GENTZEN'S HAUPTSATZ FOR THE SYSTEMS NI AND NK (*)

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In [1], [2] Gentzen sets up two systems of natural deduction NI and NK for intuitionistic and classical quantification theory. NI has the required properties to prove that all deductions can be reduced to a normal form that avoids roundabouts. NK, however, does not possess such a property because it contains the principle of excluded middle.

To overcome this difficulty Gentzen developes another system of quantification theory which, in the intuitionistic version LI as well as in the classical LK, possesses the required property. Gentzen proves the Hauptsatz for these two systems.

We intend to prove Gentzen's Hauptsatz for NI and a weaker variant for NK. Gentzen has certainly had proofs of these two theorems (1) but he has not published them.

We use a special version of NI and NK in which the negation $\neg A$ is expressed by the implication $A \supset \Lambda$ (Λ is an atomic sentential constant denoting an arbitrary inconsistent proposition). In this way the two rules introducing and eliminating negation become special cases of the rules introducing and eliminating implication (2).

The rules of NI are the following:

α)

$$\text{I}\supset \quad \frac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B} \qquad\qquad \text{I}\vee \quad \frac{A}{A \vee B} \quad \frac{B}{A \vee B}$$

$$\text{I}(x) \quad \frac{A(a)}{(x)\,A(x)} \qquad\qquad \text{I}(\exists x) \quad \frac{A(a)}{(\exists x)\,A(x)}$$

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(1) [1] p. 177; [2] p. 5.
(2) [1] p. 189; [2] p. 27.

$$\text{I}\wedge \quad \frac{A, B}{A \wedge B} \qquad\qquad \text{E}\Lambda \quad \frac{\Lambda}{A}$$

β)₁

$$\text{E}\supset \quad \frac{A, A \supset B}{B} \qquad\qquad \text{E (x)} \quad \frac{(x)\, A(x)}{A(a)}$$

$$\text{E}\wedge \quad \frac{A \wedge B}{A} \qquad\qquad \frac{A \wedge B}{B}$$

β)₂

$$\text{E}\vee \quad \frac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

$$\text{E}\,(\exists x) \quad \frac{(\exists x)\; A\;(x) \qquad \begin{matrix}[A(a)]\\ C\end{matrix}}{C}$$

NK contains besides the rule of elimination of double negation:

$$\frac{(A \supset \Lambda) \supset \Lambda}{A}$$

The rules of NI and NK allow to introduce (I) or eliminate (E) logical signs. I ⊃, E∨ and E(∃x) discharge the corresponding assumptions enclosed in square brackets. In I (x) the proper variable a does not occur either in the conclusion or in an assumption on which the conclusion depends. In E (∃ x) the proper variable a does not occur either in the left premise or in the right or in an assumption different from [A (a)] on which the right premise C depends. The assumptions enclosed in square brackets may not occur in the deduction; i.e. we may discharge assumptions which have not been really used. (3)

The degree of a formula is the number of its logical signs (Λ has

(3) [1] p. 185; [2] p. 21.

degree O). The degree of a β) rule is the degree of its single premise, of its right premise in E $\supset$ or its left premise in E$\vee$ and E ($\exists$ x).

We write the deductions in tree-form; $\mathfrak{A}$ (A), $\mathfrak{B}$ (A $\supset$ B), etc. denote trees with the corresponding endformulas. Indices are used to distinguish different occurrences of the same formula.

If A_1 in $\mathfrak{A}$ (A_1) does not depend on any assumption, then $\mathfrak{A}$ (A_1) is a deductive tree; otherwise it is a derivative tree. $\mathfrak{A}$ (A_1) is conservative if its assumptions are: a) undischarged, or b) discharged and proper subformulas of A, or c) discharged and proper subformulas of an undischarged assumption.

Lemma 1: If $\mathfrak{A}$ (A_1) is deductive and conservative all its assumptions are proper subformulas of A.

We divide the rules of NI in two groups; the α) rules transfer the conservativity of the trees ending in their premises to the tree ending in its conclusion. This is not the case in the β) rules.

Lemma 2: Given a $\mathfrak{A}$ (F_1) we can obtain a $\mathfrak{B}$ (F_1) without new undischarged assumptions and with proper variables different from each other and occurring in $\mathfrak{B}$ (F_1) only above the conclusion of the corresponding rule ([4]).

