# SOME THEOREMS ON THE RELATIVE STRENGTHS OF MANY-VALUED LOGICS

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In what follows, we adopt the definitions given in the author's papers 'On finitely many-valued logics' (*Logique et Analyse*, n° 25-26 1964) and 'On denumerably many-valued logics' (*Ibid.*,). Also, 'm', 'n', and 'p' are used as metalinguistic variables; the first two are to range over all positive integers greater than 1 and 'p' over these integers and also $\omega$.

If k is a positive integer, then the $k^{th}$ sentential constant = $\langle$the $2k^{th}$ individual constant the $2k^{th}$ 1-place predicate$\rangle$. A formula f is p-satisfiable just in case there are a p-interpreter i and assigner in Ui a such that Int ia (f) $\neq$ O. Similarly, f is sententially p-satisfiable just in case there is a v in VTp such that v(f) $\neq$ O. Obviously, f is p-satisfiable just in case $\sim$f is not p-valid and sententially p-satisfiable just in case $\sim$f is not a p-tautology.

If $s_1, \ldots, s_n$ are formulas, then $s_1 < s_2 = \sim \vdash \langle s_2 \rightarrow s_1 \rangle$ and $s_1 < \ldots < s_n = \langle \ldots \langle (s_1 < s_2) \wedge \ldots \rangle \wedge (s_{n-1} < s_n) \rangle$. Obviously,

*Theorem 1.* If $s_1, \ldots, s_n$ are distinct sentential constants, then $s_1 < \ldots < s_n$ is (sententially) (1) p-satisfiable just in case n is not greater than p.

Hence,

*Theorem 2.* If n is smaller than p, $s_1, \ldots, s_{n+1}$ are distinct sentential constants, and $t = \sim(s_1 < \ldots < s_{n+1})$, then

(1) t is n-valid and not p-valid;

(2) t is an n-tautology and not a p-tautology;

(3) for any d and e, if d is an n-valued logic and e is a p-valued logic, then t is a provable and not e-provable; and

(4) t is n-provable, but not p-provable.

For assume the antecedent. (1) and (2) follow immediately from theorem 1 and (3) follows from (1). (4) follows from (2) and (1) via

(1) We enclose a word in parentheses in a theorem to indicate that the theorem holds whether or not the word is present. Thus, every such theorem is really two theorems.

the facts that every formula which is an n-tautology is n-provable and that every p-provable formula is p-valid.

In other words, as we increase the number of truth values, we lose provable formulas.

We say that a positive integer i divides a positive integer j just in case there is a positive integer k such that k multiplied by i = j.

*Theorem 3.* If m—1 divides n—1, then, for any formula f,
(1) f is n-valid only if f is m-valid;
(2) f is an n-tautology only if f is an m-tautology [2];
(3) for any d and e, if d is an m-valued logic and e is an n-valued logic, then f is e-provable only if f is d-provable; and
(4) f is n-provable only if f is m-provable.

For assume the antecedent. Hence, for some positive integer k, k multiplied by (m—1) = n—1. Assume now that i is an m-interpreter and that v is VTm. For any positive integer k and predicate q, let Sq = the k-term sequence s such that the range of s = {i(q)}. Let j = the n-interpreter such that j(c) = i(c) for any individual constant c and j(q) = (Sq) (1) ⁀ ... ⁀ (Sq) (k) for any predicate q. Finally, let w = the w in VTn such that, for any sententially atomic formula a, w(a) = (k multiplied by (the b such that v(a) = b divided by (m—1))) divided by (k multiplied by (m—1)). By an induction among the members of TF, it can be shown that, for any term or formula t, formula g, and assigner in Uj = Ui a, Int ja (t) = Int ia (t) and w(g) = v(g). Hence, if f is n-valid, then f is i-true; and, if f is an n-tautology, then v(f) = 1. But then (1) and (2) hold. (3) and (4) follow immediately from (1) and (2) respectively.

Combining theorems 2 and 3 with the facts that a formula is n-valid if ω-valid, an n-tautology if an ω-tautology, and n-provable if ω-provable, we have

*Theorem 4.* If m is smaller than p and either m—1 divides p—1 or p = ω, then
(1) the set of all p-valid formulas is a proper subset of the set of all m-valid formulas;
(2) the set of all formulas which are p-tautologies is a proper subset of the set of all formulas which are m-tautologies;

[2] This is a generalization of the third part of theorem 17 of J. Łukasiewicz's and A. Tarski's 'Investigations into the sentential calculus' (in Tarski's book *Logic, Semantics, Metamathematics*, Oxford, 1956).

