ANOTHER PROOF OF THE STRONG LÖWENHEIM-SKOLEM THEOREM

ROLF SCHOCK

In 'Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen' (*Skrifter utgeit av Videnskapsselskapet i Kristiana,* 1920), the late T. Skolem sketched a proof of more or less the fact that, if each member of a set of sentences is true within a well-ordered universe, then each member of that set of sentences is also true within a countable universe. Nevertheless, the proof actually establishes a bit more-namely, that if each member of a set of sentences s is true with respect to a given rule i which interprets the constants occurring in members of s and the universe of discourse of i is well-ordered, then each member of s is also true with respect to a restriction of i to a countable subset of the universe of discourse of i.

The proof sketched by Şkolem is somewhat intricate and requires the association of each sentence with another in Skolem normal form (that is, consisting of universal quantifier phrases followed by existential quantifier phrases followed by a quantifier-free formula). In the present paper, what Skolem sketched the proof of is proven without the Skolem normal form and by fairly simple semantic methods reminiscent of those used by L. Henkin in 'The completeness of the first order functional calculus' (*Journal of Symbolic Logic,* vol. 14, 1949). Moreover, the metamathematical framework employed is one in which arbitrary formulas of first-order quantifier logic with both identity and descriptions are included and in which the empty set is also a universe of discourse.

We say that

(1) s is countable just in case s is a finite or denumerably infinite set;

(2) x is an r-first member of s just in case s is a set, x is in s, r is a relation, and, for any y in s, not yrx although either y = x or xry; and

(3) r well-orders s just in case s is a set and, for any q, if q is a non-empty subset of s, then there is exactly one x such that x is an r-first member of q.

Our language contains the following symbols:

(1) the logical constants ~ ('not'), → ('only if'), ∧ ('and'), ∨ ('or'), ↔ ('if and only if'), ⋀ ('for all'), ⋁ ('for some'), ℩ ('the'), E ('exists'), and I ('is identical with'); we call the first five of these sentential connectives and all of the rest except E and I variable binders;
(2) a denumerable infinity of distinct
(a) individual variables,
(b) individual constants, and
(c) predicates of any positive number of places among which E is the first 1-place predicate and I is the first 2-place predicate.

In the metalanguage, we use '⟨', '⟩' and '{', '}' to mark the boundaries of non-empty finite sequences and sets respectively and '⌢' and '$_{-1}$' as standing for the operations of concatenating two finite sequences and of removing the first term of a non-empty finite sequence respectively. Also, we use 'n' as a metalinguistic variable ranging over all positive integers. TF, terms, and formulas will be understood as follows:
(1) TF = the intersection of all sets k such that
(a) for any variable or individual constant t, the pair t, ⟨tE⟩ is in k;
(b) for any n, n-place predicate p, and n-term sequence of members of the domain of k t, the pair t(1), ⟨t(1)p⟩ ⌢ t_{-1} is in k;
(c) for any variable v and for f and g in the range of k,
(i) the pair ⟨℩vf⟩, ⟨vE⟩ is in k and
(ii) for any h in {⟨~f⟩ ⟨f→g⟩ ⟨f∧g⟩ ⟨f∨g⟩ ⟨f↔g⟩ ⟨⋀vf⟩ ⟨⋁vf⟩}, the pair v, h is in k;
(2) t is a term just in case t is in the domain of TF; and
(3) f is a formula just in case f is in the range of TF.

A symbol s occurs in a set of terms or formulas x just in case s occurs in some member of x.

In what follows, we omit sequence marks according to the usual conventions for the omission of parentheses and presuppose a correlation of the formulas with the positive integers.