We choose in $\mathfrak{A}$ (F_1) a I (x) or E ($\exists$ x) which follows no other rule of this kind and substitute in the tree ending in the premise of I (x) or in the right premise of E ($\exists$ x) the proper variable of this rule by another variable not occurring in $\mathfrak{A}$ (F_1). The undischarged assumptions of $\mathfrak{A}$ (F_1) are not affected by this substitution because the proper variable of an I (x) or E ($\exists$ x) cannot occur in them. F_1 remains unchanged because it lies beneath the endformula of the tree in which the substitution is performed. We repeat this operation with the other I (x) and E ($\exists$ x) of $\mathfrak{A}$ (F_1). The undischarged assumptions and F_1 remain unaltered. The I (x) and E ($\exists$ x) above the rule which is handled are not affected by the performed substitution because its two intervening variables must be different from the proper variables of those rules.

In the following we can restrict our considerations to trees fulfilling the lemma 2.

In a $\mathfrak{A}$ (F_1) in NI we choose an arbitrary β) rule; there are five possible cases: E $\supset$, E $\vee$, E (x), E ($\exists$ x) and E $\wedge$. An implicative (disjunctive, universal, existential, conjunctive) thread of the selected rule is a sequence of formulas (X_1, X_2 ... X_l) such that:

([4]) [1] 3.10 p. 198; [2] 3.10 p. 55.

a) $X_1 \equiv A \supset B$ ($X_1 \equiv A \vee B$; $X_1 \equiv (x)\, A(x)$; $X_1 \equiv (\exists x)\, A(x)$; $X_1 \equiv A \wedge B$).
b) If X_n is the conclusion of an α) rule or a $\beta)_1$ rule, then $n = 1$.
C) If X_n is the conclusion of an $E\vee$, then X_{n+1} is one of the right premises.
d) If X_n is the conclusion of an $E(\exists x)$, then X_{n+1} is the right premise.
e) If X_n is an assumption of $\mathfrak{A}(F_1)$, then $n = 1$.

Lemma 3: an implicative (disjunctive...) thread contains only formulas $A \supset B$ ($A \vee B$; $(x)\, A(x)$; $(\exists x)\, A(x)$; $A \wedge B$) and it splits in each $E\vee$; it goes only through $E\vee$ or $E(\exists x)$ and never through α) or $\beta)_1$ rules. Its last formula is either a) conclusion of a $\beta)_1$ rule, or b) assumption of $\mathfrak{A}(F_1)$, or c) conclusion of an α) rule.

A β) rule is normal if it has at least a thread (implicative, disjunctive...) which satisfies condition a) or b) of lemma 3. Otherwise the β) rule is abnormal. The height of a $\mathfrak{A}(F_1)$ is the maximum degree of the abnormal β) rules.

Lemma 4: Every thread of an abnormal β) rule ends in the conclusion of an $E\wedge$ or there is at least one thread ending in the conclusion of a different α) rule.

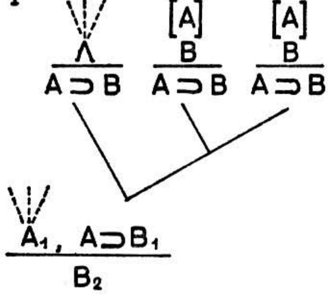
Lemma 5: Given a $\mathfrak{A}(F_1)$ in NI of height $n > O$ there is a $\mathfrak{B}(F_1)$ in NI of height O and without new undischarged assumptions.

Proof. We choose in $\mathfrak{A}(F_1)$ an abnormal β) rule of maximum degree which is not preceeded by another β) rule of the same degree. The possible cases are:

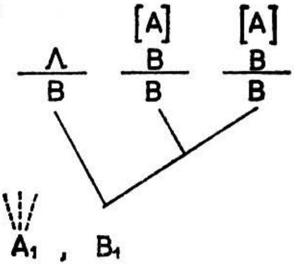
1)
$$E\supset \frac{A,\, A \supset B}{B}$$

By lemma 4 there are two alternatives. In both we replace every formula of the implicative threads of this rule by B. If $A \supset B$ does not contain the proper variable of an $E(\exists x)$, then B does not contain it. Therefore in the first alternative we obtain a new tree with an abnormal β) rule of maximum degree less and without new undischarged assumptions. In the second alternative the α) rule must be an $I\supset$; the situation is the following:

