SEMANTICS FOR DEONTIC LOGIC

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1. *Introduction*

In this paper I consider some deontic logics that are counterparts, in an obvious sense, of the alethic modal logics M, the Brouwersche system, S4, and S5. In section 2 these logics are specified and some of their relations to other modal logics, both alethic and deontic, are indicated. In sections 3 and 4 I adapt the semantical techniques of Kripke [5] for alethic modal logic to deontic logic. Specifically, section 3 contains a model-theoretic definition of validity and section 4 a corresponding method of semantic tableaux for each deontic logic under consideration. In section 5 each deontic logic is proved consistent and complete with respect to the valid formulas of its corresponding model theory, and the method of semantic tableaux is shown to constitute a decision procedure for each. Modalities are discussed in section 6. It is shown that each deontic logic has the same number of nonequivalent modalities as its corresponding alethic logic.

2. *Six deontic logics*

We define six logistic systems based on an enumerably infinite list of propositional variables and primitive connectives for negation ($\sim$), conjunction ($\wedge$), and obligation ($\Box$) (1). The well-formed formulas (wffs) of these systems are as usual. The axiom schemes and rules of inference for all six systems are contained in the following list:

A1. $A \supset (B \supset A)$
A2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A3. $(\sim B \supset \sim A) \supset (A \supset B)$
A4. $\Box (A \supset B) \supset (\Box A \supset \Box B)$
A5. $\Box A \supset \Diamond A$

(1) Familiar additional connectives are defined in terms of $\sim$, $\wedge$, and $\Box$ in the usual ways. I.e., $A \supset B$ is defined as $\sim (A \wedge \sim B)$, $A \vee B$ as $\sim (\sim A \wedge \sim B)$, $\triangle A$ as $\sim \Box \sim A$, and $A \equiv B$ as $(A \supset B) \wedge (B \supset A)$.

A6. $\Box(\Box A \supset A)$
A7. $\Box(A \supset \Box\Diamond A)$
A8. $\Box A \supset \Box\Box A$
A9. $\Diamond A \supset \Box\Diamond A$
R1. If $\vdash A$ and $\vdash A \supset B$, then $\vdash B$.
R2. If $\vdash A$, then $\vdash \Box A$.

The six deontic ([2]) logics to be studied here are called F, D, DM, DB, DS4, and DS5. They are defined as follows (where R1 and R2 are assumed for all):

F is Al — A4
D is Al — A5
DM is Al — A6
DB is Al — A7
DS4 is Al — A6, A8
DS5 is Al — A5, A8, A9

The names of the last four of these deontic logics derive from the alethic modal logics M, the Brouwersche system (hereafter referred to as "B"), S4, and S5 to which they are similar ([3]). In fact, if the axiom scheme

$$\Box A \supset A \qquad (1)$$

is added to DM (DB, DS4, DS5), the resulting system is identical with M (B, S4, S5). The result of adding this scheme to F or D is also M.

We now point out several relations between the systems just defined and others that have been studied or suggested in the literature. (a) Since none of our systems contains the axiom scheme (1), each of them is a result of following the suggestion of Kripke [5] (p. 95) for obtaining a deontic logic. (b) The systems DM, DS4, and DS5 are

([2]) Notice that it is stretching the terminology somewhat to call F a «deontic logic» since it does not contain A5. We shall nevertheless do so in this paper in order to have an easy way of referring to all the systems under consideration.

We use «F» as the name of the system based on A1-A4 since this system is fundamental for (i.e., included in) all the deontic and alethic systems considered in this paper.

([3]) M is given by von Wright in [10], p. 85. B is given by Kripke [5], p. 68. S4 and S5 are given by Lewis in [6], p. 501.

identical, respectively, with the systems OM^+, $OS4^+$, and $OS5^+$ of Smiley [8] (4). Smiley [8] has shown that these latter systems are identical, respectively, with the purely deontic parts of the mixed alethic-deontic systems OM, OM′, and OM″ of Anderson [1]. Hence the decision procedures given below for DM, DS4, and DS5 apply directly to the deontic fragments of Anderson's systems. The only previously-known decision procedures for these fragments required the use of procedures for M, S4 and S5 on formulas in which the obligation connective was expressed by Anderson's complicated definition. (c) It is easy to show, in analogy with the results given in sections 3-5, that the doxastic logic of Hintikka [3] (chapter 3) is equivalent to the system obtained by deleting A6 from our DS4, assuming that the variable a of Hintikka's connective B_a is held constant (5). (d) Finally, we observe that D and DS5 are identical, respectively, with the systems called DM and DS5 by Fitch [2]. Fitch's tree-proof method has several features in common with our use of semantic tableaux.