Given terms t an u and a term or formula f, we understand freedom and PStuf (the result of properly substituting t for u in f) as follows:
(1) if u = f, then u is free in f and PStuf = t;
(2) if u ≠ f, then

(a) if f is a variable or individual constant, then u is not free in f and PStuf = f;

(b) for any n, n-place predicate p, and n-term sequence of terms v, if f = $\langle v(1)p \rangle \frown v_{-1}$, then u is free in f just in case u is free in some member of the range of v and PStuf = $\langle$PStu v(1) p$\rangle \frown$ (the n-term sequence w such that w(k) = PStu v(k) for any k in the domain of w)$_{-1}$;

(c) for any sentential connective c and formulas g and h,

- (i) if f = cg, then u is free in f just in case u is free in g and PStuf = cPStug;
- (ii) if f = gch, then u is free in f just in case u is free in either g or h and PStuf = PStug c PStuh;

d) for any variable binder b, variable v, and formula g, if f = bvg, then

- (i) u is free in f just in case u is free in g and v is not free in u; and
- (ii) PStuf = the z such that
  - a) if u is not free in f, then z = f;
  - b) if u is free in f and v is not free in t, then z = bvPStug; and
  - c) if u is free in f, v is free in t, and w = the first variable not occurring in either f or t, then z = bwPStuPSwvg.

On the other hand, if t and u are finite sequences of terms, the domain of t = the domain of u, and f is a term or a formula, then

(1) OPStuf = f;

(2) for any n in the domain of t, nPStuf = PS t(n) u(n)n—1PStuf; and

(3) PStuf = the z such that, for some natural number m and m-term sequence w, the domain of w = the domain of t, w(n) = the nth variable not occurring in the range of t $\frown$ u $\frown$ $\langle$f$\rangle$ for any n in the domain of w, and z = mPStw mPSwuf.

Given a term or formula t, VRt and CNt are the (non-repeating) sequences of variables free in t and individual constants occuring in t respectively. We say that

(1) s is a sentence just in case s is a formula and VRf is empty;

(2) s is an existential sentence just in case s in a sentence and, for some variable v and formule f, s = $\bigvee$vf; and

(3) ESn = the nth existential sentence.

If x is a set of formulas and there are infinitely many individual constants not occurring in x, then Cxn = the first individual constant c such that

(1) c does not occur in x;

(2) there is no positive integer m smaller than n such that c = Cxm; and

(3) there is no positive integer m not greater than n such that c occurs in ESm.

By an interpreter, we mean a function i such that

(1) the domain of i = the set of all individual constants and predicates and

(2) there is a set u such that

(a) for any individual constant c, either i(c) = the empty set or, for some m in u, i(c) = {m};

(b) for any n-place predicate p, i(p) is a set of n-term sequences of members of u;

(c) i(E) = the set of all s such that, for some m in u, s = ⟨m⟩; and

(d) i(I) = the set of all s such that, for some m in u, s = ⟨mm⟩.

If i is an interpreter, then Ui (the universe of i) is the u under (2) above.

If x is a set, then a is an assigner in x just in case a is a function such that

(1) the domain of a = the set of variables and

(2) for any v in the domain of a,

(a) if x is empty, then a(v) = the empty set and

(b) if x is not empty, then, for some m in x, a(v) = {m}.

If a is an assigner in x, v is a variable, and y is an object of any kind, then $a(^{v}{}_{y})$ = a with the pair v, a(v) removed and the pair v, y added in its place.

Given an interpreter i and assigner in Ui a, we understand Int ia (the interpretation with respect to i and a of ...) as follows:

(1) for any variable v, Int ia (v) = a(v);

(2) for any individual constant c, Int ia (c) = i(c);

(3) for any n, n-place predicate p, and n-term sequence of terms t, Int ia $(\langle t(1)p\rangle^\frown t_{-1})$ = the z such that either there is a u in i(p) such that u(k) is in Int ia (u(k)) for any k in the domain of u and z = 1 or not and z = O;

(4) for any variable v and formulas f and g,
   (a) Int ia ($\sim$f) = 1-Int ia (f);
   (b) Int ia (f$\rightarrow$g) = the smallest member of $\{$1, (1-Int ia (f)) + Int ia (g)$\}$;
   (c) Int ia (f $\wedge$ g) = the smallest member of $\{$Int ia (f) Int ia (g)$\}$;
   (d) Int ia (f $\vee$ g) = the greatest member of $\{$Int ia (f) Int ia (g)$\}$;
   (e) Int ia (f$\leftrightarrow$g) = (1—the greatest member of $\{$Int ia (f) Int ia (g)$\}$) + the smallest member of $\{$Int ia (f) Int ia (g)$\}$;
   (f) Int ia (ιvf) = the z such that either there is a k in Ui such that, for any m in Ui, Int ia ($^{v}\{_{m}\}$) (f) = 1 just in case m = k and z = $\{$k$\}$ or not and z = the empty set;
   (g) Int ia ($\bigwedge$vf) = the z such that either Ui is empty and z = 1 or not and z = the smallest member of the set of all r such that, for some m in Ui, Int ia ($^{v}\{_{m}\}$) (f) = r; and
   (h) Int ia ($\bigvee$vf) = the z such that either Ui is empty and z = O or not and z = the greatest member of the set of all r such that, for some m in Ui, Int ia ($^{v}\{_{m}\}$) (f) = r.

