ON DENUMERABLY MANY-VALUED LOGICS

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In the paper F = 'On finitely many-valued logics' (*Logique et Analyse*, vol 21, 1964 [1]), the author was not sure about how to deal with ω-valued [2] semantics. In the present paper, one such semantic theory is developed and used to characterize ω-valued logics.

1. ω-VALUED SEMANTICS IN EMPTY AND NON-EMPTY UNIVERSES.

By an ω-interpreter, we mean a function i such that
(1) the domain of i = the set of all individual constants and predicates and
(2) there is a set u such that

(a) for any individual constant c, either i(c) = the empty set or, for some m in u, i(c) = $\{m\}$;

(b) for any positive integer m and m-place predicate p, i(p) is a non-empty finite sequence of sets of m-term sequences of members of u; and

(c) for any k in the domain of i(I), (i(I))(k) = the set of all t such that, for some m in u, t = $\langle mm \rangle$.

Given an ω-interpreter i, Ui is the u under (2) above. Obviously,

Theorem 1. If i is an n-interpreter, then i is an ω-interpreter.

Notice that the converse need not hold.

Given an ω-interpreter i and assigner in Ui a, Int ia is understood as it was for n-interpreters in F; however, in the case of atomic formulas, the division is by the greatest member of the domain of the sequence which is the i-value of the predicate concerned rather than by n—1 (although this objet turns out to be n—1 if i is also an n-interpreter). The clauses for formulas beginning with $\bigwedge$ and $\bigvee$ remain proper since, for any variable v and formula f, the set of all r such that, for some m in Ui, Int ia $({}^{v}\{m\})$ f = r is finite.

[1] We here adopt all the definitions and conventions of F. In particular, «n» is a metalinguistic variable ranging over all positive integers greater than 1.

[2] As usual, ω = the (denumerable) set of all finite ordinal numbers.

Given an ω-interpreter i, Ti and i-truth are also understood as they were for n-interpreters in F; similarly, Tω and ω-validity are understood as they were in F, but this time with respect to ω-interpreters. It can be shown that

Theorem 2. Tω = the set of all r such that, for some positive integer m and natural number not greater than m k, r = k divided by m [3].

Also,

Theorem 3. There is an ω-interpreter i such that Ti = Tω.

Let p = the set of all 1-place predicates. By a well-known theorem of set theory, there is a denumerable set of mutually disjoint denumerable sets q such that p = the union of q. Hence, let s be a function which correlates the positive integers with q. Also, let t be a function which assigns to any positive integer m a function which correlates the natural numbers with s(m). Finally, let x be an object and i be the ω-interpreter such that

(1) Ui = $\{x\}$;

(2) for any individual constant c, i(c) = $\{x\}$;

(3) for any 1-place predicate p, positive integer m, and natural number not greater than m k, if p = (t(m)) (k), then i(p) = the m-term sequence u such that, for any l in the domain of u, either k $\neq$ O, l is not greater than k, and u(l) = $\{(x)\}$ or not and u(l) = the empty set;

(4) i(I) is a 1-term sequence; and

(5) for any n-place predicate p, if p $\neq$ I, then i(p) = $\langle$the empty set$\rangle$.

Hence, for any individual constant c, 1-place predicate p, assigner in Ui a, positive integer m, and natural number not greater than m k, if p = (t(m)) (k), then Int ia (cp) = the number of members of the set of all l in the domain of i(p) such that $\langle x\rangle$ is in (i(p)) (l) divided by the greatest member of the domain of i(p) = k divided by m. Hence, by theorem 2 and the definition of Tω, Ti = Tω and the theorem holds.

Theorems 2 and 3 justify the present treatment of ω-valued semantics.

Theorem 4. If f is a formula and f is ω-valid, then f is valid.

(3) Thus, our ω truth values are just those given in J. Łukasiewicz's and A. Tarski's «Investigations into the sentential calculus» (in Tarski's book *Logic, Semantics, Metamathematics*, Oxford, 1956).

This follows from theorem 1.

Theorem 5. The set of all ω-valid formulas is a proper subset of the set of all 2-valid formulas.