3. *Models*

We give modellings for the deontic systems of section 2 that closely parallel those given by Kripke [5] for the corresponding alethic systems (i.e., M, B, S4, S5). Specifically, we define a *model structure* as an ordered triple (G, K, R), where K is a non-empty set, $G \varepsilon K$ and R is a relation defined over K. We also stipulate that G never bears the relation R to itself. If no further restrictions are placed on R we call the model structure an F *model structure*. A model structure is called a D *model structure* if it has the following property: If $H_1 \varepsilon K$, then there is an $H_2 \varepsilon K$ such that $H_1 R H_2$. We call a D model structure a DM *model structure* if R is reflexive over {K-G} (6), a DB *model structure* if R is reflexive and symmetric over {K-G}, and a DS4 *model structure* if R is reflexive over {K-G} and transitive over K. A D model structure is called a DS5 *model structure* if R is transitive over K and has the following property: If $H_1, H_2, H_3 \varepsilon K$, $H_1 R H_2$,

(4) That DS5 is identical with OS5+ follows from the fact that A6 is a theorem of DS5. This is proved as T14 in section 5.

(5) It is also interesting to notice that Hintikka's epistemic logic ([3], chapter 3) is identical with S4, assuming that the variable a of the connective K_a is held constant.

(6) By {K-G} we mean the set consisting of all member of K except G.

and $H_1 \, R \, H_3$, then $H_2 \, R \, H_3$. It must be emphasized that DM, DB, DS4, and DS5 model structures are all D model structures. Hence they all have the property that each member of K bears the relation R to some member of K.

An F (D, DM, DB, DS4, DS5) *model* for a wff A of F (D, DM, DB, DS4, DS5) is a binary function Φ (P, H) associated with a given F (D, DM, DB, DS4, DS5) model structure (G, K, R). The variable P ranges over wellformed subformulas of A, while the variable H ranges over members of K. The function Φ takes values in the set $\{T, F\}$.

Assume that for a given model Φ associated with a model structure (G, K, R) the value of Φ (P, H) is specified for all P that are propositional variables of A, for all $H \varepsilon K$. The value of Φ (P, H) for all subformulas P of A can then be defined by induction as follows.

If $\Phi(B, H) = \Phi(C, H) = T$, then $\Phi(B \wedge C, H) = T$; otherwise $\Phi(B \wedge C, H) = F$. If $\Phi(B, H) = T$, then $\Phi(\sim B, H) = F$; otherwise $\Phi(\sim B, H) = T$. Finally, if $\Phi(B, H') = T$ for every H' in K such that $H R H'$, then $\Phi(\Box B, H) = T$; but if there exists an H' such that $H R H'$ and $\Phi(B, H') = F$, then $\Phi(\Box B, H) = F$.

We say that a wff A is *true* in a model Φ associated with a model structure (G, K, R) if $\Phi(A, G) = T$; *false* if $\Phi(A, G) = F$. We say that A is *valid* in F (D, DM, DB, DS4, DS5) if and only if it is true in all its F (D, DM, DB, DS4, DS5) models. It will be shown in section 5 that a wff is valid in a given system if and only if it is provable in that system.

Informally, the modellings given above can be explained as follows. We take $\{K\text{-}G\}$ to be a set of «permitted worlds», and G to be the «real world». The function Φ assigns a truth-value to each propositional variable of a wff A (and ultimately to each subformula of A including A itself) in each world H. If $H_1, H_2 \varepsilon K$, we interpret $H_1 \, R \, H_2$ as «H_2 is permitted with respect to H_1», or «every state of affairs described by a proposition that is true in H_2 is permitted in H_1». Hence, in view of the definition given above, «A is obligatory in the world H» (i.e., $\Phi(\Box A, H) = T$) is interpreted as «A is true in every world that is permitted with respect to H». The stipulation that G does not bear R to itself amounts to saying that some states of affairs existing in the real world are not permitted.