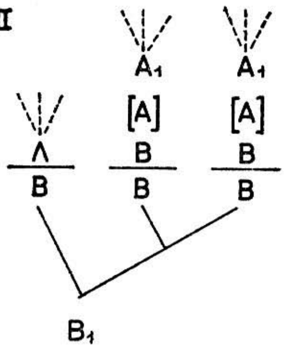
I



II



III



In II we insert $\mathfrak{A}$ (A_1) above every [A] and obtain III. The sentential rules, the E (x) and the I (∃ x) in the branches of III starting in [A] and ending in B_1 remain correct. The I (x) also because by lemma 2 and the structure of I its proper variables cannot occur in $\mathfrak{A}$ (A_1) and therefore they not occur in the new undischarged assumptions above [A]. The E (∃ x) remain also correct because the new undischarged assumptions above [A] by the same reason as before cannot contain its proper variables. A ⊃ B has a higher degree than A and therefore the insertion of $\mathfrak{A}$ (A_1) above [A] can only introduce new abnormal β) rules of lesser degree than the maximum.

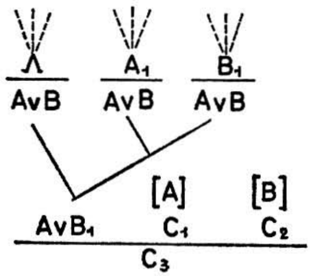
Consequently in the second alternative we obtain a $\mathfrak{B}$ (F_1) without new undischarged assumptions and with an abnormal β) rule of maximum degree less.

2)

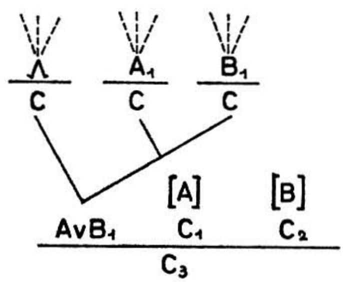
$$E\vee \quad \frac{A \vee B \quad \overset{[A]}{C} \quad \overset{[B]}{C}}{C}$$

By lemma 4 there are two alternatives; in both we replace every formula of the disjunctive threads of this rule by C. By lemma 2 and the structure of I, C cannot contain the proper variable of a E ($\exists$ x) occurring in $\mathfrak{A}$ ($A \vee B_1$); therefore in the first alternative we get a new tree without new undischarged assumptions and an abnormal β) rule of maximum degree less. In the second alternative the situation is this:

I



II



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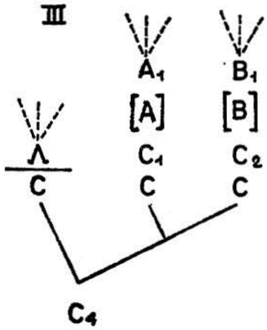
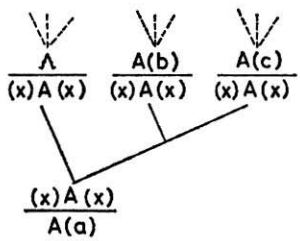


In II we put beneath A_1 and B_1 the $\mathfrak{A}$ (C_1) and $\mathfrak{B}$ (C_2) of I and obtain III. The disappearence of the assumptions [A] and [B] can only affect β) rules of degree less than the maximum. The sentential rules, the E (x) and I ($\exists$ x) in the branches of III going from [A] and [B] to C_4 are valid. The I (x) also because they can only occur between [A] and C_1, and [B] and C_2; but the $\mathfrak{A}$ (A_1) and $\mathfrak{B}$ (B_1) by lemma 2 and the structure of I cannot contain its proper variable. The E ($\exists$ x) occurring between [A] and C_1, and [B] and C_2 are valid by the same reason. If they occur between Λ, C_1, C_2 and C_4 they remain correct because by lemma 2 and the structure of I neither $\mathfrak{A}$ (C_1) nor $\mathfrak{B}$ (C_2) of I can contain its proper variables. By this way we obtain in the second alternative a $\mathfrak{B}(F_1)$ without new undischarged assumptions and an abnormal β) rule of maximum degree less.