If f is a formula, then f is i-true just in case, for any assigner in Ui a, Int ia (f) = 1. Also, if t is a term or a formula and j is an interpreter, then t is absolute in i from j just in case Uj is included in Ui and, for any assigner in Uj a, Int ia (t) = Int ja (t).

If x is a set of formulas and t is an interpreter, then j is an x-restriction of i just in case j is an interpreter such that
(1) Uj is included in Ui;
(2) for any individual constant which occurs in x c, j(c) = the intersection of i(c) and Uj; and
(3) for any n and n-place predicate which occurs in x p, j(p) = the intersection of i(p) and the set or all n-term sequences whose ranges are included in Uj.

Obviously, if j is an x-restriction of i and k is an x-restriction of j, then k is an x-restriction of i.

If i is an interpreter, r well-orders Ui, x is a set of formulas, and there are infinitely many individual constants which do not occur in x, then Mirx (the modification of i by r and x) = the interpreter m such that, for any c in the domain of m,
(1) if there is no n such that c = Cxn, then m(c) = i(c); and
(2) for any n, if c = Cxn, then m(c) = the z such that, for some variable v and formula f, ESn = $\bigvee$vf and

(a) if $\sim \bigvee$vf is m-true, then m(c) = the empty set and

(b) if $\bigvee$vf is m-true, then m(c) = {the r-first member of the set of all y in Um such that, for some assigner in Um a, Int ma $(^{v}\{_{y}\})$ (f) = 1}.

The properness of(a) and (b) above follows from the theory of recursive definitions since UMirx = Ui and Int Mirx a (ESn) is fixed independently of Cxn for any n and assigner in UMirx a.

Theorem 1. If i is an interpreter, r well-orders Ui, x is a set of formulas, and there are infinitely many individual constants which do not occur in x, then

(1) Mirx is an x-restriction of i and i is an x-restriction of Mirx; also,

(2) for any term or formula t, if every individual constant which occurs in t occurs in x, then t is absolute in i from Mirx and t is absolute in Mirx from i.

Assume the antecedent. (1) follows immediately from the facts that UMirx = Ui and that (Mirx) (c) = i(c) for any c in the domain of Mirx which occurs in x and (2) from these same facts via an induction among the members of TF.

If i, r, and x are as in the antecedent of theorem 1, then Rirx (the reduction of i by r and x) = the interpreter s such that

(1) for any individual constant c, s(c) = (Mirx) (c); and

(2) for any n and n-place predicate p, s(p) = the intersection of (Mirx) (p) and the set of all n-term sequences whose ranges are included in the set of all y such that, for some individual constant c, y is in (Mirx) (c).

Theorem 2. If i is an interpreter, r well-orders Ui, x is a set of formulas, and there are infinitely many individual constants which do not occur in x, then

(1) Rirx is an x-restriction of Mirx;

(2) URirx is countable; and

(3) for any term or formula t, t is absolute in Mirx from Rirx.

Assume the antecedent. (1) and (2) obviously hold. To show (3), let m = Mirx and s = Rirx. Obviously,

(a) if v is a variable or an individual constant, then v is absolute in m from s.