As was pointed out in section 2, the difference between our six deontic systems and the corresponding alethic systems (i.e., M, B, S4, and S5) is that the former lack the axiom (1). An exactly analogous situation obtains in the model-theoretic approach. Specifically, (1) is equivalent to the assumption that R is reflexive over K. For suppose that R is reflexive over K and that $\Box A$ is true in some member of K, call it H. Then by the definition of $\Phi(\Box A, H)$ and the reflexivity of R, it follows that A is true in H. $\Box A \supset A$ is therefore true in every member of K, since H was chosen arbitrarily. Hence if R is assumed to be reflexive over K, a deontic interpretation of the modellings becomes untenable. We then must adopt something like Kripke's alethic interpretations of his modellings for M, B, S4, and S5 [5] in which K is a set of possible worlds, and «A is necessary in world H» (i.e., $\Phi(\Box A, H) = T$) is interpreted as «A is true in every world (including H itself) that is possible with respect to H».

Consider now the differences among the deontic modellings themselves. If no restriction is placed on R then we have the simplest of these modellings, the one for F. Strictly speaking this modelling can hardly be given a deontic interpretation since it is possible to specify an F model in which $\Box A \supset \Diamond A$ is false. Indeed a satisfactory intuitive interpretation of F is difficult to give. Since $\Box A \supset \Diamond A$ is valid in the modellings for each of the remaining systems, they may all be properly characterized as deontic. Among them, DM, DB, DS4, and DS5 are significantly different from D in that R is reflexive over $\{K\text{-}G\}$ in each of the former. This restriction is equivalent to A6 and amounts to saying that every proposition true in a permitted world is also permitted in that world, or that each permitted world is permitted with respect to itself. Hence if we think of a permitted world as a state of affairs in which everything that is actual is permitted, then one of the systems DM, DB, DS4, DS5 is the deontic logic we want. If, on the other hand, we feel that even permitted worlds may have (with respect to themselves) some undesirable aspects, then our deontic logic is D. Among the systems DM, DB, DS4 and DS5, DM is the simplest. Additional assumptions about the relations among the various worlds — both real and permitted — give the systems DB, DS4, and DS5. In particular, by arguments analogous to those given by Kripke [5] for the various alethic reduction axioms, it can be shown that A7 is equivalent to symmetry of R over $\{K\text{-}G\}$, that A8 is equivalent to transitivity of R over K, and that A9 is equivalent to the special condition placed on R by a DS5 model structure.

4. *Semantic tableaux*

The method of semantic tableaux has been developed for modal logic by Kripke [4], [5]. We give here an analogue, for our deontic systems, of what Kripke [5] calls the «S-formulation» of the method of semantic tableaux for alethic modal logic. Familiarity with this method is presupposed.

The tableaux rules for negation and conjunction and the rule for obligation on the right are the same, for all our deontic systems, as those given in [5] (7). The rules for obligation on the left are different for each of the six systems. For F the rule is:

Yl. If $\Box$A appears on the left of a tableau t_1, put A on the left of each tableau t_2 such that t_1 S t_2.

For D we add a stipulation to assure that each tableau will always bear the relation S to some other tableau:

Yl. If $\Box$A appears on the left of a tableau t_1, put A on the left of each tableau t_2 such that t_1 St_2. If there is no such t_2, then start a new tableau t_2 with A on the left such that t_1 S t_2.

For the remaining systems the rules distinguish between the main tableau and the auxiliary tableaux of a set. We use «Ylm» to designate the part of the rule that applies to a main tableau and «Yla» to designate the part of the rule that applies to auxiliary tableaux.

From Dm we have:

Ylm. Same as Yl for D.

