THEORY OF MULTIPLICITIES

F. G. ASENJO

I. *Purpose*

Cantor introduces sets as «collections of definite and separate objects into wholes» in one paper [5] and as «multiplicities taken as units» in another [6] (Cf. also [7]). These are not definitions but informal characterizations; however, presumably sets comply with them. Indeed, elements of a set are meant to be separated objects, but we are never told how such separation should be understood: a few from all the others, or each one from any other. Since multiplicities exist in which separation cannot be perfected (see final section), while at the same time the theory of (cardinal) sets actually assumes the existence of multiplicities of completely separable elements, it would seem that sets develop from a rather special kind of multiplicity.

In pragmatic terms, before composing elements we must produce them through a process of division. Bearing in mind a given property, we separate entities from one another and from the continuum of things. But in order to obtain a multiplicity with which to form a unit, a rather advanced stage of preliminary analysis must already be underway, an analysis that turns amorphous bundles of objects into clear-cut multiplicities of elements. Although one may legitimately take this analysis for granted, we believe it is useful to systematize the formal features of such a process of division.

Or to put it still another way. We are told to distinguish the element e from the set $\{e\}$, the reason for this distinction being well known. But how should we distinguish between a, b and the set $\{a, b\}$ when a is different from b ? This paper will answer this question by showing how to get to what remains of the set after removing the brackets. The study will demonstrate that multiplicities are abstracts entities which precede any theory of sets — «precede» being used here in the same sense we mean when we say that non-associative systems precede the theory of semi-groups.

2. *Conglomerates and Divisions*

We shall refer to the following symbols. a) *Terms*: 0, ±1, ±2, ... b) *Functions*: **0**, **±1**, **±2**, ... c) *Equality*: =. d) *Parentheses* and *edges*.

Some strings of symbols will be called «*conglomerates*» in accordance with the formation rules that follow. c1) A term is a conglomerate. c2) If x and y are conglomerates and **w** is a function, then x**w**y is a conglomerate. Other expressions will be called «*divisions.*» d1) A function is a division. d2) A directed tree of degree two (not necessarily finite), in which the vertices are terms and the two edges issuing from each vertex are ordered, is a division. We now establish a one-to-one correspondence between divisions formed according to d1 and divisions formed according to d2. This correspondence in effect labels some trees with a function symbol. A collection of trees and their labels (under a given correspondence) will be called a «*system S of conglomerates and divisions*». Given a system S, the expression «*division tree* **w**» names a well-determined tree for each value of **w**. Each division tree is the product graph of elementary trees of the form

$$\begin{array}{c} x \\ \swarrow \; \searrow \\ y \quad z \end{array}$$

. With specific values for x, y, z, **w**, if

$$\begin{array}{c} x \\ \swarrow \; \searrow \\ y \quad z \end{array}$$

occurs in the division tree **w** then, by definition, the predicate formula x = y**w**z will be said to be «*satisfied*» in S by the given values of x, y, z, and **w**. In such a case we shall call y and z the «**w**-*components*» of x. Since

$$\begin{array}{c} x \\ \swarrow \; \searrow \\ y \quad z \end{array}$$

may occur in different division trees, then x = y**w**z may be satisfied by several values of **w** while x, y, and z retain the same values. However, x = y**w**z does not necessarily imply x = z**w**y. When it is possible to determine whether or not x = y**w**z is satisfied in S for any given values of x, y, z, and **w**, we shall say that the system S is «*well defined*».

A term x will be called «**w**-divisible» in S if and only if x occurs in the division tree **w** without being a terminal vertex. If x is **w**-divisible in S and x = y**w**z is satisfied in S by particular values of x, y, z, and **w**, then we shall say that y is «**w**-*distinguishable*» from z in S. If in addition z is **w**-distinguishable from y in S, then y and z will be called «*separable*» *in* S. If x is not **w**-divisible for every value of **w** in S, then x will be called «*indivisible*» in S.

A conglomerate will be called a «*section*» of a division tree **w** in S if and only if (i) it is of the form y**w**z where y and z are terms and x = y**w**z is satisfied in S by the particular values of x, y, z, **w**, or (ii) it is of the form u**w**v where u and v are sections.