This follows from theorem 4 and theorem 2 of F.

Theorem 6. There is a nonzero formula which is not ω-valid.

This follows from theorem 4 and theorem 4 of F. Also,

Theorem 7. If i is an ω-interpreter and t is a term or a formula, then there are an n and n-interpreter j such that Uj = Ui and, for any assigner in Uj a, Int ja (t) = Int ia (t).

Assume the antecedent. Obviously, there is a non-repeating finite sequence s such that the range of s = the set of all predicates occurring in t. Let d be a function such that the domain of d = the domain of s and, for any k in the domain of d, d(k) = the greatest member of the domain of i(s(k)). Also, let p = the p such that either d is empty and p = 1 or not and p = d(1) multiplied by ... multiplied by d (the greatest member of the domain of d) and let e be a function such that domain of e = the domain of d and, or any k in the domain of e, e(k) = p divided by d(k). Finally, let j = the p+1 - interpreter j such that

(1) for any individual constant c, j(c) = i(c);

(2) j(I) is a p-term sequence and the range of j(I) = the range of i(I);

(3) for any predicate q,

(a) if q does not occur in t and q $\neq$ I, then every member of the range of j(q) is empty and

(b) if q occurs in t, then, for some k and r, q = s(k), r is an e(k)-term sequence, the range of r = $\{$i(s(k))$\}$, and j(q) = r(1) $^\frown$ ... $^\frown$ r(e(k)).

Obviously, Uj = Ui. Also, for any positive integer m, m-place predicate which occurs in t q, m-term sequence of terms which occur in t u, assigner in Uj a, natural number l, and positive integer k, if Int ia ($\langle$u(l)$\rangle$ $^\frown$ $\langle$q$\rangle$) $^\frown$ u_{-1}) = l divided by the greatest member of the domain of i(q), s(k) = q, and Int ja (u(h)) = Int ia (u((h)) $\neq$ the empty set for any h in the domain of u, then Int ja ($\langle$u(l)$\rangle$ $^\frown$ $\langle$q$\rangle$) $^\frown$ u_{-1}) = (e(k) multiplied by l) divided by p = ((p divided by d(k)) multiplied by I) divided by p = l divided by d(k) = l divided by the greatest member of the domain of i(q). Hence, by an induction among the members of TF, the theorem holds.

But then

Theorem 8. The set of all ω-valid formulas = the set of all valid formulas.

For assume that f is a formula and valid and that i is an ω-interpreter. By theorem 7, there are an n and n-interpreter j such that f is j-true just in case f is i-true. Since f is valid, f is n-valid and so j-true. Hence, f is ω-valid and so the theorem holds by theorem 4.

Theorem 9. If f is a formula, then f is 2-valid just in case Af is ω-valid.

This follows from theorem 8 and theorem 6 of F.

If t is a set of real numbers not smaller than O and not greater than 1, then Vt = the set of all functions v such that the domain of v = the set of all formulas, the range of v is included in t, and, for any formulas f and g,

(1) v(IIf) = the z such that either v(f) is in $\{01\}$ and z = 1 or not and z = O;
(2) v($\vdash$f) = the z such that either v(f) = 1 and z = 1 or not and z = O;
(3) v($\sim$f) = 1—v(f);
(4) v(f$\rightarrow$g) = the smallest member of $\{1, (1—v(f))+v(g)\}$;
(5) v(f$\wedge$g) = the smallest member of $\{v(f)\ v(g)\}$;
(6) v(f$\vee$g) = the greatest member of $\{v(f)\ v(g)\}$; and
(7) v(f$\leftrightarrow$) = (1—the greatest member of $\{v(f)\ v(g)\}$) + the smallest member of $\{v(f)\ v(g)\}$.

Obviously,

Theorem 10. If f is a formula, then f is an n-tautology just in case v(f) = 1 for any v in VTn.

Theorem 11. VTn is a proper subset of VTω.

We say that a formula f is an ω-tautology just in case v(f) = 1 for any v in VTω and that f is sententially atomic just in case there are no sentential connective c and formulas g and h such that either f = cg or f = gch.