Yla. If $\Box$A appears on the left of an auxiliary tableau t_1, put A on the left of t_1 and on the left of each tableau t_2 such that t_1 S t_2, if any such t_2 exists.

Notice that we get the effect of reflexivity over the auxiliary tableaux by specifying in Yla that A is placed on the left of t_1.

The rule Yl for DB is:

Ylm. Same as Yl for D.

Yla. If $\Box$A appears on the left of an auxiliary tableau t_1, then: (1) put A on the left of t_1; and (2) put A on the left of each tableau t_2 such that t_1 S t_2, if any such t_2 exists; and

(7) We assume here, of course, that our obligation connective takes over the role of Kripke's necessity connective.

(3) put A on the left of the unique *auxiliary* tableau t_3 such that $t_3 S t_1$, if such an auxiliary tableau exists.

Here Yla gives us the effect of both reflexivity and symmetry over the auxiliary tableaux. It must be emphasized that clause (3) of Yla can be applied only if the unique t_3 such that $t_3 S t_1$ is also an auxiliary tableau. Failure to observe this restriction would give the effect of symmetry over all tableaux, and this is certainly not what we want in a deontic logic.

For DS4 Yl is as follows:

Ylm. If $\Box A$ appears on the left of a main tableau t_1, put $\Box A$ on the left of each tableau t_2 such that $t_1 S t_2$. If there is no such t_2, then start a new tableau t_2 with $\Box A$ on the left such that $t_1 S t_2$.

Yla. If $\Box A$ appears on the left of an auxiliary tableau t_1, put A on the left of t_1 and $\Box A$ on the left of each tableau t_2 such that $t_1 S t_2$, if any such t_2 exists.

Notice that this version of Yl gives the effect of transitivity over all the tableaux of a set. For if a tableau t_3 is introduced such that $t_2 S t_3$, subsequent applications of Yl will put $\Box A$ and hence ultimately A on the left of t_3.

Finally, we give Yl for DS5:

Ylm. Same as Ylm for DS4.

Yla. If $\Box A$ appears on the left of an auxiliary tableau t_1, then: (1) put A on the left of t_1; and (2) put $\Box A$ on the left of each tableau t_2 such that $t_1 S t_2$ if any such t_2 exists; and (3) put $\Box A$ on the left of the unique tableau t_3 (either main or auxiliary) such that $t_3 S t_1$.

In analogy to the previous cases, it can be verified that this version of Yl gives the effect, for a set of tableaux, of the conditions placed on R by a DS5 model structure.

We define *closure* for tableaux, sets of tableaux, systems of tableaux, and constructions as in [5]. It can be shown, in analogy with the proofs of Lemma 1 and Lemma 2 of [5], that the following result holds ([8]).

([8]) It should be pointed out that the stipulation that the element G of a model structure (G, K, R) never bears the relation R to itself plays a crucial

The F (D, DM, DB, DS4, DS5) construction for a wff A is closed if and only if A is valid in F (D, DM, DB, DS4, DS5).

Hence we can make use of the method of semantic tableaux to determine validity in the modellings of section 3.

5. *Consistency and Completeness*

We now use the method of semantic tableau to prove that a wff is a theorem of one of our deontic systems if and only if it is valid in the modelling of that system. The consistency part of this result is immediate, since it is easy to verify that the appropriate construction for each axiom is closed and that the rules preserve validity. It will also be apparent to those familiar with [5] that it is sufficient to prove Case Yl of the Lemma of section 4.2 of [5] for each of our deontic systems in order to establish the completeness of each.

We begin by giving a number of theorems and derived rules of the systems under consideration. Each of these theorems and rules is stated for the simplest system in which it holds (e.g., if a theorem holds in both DB and DS5 we will state it as a theorem of DB). Proofs are given only where they are not well-known or obvious.

Notice that derived rules 1-3 and T1-T8 — all well-known in modal logic — can be established in the very weak system F. Notice too that theorems such as T9 and T11 whose proofs are trivial in alethic modal logic have simple yet nontrivial proofs in deontic logic (in fact, they are not provable at all in some of our deontic systems).