3. PRODUCT GRAPH OF DIVISION TREES, DIVISION SKELETONS, MULTIPLICITIES

Let us now consider a sequence of division trees **w**, **u**, **v**, ... from S and form with them the *product graph* P. Next let us examine the following subgraphs of P. *If* x occurs in **w** and y occurs

```
x              y
↙ ↘            ↙ ↘
y   z          t   s
```

in **u**, then both will combine in P as the single subgraph P_x:

```
    x
   ↙ ↘
  y   z
 ↙ ↘
t   s
```

(Although the subgraph P_x is written in tree form, it may be cyclic and hence not necessarily a tree.) With specific values of x, y, z, t, s, whenever P_x is a subgraph of P we shall say that the predicate formula x = (t**u**s)**w**z is *satisfied* in P by the values of x, t, s, z, **u**, **w**. Combining elementary trees from eventually different division trees into a single subgraph can be extended by connecting new elementary trees of S to the terminal vertices of P_x. Then new and larger graphs in tree form are obtained. Between all these graphs and the collection of predicate formulas x = m**w**n, with m and n conglomerates, there is a correspondence that is many-many. The correspondence between P_x and x = (t**u**s)**w**z mentioned earlier makes obvious all the changes that can be made in either P_x or x = (t**u**s)**w**z without affecting the correspondence. Each expression of the form x = m**w**n, with m and n conglomerates, will be said to be *satisfied* in P by particular terms and functions if and only if one of the graphs that corresponds to x = m**w**n is a subgraph to P.

Given a conglomerate m**w**n, we now disregard all terms and retain functions and parentheses. We then obtain a «*division skeleton.*» A division skeleton may be represented by a graph in tree form where vertices are functions. Division skeletons can be considered as operators which, when applied to different terms of P, produce different subgraphs in tree form of P.

If we extend in the obvious way the definition of section of a division tree to «*section* of a subgraph of P in tree form», then «*multitiplicities*» may be defined as any section of a subgraph of P obtained by a division skeleton applied to a vertex of P. Multiplicities may be written as conglomerates: for example, $((x_1\mathbf{y}_1x_2)\ \mathbf{y}_2x_3)\ \mathbf{y}_3(x_4\mathbf{y}_4x_5)$, with the x_i representing terms and the $\mathbf{y}_j$, representing functions (or more briefly, if divisions are unimportant: $((x_1, x_2), x_3), (x_4, x_5))$. If this example represents a multiplicity in P, then $x = ((x_1y_1x_2)y_2x_3)y_3 (x_4y_4x_5)$ is satisfied in P by the particular terms and functions represented. In both of the ways that a multiplicity can be written the parentheses indicate direction and degree of separability of terms. In the example just given, x_1 is y_1-distinguishable from x_2; but if each variable represented a different term, this multiplicity would not provide information as to whether or not x_1 and x_2 were separable. To get this information another multiplicity that contains (x_2, x_1) must be obtainable in P. If we define a «*sequence of elements*» as a sequence of perfectly separable terms, that is, a sequence of terms in which each term is separable from any other term (in the sense of section 2), then it is clear that, to obtain n elements, at least (2n-2) ! / (n-1) ! different multiplicities of n terms must be produced in P, multiplicities which should provide for each x_i and x_j both (x_i, x_j) and (x_j, x_i). If all these multiplicities can be obtained in P, then we may drop the parentheses and write simply $x_1, x_2 \ldots, x_n$. We may consider that Cantor's theory of cardinal sets starts at his point, the point at which sequences of perfectly separable terms are obtained within a given system of conglomerates and divisions.

4. *Multiplicities Redefined*

Theories stated in terms of graph concepts have the advantage of being intuitive but the disadvantage of lacking the methods of proof necessary to build a general theory. For this reason a redefinition of multiplicities in less intuitive and more workable terms is desirable. The definition that follows reveals the algebraic nature of multiplicities.

Consider a non-empty set of terms S and a sequence of single-valued binary operations on S. These operations need not be defined for all the ordered pairs of elements of S. In this way, we obtain a family of halfgroupoids $F_1 = \{S_i : i \varepsilon S\}$ in which the index set S is the given set of terms. This family is not necessarily compatible (ab = c in S_i and ab = d in S_j, with $i \neq j$, does not necessarily imply c = d), nor is it necessarily disjoint ($a \varepsilon S_i$ and $a \varepsilon S_j$ does not imply i = j).