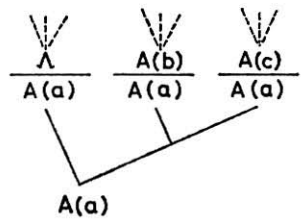
$$3) \qquad E(x)\ \frac{(x)\,A(x)}{A(a)}$$

By lemma 4 there are two alternatives; in both we replace every formula of the universal threads of this rule by A (a). By lemma 2 A (a) cannot contain the proper variable of an E ($\exists$ x) occurring in $\mathfrak{A}$ (A (a)) of I. Therefore in the first alternative we get a new tree with the required properties. In the second the situation is the following:

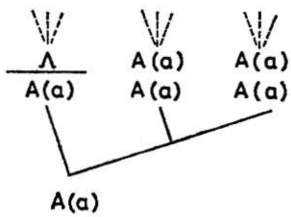
I



II



III



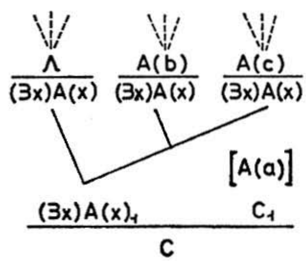
Since b cannot occur in (x) A (x), A (x) is the result of substituting x for b in A (b). Furthermore A (a) is obtained by substituting a for x in A (x). Therefore A (a) is the result of substituting a for b in A (b). By the same reason A (a) is the result of substituting a for c in A (c). If in $\mathfrak{A}$ (A(b)) and $\mathfrak{B}$ (A(c)) of II we substitute a for b and for c respectively we obtain III. In this tree there are no new undischarged assumptions because by the restriction on the proper variable of an I (x) neither b nor c can occur in undischarged assumptions of $\mathfrak{A}$ (A (b)) and $\mathfrak{B}$ (A (c)) respectively. By lemma 2 and the structure of I it can easely be proved that all rules of III are correct. Therefore we get a new tree with the desired properties.

4)

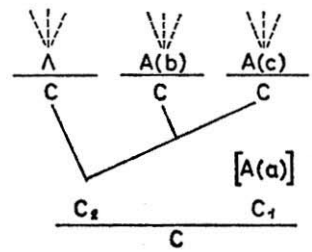
$$E(\exists x)\ \frac{(\exists x)\,A(x) \qquad \begin{array}{c}[A(a)]\\ C\end{array}}{C}$$

By lemma 4 there are two alternatives; in both we replace every formula of the existential threads of this rule by C. By lemma 2 and I, C cannot contain the proper variable of an $E(\exists x)$ occurring in $\mathfrak{A}((\exists x)\,A(x)_1)$ of I. This settles the first alternative. In the second the situation is the following:

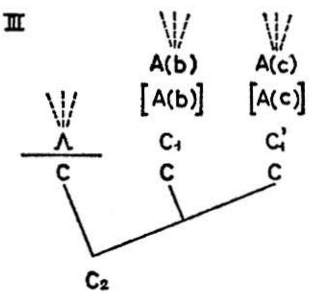
I



II



III



By similar reasons as in case 3) A (b) (A (c)) is the result of substituting b (c) for a in A (a). In II we insert beneath A (b) and A (c) the result of substituting in $\mathfrak{A}(C_1)$ of I b for a and c for a respectively and we obtain III. The sentential rules, the E (x) and the $I(\exists x)$ in the branches of III going from [A (b)] and [A (c)] to C_2 are correct. The I (x) can only occur between [A (b)] and C_1, and [A (c)] and C'_1, but then the proper variables can be neither b nor c nor a, because in the corresponding I (x) of $\mathfrak{A}(C_1)$ of I the proper variable was neither a (a is the proper variable of the $E(\exists x)$ in question) nor b nor c (by lemma 2 and the structure of I). Therefore the substitution in $\mathfrak{A}$ (C_1) of I of b for a and c for a has not altered the valididty of these I (x). Furthermore in $\mathfrak{A}$ (A (b)) or $\mathfrak{B}$ (A (c)) of III there are no undischarged assumptions containing the proper variable of such I (x), because otherwise I would violate lemma 2. On the other hand the E ($\exists$ x) in III are coming from $\mathfrak{A}$ (($\exists$ x) A $(x)_1$) or from $\mathfrak{A}$ (C_1) of I and by similar reasons as used before we can prove their correctness. Since the degree of A (a) is less than the one of ($\exists$ x) A (x) we obtain a new tree with an abnormal β) rule of maximum degree less and without new undischarged assumptions.