Theorem 12. If f is a formula and f is an ω-tautology, then f is ω-valid and so valid.

The proof is similar to that of theorem 11 of F and via theorem 4.

Theorem 13. If f is a formula and f is an ω-tautology, then f is a tautology.

This follows from theorems 10 and 11. Also, *Theorem 14.* If v is in VTω and f is a formula, then there are an n and a w in VTn such that w(f) = v(f).

Assume the antecedent. Obviously, there is a non-repeating and non-empty finite sequence s such that the range of s = the set of all sententially atomic formulas occurring in f. Let d be a function such that the domain of d = the domain of s and, for any k in the domain of d, d(k) = the smallest b such that, for some a, v(s(k)) = a divided by b. Also, let p = d(1) multiplied by ... multiplied by d (the greatest member of the domain of d) and let e be a function such that the domain of e = the domain of d and, for any k in the domain of e, e(k) = p divided by d(k). Finally, let w = the w in VTp+1 such that, for any sententially atomic formula g,
(1) if g goes not occur in f, then w(g) = O divided by p and
(2) if g occurs in f, then, for some k in the domain of s, g = s(k) and w(g) = (e(k) divided by p) multiplied by (the a such that v(g) = a divided by d(k)).

Hence, for any sententially atomic formula which occurs in f g and k in the domain of s, if g = s(k), then w(g) = (e(k) divided by p) multiplied by (the a such that v(g) = a divided by d(k)) = (1 divided by d(k)) multiplied by (the a such that v(g) = a divided by d(k)) = v(g).

Now let UG = the set of all pairs t, g in TF such that, if g is a formula which occurs in f, then w(g) = v(g). If the pairs t, g and u, h are in UG and c is a sentential connective, then w(cg) = v(cg) if cg is a formula occurring in f and w(gch) = v(gch) if gch is a formula occuring in f. Hence, TF is included in UG and so, since f occurs in f, the theorem holds. It follows that

Theorem 15. The set of all formulas which are ω-tautologies = the set of all formulas which are tautologies [4].

For assume that f is a formula which is a tautology and that v is in VTω. By theorem 14, there are an n and a w in Tn such that w(f) = v(f). Since f is an n-tautology, w(f) = 1 and so f is an ω-tautology. Hence, by theorem 13, the theorem holds.

2. ω-VALUED LOGICS

By an ω-valued logic, we mean a deductive system d such that the set of all d-provable formulas = the set of all ω-valid formulas.

[4] This is a generalization of the fourth part of theorem 17 of the paper cited in note 3.

Theorem 16. If d is an ω-valued logic and f is a formula, then f is d-provable just in case f is e-provable for any n-valued logic e.
This follows from theorem 8.

Lω and ω-provability are understood as they were in F, but this time with respect to ω-tautologies. By theorem 13,

Theorem 17. If f is a formula and f is ω-provable, then f is provable.
Also, by theorems 12 and 8 and theorems 7 through 11, 15, 16, and 18 through 24 of F, it follows that

Theorem 18. If f is a formula, then f is ω-provable only if f is ω-valid.

Because of theorems 18 and 8 and theorems 25 and 32 through 40 of F, it seems likely that Lω is an ω-valued logic; nevertheless, the proof cannot be given in quite the usual way for the reason given for the case of the systems Ln with n greater than 2 in F.

The ω-valued logics are more adequate than the finitely many-valued ones in that they contain the finitely many-valued ones in the sense of theorem 16. Nevertheless, except with respect to the problem of the number of truth values to be employed, the ω-valued logics are less adequate than the 2-valued ones in all the ways in which the higher-valued logics are.

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ERRATA FOR «ON FINITELY MANY-VALUED LOGICS» of the same author, printed in the 25-26 n., April 1964

p. 54, line 4, replace the first «⟨» by «(».
p. 55, line 10, replace the first «v» by «∧».
p. 56, line 9 from bottom, replace the phrase: «, then x is n-consistent just in case x does not n-imply ∼f» by the phrase: «. Hence, x does not n-imply g.»
p. 57, bottom line: replace «vol 4, 1963» by «vol. 5, 1964».