Next, let us form the families of Cartesian products:

$$F_2 = \{S_i \times S_i : i \varepsilon S\} = \{S \times_i S : i \varepsilon S\},$$

$$F_3 = \{(S \times_i S) \times_j S \cup S \times_k (S \times_l S) : i, j, k, l \varepsilon S\},$$

$F_4 = \{S \times_i (S \times_j (S \times_k S)) \cup \ldots \cup ((S \times_r S) \times_s S) \times_t : i, j, k, \ldots, r, s, t \varepsilon S\}$, and in general

$F_n = \{S \times_i T \cup U \times_j V \cup W \times_k S$: T and W are any members of F_{n-1}, U and V are members of F_x and F_y respectively, with $x + y = n$ and i, j, k ε S$\}$. An element of a member of F_2 is indicated by **aib**, and we will call it *of order two*; an element of a member of F_3 is indicated by either **(aib)jc** or **ak(bic)**, and we will call it *of order three* (the order of an element of a member of F_n being in general n, its number of terms). An element of a member of F_n is of the form **xyz**, with the order of x plus the order of z being n (with an appropriate parenthesis structure). We now call a «*component* of an element of a member of F_n» any expression of the form **abc** that occurs in **xyz** with the order of a plus the order of c being k with $2 \leqslant k \leqslant n$.

Finally, let us form the union of all families defined above: $F = \cup_n F_n$. Then a «*multiplicity*» can be defined as any element of a member of F such that every component **abc** of the element has the property that a . c is defined in S_b. This definition is algebraically equivalent to the one given in section 3. Interesting problems regarding the algebra of multiplicities arise from imposing special algebraic conditions on the family of halfgroupoids F_1.

5. *Final remarks*

We invite the reader to compare the theory presented here with Leonard and Goodman's theory of individuals [8], noting that the discreteness symbol employed by Leonard and Goodman is a predicate symbol, while **0**, **±1**, **±2**, ... are function symbols. Also, it is useful to interpret conglomerates and multiplicities as term-relation numbers [1], [2]. It is clear that divisions can be interpreted as internal relations.

The theory of multiplicities, instead of proceeding from the simple to the complex, starts by dividing complex conglomerates into simpler ones, in accordance with the principle that division is prior to composition. In physics, for example, we know that at times several particles exist in a given volume, although it is impossible to distinguish more than one particle from the others. Nevertheless, we accept the idea that each particle is a «definite and separate» element: a convenient simplification, since we do not know any better way of considering the matter. Also, in the midst of a horizontal train of waves, some barriers work only in one direction and not in the other, permitting us to distinguish left from right, for instance, but not vice versa. The idea of multiplicity introduced here calls attention to the physical character of these facts, a character that is violated by superimposing the idea of element, which in turn is tantamount to the introduction of the physical hypothesis of complete separation — a hypothesis we have no grounds for transferring to the microscopic world from our macroscopic observations.

University of Pittsburgh F. G. ASENJO

REFERENCES

1. F. G. ASENJO, «Relations irreducible to classes», *Notre Dame J. of Formal Logic*, IV (1963).
2. ——, «The arithmetic of the term-relation number theory», *Notre Dame J. of Formal Logic*, V (1964).
3. ——, «One and many», *Philosophy and Phenomenological Research* (to appear).
4. R. H. BRUCK, *A Survey of Binary Systems*, Springer (1958).
5. G. CANTOR, «Beiträge zur Begründung der transfiniten Mengenlehre», *Math. Annalen*, XLVI (1895), pp. 481-512.
6. ——, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*, Leipzig (1883).
7. I. KANT, «Totality is nothing else but plurality contemplated as unity», *Critique of Pure Reason*, Transc. Logic, I, § 7.
8. H. S. LEONARD and N. GOODMAN, «The calculus of individuals and its uses», *J. of Symbolic Logic*, 5 (1940), p. 45.
9. O. ORE, *Theory of Graphs*, Amer. Math. Soc. Coll. Pub. XXXVIII (1962).