Derived rule 1 of F: Substitution of equivalents (subs $\equiv$)

If $\vdash A \equiv B$, and if D is the result of substituting B for one or more occurrences of A in C, then $\vdash_{F} C \equiv D$ [9].

Derived rule 2 of F (DR2)

If $\vdash_{F} A \supset B$, then $\vdash_{F} \Box A \supset \Box B$.

T1. $\vdash_{F} \Box (A \supset B) \supset (\Diamond A \supset \Diamond B)$

role in this proof for the system DB. Without this stipulation it is easy to construct a DB countermodel to A7. The stipulation is apparently not needed for the other systems, but since it is natural for deontic logic we include it.

[9] We use the notation $\vdash_{a} A$ to mean «A is a theorem of the system a».

Derived rule 3 of F (DR3)

If $\vdash_F A \supset B$, then $\vdash_F \Diamond A \supset \Diamond B$.

T2. $\vdash_F \Box (A \wedge B) \equiv (\Box A \wedge \Box B)$

T3. $\vdash_F \Diamond (A \vee B) \equiv (\Diamond A \vee \Diamond B)$

T4. $\vdash_F \Box A \supset \Box (A \vee B)$

T5. $\vdash_F \Diamond (A \wedge B) \supset \Diamond A$

T6. $\vdash_F \Box (A \supset B) \supset [\Box (B \supset C) \supset \Box (A \supset C)]$

T7. $\vdash_F \Box (A \supset B) \supset [\Box (C \supset D) \supset \Box ((A \wedge C) \supset (B \wedge D))]$

T8. $\vdash_F (\Box A \wedge \Diamond E)) \supset \Diamond (E \wedge A)$

Proof: 1. $E \supset (\sim A \vee (E \wedge A))$ propositional calculus (pc)

2. $\Diamond E \supset \Diamond (\sim A \vee (E \wedge A))$ 1, DR3

3. $\Diamond E \supset (\Diamond \sim A \vee \Diamond (E \wedge A))$ 2, T3, pc

4. $(\Box A \wedge \Diamond E) \supset \Diamond (E \wedge A)$ 3, pc, def

T9. $\vdash_{DM} \Box \Box A \supset \Box A$

By A4, A6.

T10. $\vdash_{DM} \Box (B \supset C) \supset [\Diamond (\Box A \wedge B) \supset \Diamond (A \wedge C)]$

Proof: 1. $(\Box A \supset A) \supset [(\Box A \wedge B) \supset (A \wedge B)]$ pc

2. $\Box (\Box A \supset A) \supset \Box [(\Box A \wedge B) \supset (A \wedge B)]$ 1, DR2

3. $\Box [(\Box A \wedge B) \supset (A \wedge B)]$ A6, 2, R1

4. $\Diamond (\Box A \wedge B) \supset \Diamond (A \wedge B)$ T1, 3, R1

5. $(B \supset C) \supset [(A \wedge B) \supset (A \wedge C)]$ pc

6. $\Box (B \supset C) \supset \Box [(A \wedge B) \supset (A \wedge C)]$ 5, DR2

7. $\Box (B \supset C) \supset [\Diamond (A \wedge B) \supset \Diamond (A \wedge C)]$ T1, 6, pc

8. $\Box (B \supset C) \supset [\Diamond (\Box A \wedge B) \supset \Diamond (A \wedge C)]$ 4, 7, pc

T11. $\vdash_{DS4} \Box A \supset \Diamond \Box A$

By A5, A8

T12. $\vdash_{DS4} (\Box A \wedge \Diamond E) \supset \Diamond (E \wedge \Box A)$

By T8, A8

T13. $\underset{DS5}{\vdash} \Diamond(\Box A \wedge B) \supset \Box A$

By T5, A9

T14. $\underset{DS5}{\vdash} \Box(\Box A \supset A)$

Proof: 1. $\Diamond(\Box A \wedge \sim A) \supset (\Diamond \Box A \wedge \Diamond \sim A)$ T5, pc

2. $(\Diamond \Box A \supset \Box A) \supset \Box(\Box A \supset A)$ 1, pc, def.

