be defined. Despite its superiority as an analysis, the P-C analysis of course does not furnish synonyms for numerals: D3 does not give a synonym for '2', for a person can believe $2 > 1$ without believing its analysis. To expect a non-numerical synonym for '3' is to expect the impossible: to request a non-numerical analysis of the number 3, that is one which does not, like the *Oxford English Dictionary* definition of 'three' as 'one more than two', presuppose or appeal to other numbers and their relations, is to exclude the giving of synonyms for '3'.

III

The P-C thesis and analysis also gain support from other directions. Support from one direction derives from Benacerraf's concluding arguments ([16]), where it is stressed that «number theory is the elaboration of the properties of *all* structures of the order type of the numbers». I concur: because number theory is the theory of certain properties and relations of number properties, all structures of the order-type of the numerals exhibit these properties and their relation-instances. What usual explications of number do is to represent numbers by samples which instantiate these properties. Much as property-instances of colours in objects and samples can and do represent colour properties, so property-instances of numbers in standard manifolds and of membership numbers in paradigm sets can represent numbers. So also sets which instantiate a number property can indirectly represent and play *some* of the roles of the property (once the recursive progression to which the sets belong is specified). There are indefinitely many such representations none of which is uniquely singled out. It is not, however, the fact that there is no reason for preferring one representation to another that stops sensible equation of numbers with sets or with objects. There might

([16]) BENACERRAF, *op.cit.*, 69-71. These arguments can readily be turned against Benacerraf and against his earlier rejection of property theses.

be such reasons. Russell's cardinal classes, though not uniquely singled out, can claim advantages over certain other representations. No, it is simply not significant to *identify* number properties with their representatives; for sentences like 'That set is the number four' and 'The author of Waverley is the number seven' are not significant. So although any recursive progression can be adjusted ([17]) to provide an adequate representation of finite numbers, because such an adjusted progression instantiates the finite numbers, still no such progression of classes or objects is itself adequate insofar as it cannot be significantly identified with the numbers. No object can fully play the role of 3, and not every object, e.g. an Oxford Dictionary, can even represent 3 without some abstractions and adjustment. Indeed it is not particular objects (or even progressions) which matter; it is the (number) properties progressions of these objects exhibit which are important. Not that number properties are relational properties. The cardinal number 3, though necessarily the successor of 2 and predecessor of 4, can be introduced independently of the cardinals 2 and 4.

For similar reasons the Benacerraf-Goddard thesis that numbers are number-words ([18]) has to be rejected. Number-words are either word-tokens or word-types, i.e. sets of tokens [or else property-abstracts of these sets when they are extended to include possible tokens]. Thus the identification [if not a reduced property thesis] amounts to but a special case of the defective thesis that numbers are certain objects or sets. Thus it is not true ($\sim$T) that numbers are number-words. Benacerraf, despite his earlier arguments, has here opted for a nominalistically favoured representation of numbers. Nor does either Benacerraf or Goddard succeed in escaping all Frege's objections to formalism ([19]); even to avoid more familiar objections

([17]) Otherwise Russell's cardinality requirements will not be satisfied: see Russell's criticism of the Peano characterisation of natural numbers, *Introduction to Mathematical Philosophy,* London (1919), 10. See also BENACERRAF's criticism of Quine, *op.cit.,* 51, 72.

([18]) BENACERRAF, *op.cit.,* 71-3. L. GODDARD, 'Counting', *Australasian Journal of Philosophy* 39 (1961), 223-240. Strictly Goddard's thesis (P. 227) is that «a number is a numeral which is used in controlled counting», a much more difficult thesis to formalise. Naturally such nominalistic theses have a much longer history.

([19]) FREGE against the Formalists: in P. GEACH & M. BLACK (eds.), *Translations from the Philosophical Writings of Gottlob Frege,* Blackwell, Oxford (1960), 182-233.

Benacerraf has to appeal to the function, or sense, of numerals, and Goddard to possible uses of numerals. For manifolds which are not enumerated or counted may have numbers. But then universals are as good as back.

To conclude: on the basis of (1)—(4) and of the further support for the P-C thesis, I claim that there is an analysis of cardinal number, the P-C analysis, which within limits can be established to the exclusion of all other analyses.

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