5) $\quad$ E $\wedge$ $\quad \dfrac{A \wedge B}{A}$

This case is easily handled in the same way as the others.

Repeating the operations described in 1) — 5) we can eliminate all abnormal β) rules of maximum degree and we obtain another tree without new undischarged assumptions and smaller height than $\mathfrak{A}$ (F_1). Repeating this procedure we get finally a $\mathfrak{B}$ (F_1) fulfilling the conditions of lemma 5.

Lemma 6: Every $\mathfrak{A}$ (F_1) in NI of height O is conservative.

Proof by induction. The assumptions of $\mathfrak{A}$ (F_1) are trivially conservative trees. Let us assume that the trees ending in the premises of a rule are conservative. If it is an α) rule the tree ending in the conclusion is also conservative (in I $\supset$ the assumption [A] is a proper subformula of A $\supset$ B and in E $\wedge$ the premise has no proper subformulas). If it is a β) rule there are five possible cases:

1) $\quad$ E $\supset$ $\quad \dfrac{A_1, A \supset B_1}{B_2}$

Since the height of $\mathfrak{A}$ (F_1) is O this rule has at least an implicative thread ending in a) the conclusion of a $\beta)_1$ rule or b) an assumption of $\mathfrak{A}$ (F_1). In the first alternative the $\beta)_1$ rule has again a thread whose formulas contain properly A $\supset$ B and ends in a) the conclusion of a $\beta)_1$ rule or b) an assumption of $\mathfrak{A}$ (F_1). a) cannot indefinitely repeat and therefore there must be a branch in $\mathfrak{A}$ (F_1) going through A $\supset$ B_1 and beginning by an assumption S $\equiv$ φ (A $\supset$ B). But by the inductive hypothesis $\mathfrak{B}$ (A $\supset$ B_1) is conservative and since A $\supset$ B is a proper subformula of S, S must be an undischarged assumption of $\mathfrak{C}$ (B_2) or a proper subformula of an undischarged assumption T $\equiv$ ψ (A $\supset$ B) of $\mathfrak{C}$ (B_2). In spite of the fact that the discharged assumptions of $\mathfrak{A}(A_1)$ and $\mathfrak{B}(A \supset B_1)$ may not be subformulas of B_2 they are proper subformulas of S or T and this assures the conservativity of $\mathfrak{C}(B_2)$.

The proof of the other cases is similar.

Theorem 1: a) Given a $\mathfrak{A}(F_1)$ in NI there is a conservative $\mathfrak{B}(F_1)$ in NI without new undischarged assumptions (proof by lemmas 5 and 6).

b) Given a deductive $\mathfrak{A}(F_1)$ in NI there is a deductive and conservative $\mathfrak{B}(F_1)$ in NI whose assumptions are proper subformulas of F. All the formulas of $\mathfrak{B}(F_1)$ are subformulas of F.

The first part follows from lemma I and theorem 1 a). To prove the second let us take in $\mathfrak{B}(F_1)$ a formula $D \neq \Lambda$. Either D is subformula of F or in the branch starting in D and ending in F_1 there is a first formula C which does not contain D. C must be the conclusion of a β) rule, but then it is subformula of an assumption of $\mathfrak{B}(F_1)$ which in turn is subformula of F. On the other hand if $D \equiv \Lambda$ then either D is an assumption or a conclusion of a β) rule. Again in both cases D is subformula of F.

From theorem 1 b) it easily follows the consistency of NI. Furthermore given a deductive $\mathfrak{A}(F_1)$ in NI we can construct a deductive $\mathfrak{B}(F_1)$ in NI whose last rule is an α) rule different from E Λ (otherwise $\mathfrak{B}(F_1)$ would not be deductive or NI would be inconsistent). Now if $F_1 \equiv A \vee B$ the last rule must be an I $\vee$ and then either A or B is intuitionistically provable.

Theorem 2: Let us assume that F' is the result of replacing in F all its universal quantifiers by their definitions in terms of existential quantifiers and negations. If $\mathfrak{A}(F_1)$ is deductive in NK, then there is a deductive $\mathfrak{B}(F'_1)$ in NK whose assumptions are subformulas of $F' \supset \Lambda$ (proof by the extended theorem of Glivenko and theorem 1 [5]).

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(5) [3] p. 492.

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