3. $\Box(\Box A \supset A)$ A9, pc, def, 2, R1

Case Yl of Kripke's Lemma can now be proved for each of our deontic systems as follows:

F, Case Yl

Before: (10) $\Box A \wedge X \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots$

After: $\Box A \wedge X \wedge \Diamond(E_1 \wedge A) \wedge \Diamond(E_2 \wedge A) \wedge \ldots$

Justified by T8, pc.

D, Case Yl

Subcase 1. Before: $\Box A \wedge X \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots$

After: $\Box A \wedge X \wedge \Diamond(E_1 \wedge A) \wedge \Diamond(E_2 \wedge A) \wedge \ldots$

Justified by T8, pc.

Subcase 2. Before: $\Box A \wedge X$

After: $\Box A \wedge X \wedge \Diamond A$

Justified by A5, pc.

DM, Case Ylm

Same as the two subcases for D.

DM, Case Yla

Before: $\Diamond(\Box A \wedge X \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots)$

After: $\Diamond(\Box A \wedge A \wedge X \wedge \Diamond(E_1 \wedge A) \wedge \Diamond(E_2 \wedge A) \wedge \ldots)$

Let B stand for $\Box A \wedge X \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots$

Let C stand for $\Box A \wedge X \wedge \Diamond(E_1 \wedge A) \wedge \Diamond(E_2 \wedge A) \wedge \ldots$

Hence by D, subcase 1, we have $\underset{D}{\vdash} B \supset C$, and by R2, $\underset{D}{\vdash} \Box(B \supset C)$.

From this by T10 and R1 we get $\underset{DM}{\vdash} \Diamond(\Box A \wedge B) \supset \Diamond(A \wedge C)$, which is the result we want.

(10) By «before» and «after» we mean the relevant part of the characteristic formula of the alternative set of tableaux in question, before and after the application of Yl.

DB, Case Ylm

Same as the two subcases for D.

DB, Case Yla

Before: $\Diamond [X \wedge \Diamond (\Box A \wedge Y \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots)]$

After: $\Diamond [X \wedge A \wedge \Diamond (\Box A \wedge A \wedge Y \wedge \Diamond (E_1 \wedge A) \wedge \Diamond (E_2 \wedge A) \wedge \ldots)]$

Justified as follows:

1. $\Box [\Diamond (\Box A \wedge Y \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots) \supset \Diamond (\Box A \wedge A \wedge Y \wedge \Diamond (E_1 \wedge A) \wedge \Diamond (E_2 \wedge A) \wedge \ldots)]$
   Case Yla of DM, R2
2. $\Box (X \supset X)$ — pc, R2
3. $\Box [\Diamond (\Box A \wedge Y \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots) \supset \Diamond \Box A]$ — T5, R2
4. $\Box (\Diamond \Box A \supset A)$ — A7, pc, subs $\equiv$, def
5. $\Box [\Diamond (\Box A \wedge Y \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots) \supset A]$ — 3, 4, T6, R1
6. (Before) $\supset$ (After) — 1, 2, 5, T7, T1, R1

DS4, Case Ylm

Subcase 1. Before: $\Box A \wedge X \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots$
After: $\Box A \wedge X \wedge \Diamond (E_1 \wedge \Box A) \wedge \Diamond (E_2 \wedge \Box A) \wedge \ldots$
Justified by T12, pc.

Subcase 2. Before: $\Box A \wedge X$
After: $\Box A \wedge X \wedge \Diamond \Box A$
Justified by T11, pc.

DS4, Case Yla

Before: $\Diamond (\Box A \wedge X \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots)$

After: $\Diamond (\Box A \wedge A \wedge X \wedge \Diamond (E_1 \wedge \Box A) \wedge \Diamond (E_2 \wedge \Box A) \wedge \ldots)$

Analogous to DM, Case Yla, except that T12 is used in place of T8.

DS5, Case Ylm

Same as Case Y1m for DS4.