To show that

(b) if p is an n-place predicate and t is an n-term sequence of terms such that t(k) is absolute in m from s for any k in the domain of t, then $\langle t(1)p \rangle \frown t_{-1}$ is absolute in m from s, assume the antecedent, that a is an assigner in Us, and that $f = \langle t(1)p \rangle \frown t_{-1}$. Hence, for any k in the domain of t, Int sa (t(k)) = Int ma (t(k)). If there is a k in the domain of t such that Int sa (t(k)) is empty, Int ma (t(k)) is empty and so Int sa (f) = O = Int ma (f). Assume then that u is an n-term sequence and, for any k in the domain of u, u(k) = the member of Int sa (t(k)). Obviously, u is in s(p) just in case u is in m (p) and so Int sa (f) = Int ma (f) again. But then f is absolute in m from s and (b) holds.

By an even simpler inductive argument of the same kind, we have

(c) if f and g are formulas which are absolute in m from s, c is a sentential connective, and fcg is a formula, then $\sim$f and fcg are absolute in m from s.

To show

(d) if f is a formula absolute in m from s and v is a variable, then $\bigwedge$vf is absolute in m from s,

assume the antecedent and that a is an assigner in Us. If Int ma ($\bigwedge$vf) = 1, then, since Us is included in Um and Int sa $({}^{v}\{_{y}\})$ (f) = Int ma $({}^{v}\{_{y}\})$ (f) for any y in Us by assumption, it follows that Int sa ($\bigwedge$vf) = 1 as well. Assume on the other hand that Int ma ($\bigwedge$vf) = O and so that Int ma ($\bigvee$v $\sim$f) = 1. Let c be the finite sequence such that the domain of c = the domain of VR$\bigwedge$vf and c(k) = the first individual constant d such that a((VR$\bigwedge$vf) (k)) = m (d) for any k in the domain of c; also, let g = PSc(VR$\bigwedge$vf)f. Obviously, Int ma $({}^{v}\{_{y}\})$ ($\sim$g) = Int ma $({}^{v}\{_{y}\})$ ($\sim$f) for any y in Um and, since $\bigvee$v$\sim$g is a sentence, there is an n such that $\bigvee$v$\sim$g = ESn. Hence, m (Cxn) = {the r-first member of the set of all y in Um such that Int ma $({}^{v}\{_{y}\})$ ($\sim$f) = 1} = s(Cxn). Hence, there is a y in Us such that Int ma $({}^{v}\{_{y}\})$ ($\sim$f) = 1 and so, since Int sa $({}^{v}\{_{y}\})$ ($\sim$f) = Int ma $({}^{v}\{_{y}\})$ ($\sim$f) by assumption and (c), Int sa ($\bigvee$v$\sim$f) = 1. But then Int sa ($\bigwedge$vf) = Int ma ($\bigwedge$vf) again and so $\bigwedge$vf is absolute in m from s and (d) holds.

From (c), (d), and the fact that Int ia ($\bigvee$vf) = Int ia ($\sim\bigvee$v$\sim$f) for any variable v, formula f, interpreter i, and assigner in Ui a, we have also

(e) if f is a formula absolute in m from s and v is a variable, then $\bigvee$vf is absolute in m from s.

Finally, to show that

(f) it f is a formula absolute in m from s and v is a variable, then $\imath$vf is absolute in m from s,
assume the antecedent, that a is an assigner in Us, and that w is a variable which does not occur in $\imath$vf. From(a) through(e), it follows that Int sa ($\bigvee$w$\bigwedge$v $\langle$f$\leftrightarrow$vIw$\rangle$) = Int ma ($\bigvee$w$\bigwedge$v$\langle$f$\leftrightarrow$vIw$\rangle$) and that, for any y in Us, Int sa ($^{w}\{_{y}\}$) ($\bigwedge$v$\langle$f$\leftrightarrow$vIw$\rangle$) = Int ma ($^{w}\{_{y}\}$) ($\bigwedge$v$\langle$f$\leftrightarrow$vIw$\rangle$). Hence, if Int sa ($\imath$vf) is empty, so is Int ma ($\imath$vf); and, for any y in Us, if Int sa ($\imath$vf) = {y}, then Int ma ($^{w}\{_{y}\}$) ($\bigwedge$v$\langle$f$\leftrightarrow$ vIw$\rangle$) = 1 and so Int ma ($\imath$vf) = {y}. Hence, $\imath$vf is absolute in m from s and (f) holds.