(3) for any d and e, if d is an m-valued logic and e is a p-valued logic, then the set of all e-provable formulas is a proper subset of the set of all d-provable formulas; and
(4) the set of all p-provable formulas is a proper subset of the set of all m-provable formulas.

The question now arises if there are other conditions for the consequences of theorem 4 than that m—1 divides p—1 when p is finite. We shall see that the answer is no.

If f and g are formulas, then $f+g = {\sim}f \rightarrow g$ and $f \dot{+} g = \langle {\sim} \vdash (f+g) \wedge (f+g) \rangle \vee \langle \vdash \langle f \leftrightarrow {\sim} g \rangle \wedge (f+g) \vee \langle {\sim} \vdash \langle f \leftrightarrow {\sim} g \rangle \wedge \vdash (f+g) \wedge {\sim} (f+g) \rangle$.

Notice that, for any v in VTp, $v(f+g) = v(f \dot{+} g) = v(f) + v(g)$ when this sum is not greater than 1. On the other hand, when $v(f) + v(g)$ is greater than 1, $v(f+g) = 1$ while $v(f \dot{+} g) = 0$. A corresponding situation holds with respect to p-interpreters.

If f is a formula, k is a positive integer, s is a k-term sequence, and the range of $s = \{f\}$, then $k \cdot f = (\ldots(s(1) \dot{+} s(2)) \dot{+} \ldots) \dot{+} s(k)$. Notice that, for any v in VTp, $v(k \cdot f)$ = k multiplied by v(f) if this product is not greater than 1 and 0 otherwise. A corresponding situation holds with respect to p-interpreters. Now,

*Theorem 5.* If s is a sentential constant, then $\vdash (m—1) \cdot s$ is (sententially) n-satisfiable just in case m—1 divides n—1.

For assume the antecedent. If v is in VTn, $v(\vdash (m—1) \cdot s) \neq 0$, and k = the natural number k such that v(s) = k divided by (n—1), then (m—1) multiplied by (k divided by (n—1)) = 1; that is, k multiplied by (m—1) = n—1. Hence, $k \neq 0$ and so m—1 divides n—1. Similarly, if k is a positive integer and k multiplied by (m—1) = n—1, then (m—1) multiplied by (k divided by (n—1)) = 1. But there is a v in VTn such that v(s) = k divided by (n—1) and so $v(\vdash (m—1) \cdot s = 1$. Hence, $\vdash (m—1) \cdot s$ is sententially n-satisfiable just in case m—1 divides n—1. The proof with respect to n-satisfiability is analogous.

Hence,

*Theorem 6.* If m is smaller than n, m—1 does not divide n—1, s is a sentential constant, and $t = {\sim} \vdash (m—1) \cdot s$, then
(1) t is n-valid and not m-valid;
(2) t is an n-tautology and not an m-tautology;
(3) for any d and e, if d is an m-valued logic and e is an n-valued logic, then t is e-provable and not d-provable; and

(4) t is n-provable, but not m-provable.

For assume, the antecedent. (1) and (2) follow from theorem 5 and the fact that m—1 divides m—1 and (3) follows from (1). (4) follows from (2) and (1) via the facts that every formula which is an n-tautology is n-provable and that every m-provable formula is m-valid.

Combining theorems 4 and 6, we obtain

*Theorem 7*. If m is smaller than n, then the following conditions are equivalent:

(1) m—1 divides n—1;

(2) the set of all n-valid formulas is a (proper) subset of the set of all m-valid formulas;

(3) the set of all formulas which are n-tautologies is a (proper) subset of the set of all formulas which are m-tautologies ([3]);

(4) for any d and e, if d is an m-valued logic and e is an n-valued logic, then the set of all e-provable formulas is a (proper) subset of the set of all d-provable formulas; and

(5) the set of all n-provable formulas is a (proper) subset of the set of all m-provable formulas.

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([3]) The statement of the equivalence of (1) and (3) without the parenthesized word is a generalization of theorem 19 (due to J. Łukasiewicz and A. Lindenbaum) of the paper cited in note 2.