DS5, Case Y1a

Before: $X \wedge \Diamond (\Box A \wedge Y \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots)$

After: $X \wedge \Box A \wedge \Diamond (\Box A \wedge A \wedge Y \wedge \Diamond (E_1 \wedge \Box A) \wedge \Diamond (E_2 \wedge \Box A) \wedge \ldots)$

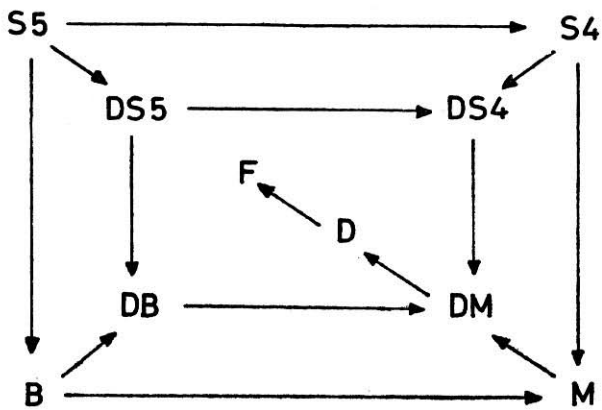
Justified by Case Yla for DS4, T13, pc. This depends on T14 (i.e., on the fact that A6 can be derived as a theorem of DS5).

All remaining steps in the completeness proof can be established exactly as in [5]. Hence we have the following metatheorem:

> A wff A is a theorem of F (D, DM, DB, DS4, DS5) if and only if it is valid in F (D, DM, DB, DS4, DS5).

It should also be pointed out that the method of semantic tableaux can be used to give decision procedures for each of our deontic systems. This follows easily for F, D, DM, and DB, since each of the tableaux rules for these systems eliminates a connective. The rules for DS4 and DS5 do not all have this property, but constructions in these systems can also be made to terminate by adopting a procedure similar to that of [5] for eliminating repetitive tableaux.

The relations between our six deontic systems and the corresponding alethic system are expressed by the following diagram:



Here an arrow means that the system from which it originates contains all the theorems of the system to which it points, and more besides. If it is not possible to move from one system to another by following arrows, then the latter system contains theorems that are not contained in the former.

6. *Deontic modalities*

Since each of the systems F, D, and DM is contained in M, it follows that each has an infinite number of nonequivalent modalities. Similarly, that the number of nonequivalent modalities in DS4 (DS5) is no smaller than fourteen (six) follows from the fact that S4 (S5) contains DS4 (DS5). The decision procedures for DS4 and DS5 can be used to show that all the reduction equivalences of S4 and S5 are theorems of the corresponding deontic systems. In particular, the converse of A8 and

$$\Box \sim \Box \sim \Box \sim \Box A \equiv \Box \sim \Box A$$

are theorems of DS4, and the converse of A9 is a theorem of DS5. Hence there are no more than fourteen (six) non-equivalent modalities in DS4 (DS5).

Since DB is contained in B, it follows that DB has at least as many nonequivalent modalities as B. That the number of nonequivalent modalities in B is infinite can be shown as follows. Consider wffs of the form

$$\Box_n A \supset \Box_m A,$$

where $\Box_n$ stands for a string of n$\Box$'s. By means of Kripke's semantic-tableaux decision procedure for B (given in [5]) it can be shown that, in general, a wff of this form is not a theorem of B unless $n \geqslant m$. As an example, let $n = 2$ and $m = 3$. The construction is then:

	$\Box\Box A \supset \Box\Box\Box A$
$\Box\Box A$	$\Box\Box\Box A$
$\Box A$	
A	
↓	
$\Box A$	$\Box\Box A$
A	
↓	
A	$\Box A$
↓	
	A

Obviously, whenever $n < m$ the left sides of the tableaux in the construction will always be «behind» the right sides, so that closure will not occur. Hence, in general, a wff of the form

$$\Box_n A \equiv \Box_m A$$

is not a theorem of B unless n = m. From this it follows easily that the number of nonequivalent modalities in B (and hence in DB) is infinite.

It should be pointed out that an exactly analogous argument can be used to show that M has infinitely many nonequivalent modalities. This result was originally established by Sobocinski [9], using an infinite matrix of McKinsey [7]. The argument from semantic tableaux is simpler and, since it is based on an intuitively plausible interpretation of M, more natural than the argument from McKinsey's matrix.

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