From (a) through (f), it follows via an induction among the members of TF that (3) holds and so that the theorem holds.

We say that
(1) if c = the n^{th} individual constant, then Kc = the $2n^{th}$ individual constant;
(2) if t is a term or a formula, k is a finite sequence, the domain of k = the domain of CNt, and k(j) = K(CNt) (j) for any j in the domain of k, then Kt = PSk(CNt) t;
(3) if x is a set of formulas, then Kx = the set of all g such that, for some f in x, g = Kf;
(4) if i is an interpreter, then
(a) Ki = the interpreter j such that, for any c in the domain of j, either there is an individual constant d such that c = Kd and j(c) = i(d) or not and j(c) = i(c); and
(b) Li = the interpreter j such that, for any c in the domain of j, either c is an individual constant and j(c) = i(Kc) or not and j(c) = i(c).

Theorem 3. If i is en interpreter, then
(1) Ui = UKi = ULi and
(2) for any term or formula t and assigner in Ui a, Int ia (t) = Int Ki a (Kt) and Int ia (Kt) = Int Li a (t).

For assume the antecedent. (1) obviously holds and (2) follows via an induction among the members of TF.

Theorem 4. If i is an interpreter, r well-orders Ui, and x is a set of formulas, then there is an x-restriction of i j such that Uj is countable and, for any term or formula t, if every individual constant which occurs in t occurs in x, then t is absolute in i from j.

Assume the antecedent. Since there are infinitely many individual constants which do not occur in Kx, we have by theorems 1 and 2 that R(Ki)r(Kx) is a Kx-restriction of Ki, UR(Ki)r(Kx) is countable, and, for any term or formula t, if every individual constant which occurs in t occurs in Kx, then t is absolute in Ki from R(Ki) r (Kx). Also, by theorem 3, Ui = UKi, UR(Ki) r (Kx) = ULR(Ki) r (Kx), and, for any term or formula t and assigner in ULR(Ki) r (Kx) a, Int ia (t) = Int Ki a (Kt) and Int R (Ki) r (Kx) a (Kt) = Int LR (Ki) r (Kx) a (t). It follows that ULR (Ki) r (Kx) is countable and, for any term or formula t, if every individual constant which occurs in t occurs in x, then Int ia (t) = Int Ki a (Kt) = Int R (Ki) r (Kx) a (Kt) = Int LR (Ki) r (Kx) a (t) for any assigner in ULR (Ki) r (Kx) a and so t is absolute in i from LR(Ki) r (Kx). Finally, if c is an individual constant which occurs in x and p is an n-place predicate, then (LR(Ki) r (Kx)) (c) = (R(Ki) r (Kx)) (Kc) = (Ki) (Kc) = i (c) and (LR (Ki) r (Kx)) (p) = (R(Ki) r (Kx)) (p) = the intersection of (Ki) (p) and the set of all n-term sequences whose ranges are included in ULR(Ki) r (Kx) = the intersection of i(p) and the set of all n-term sequences whose ranges are included in ULR(Ki) r (Kx). Hence, LR(Ki) r (Kx) is an x-restriction of i and so the theorem holds.

It should be noted that theorem 4 is a weak analogue to theorem 2.1 of A. Tarski's and R. Vaught's 'Arithmetical extensions of relational systems' (*Compositio Mathematica*, vol. 13, 1957).

Theorem 5 (strong Löwenheim-Skolem theorem). If i is an interpreter, r well-orders Ui, x is a set of formulas, and, for any f in x, f is i-true, then there is an x-restriction of i j such that Uj is countable and, for any f in x, f is j-true.

For assume the antecedent. By theorem 4, there is an x-restriction of i j such that Uj is countable and, for assigner in Uj a and f in x, Int ja (f) = 1. Hence, every f in x is j-true and the theorem holds.

Theorem 5 is sometimes felt to be paradoxical in the case of a set of formulas x which implies sentences asserting the existence of sets which are more than denumerably infinite. This feeling of paradox should disappear when it is recalled that, for any interpreter j, any function which correlates Uj with the set of all natural numbers is a function of the metalanguage which, via the axiom of regularity of the metalanguage, cannot be a member of its own domain.

Stockholm Rolf Schock