A MATHEMATICAL MODEL FOR A THEORY OF ACTS

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In this paper we are concerned with the study of acts. Our approach is through the use of set theory. We describe a class of mathematical systems and attempt to approximate a characterization of the act in terms of the structure of this class. We use the structure of these systems to define certain operations and relations on acts. What we obtain is a body of mathematical theory. It is hoped that this theory will contribute to the understanding of the notion of the act and its formal description.

Our program is to treat an abstract mathematical system and to utilize its structure in order to define a few formal notions which approximate certain of the formal aspects of corresponding notions used in natural language. We define a class of mathematical objects we call acts. We are interested in studying these 'acts' to determine whether their formal properties in some measure satisfy our intuition with regard to the act.

It is in the author's opinion unfortunate that with the relatively recent development of deontic logics there have been no corresponding formal studies of acts. Deontic logics deal with the notions of permission and obligation, and these notions are directly dependent upon the notion of the act. The act in deontic logic is ususally treated as a proposition, and the formal treatment of acts reduces to the formal treatment of propositions. Thus for example the operation of the negative of an act is formally identical to the operation of negation in the propositional logic. The negative of an act in this paper is defined in terms of the structure of a mathematical system. It is discovered that our negative operation while sharing certain of the formal properties of logical negation does not usually share all of these properties. It is shown that, whenever our negative operation does share one crucial property with logical negation, it then reduces to an operation with strongly undesirable characteristics.

In addition to the definition of act negation we characterize permission and obligation and define a causal relation on the family of acts.

We begin with a heuristic description of certain of the features of our systems.

Each system we study will consist of a set U and a collection of certain distinguished subsets. We shall refer to U as a *universe of actions*. Each element of U will be called an *action*. Any subset of U will be called an *act*. If A is an act then any element of A (any action in A) will be called an *instance* of A.

We now relate these definitions to natural language. We view the act of walking as a collection of all instances of walking. An instance of the act of walking is a particular action involving walking. If Mr. Jones takes a walk around the block on June 30, 1960 at 6 in the evening then we would like to say that this action of Mr. Jones is an instance of the general act of walking. We view this instance as an element of the set of all instances of walking. If while walking around the block Mr. Jones is also whistling a tune we may say that the action of Mr. Jones is also an instance of the general act of whistling. Thus if x represents the action of Mr. Jones and if A is the act of walking and B the act of whistling we may claim that x is an element of $A \cap B$, the act of walking while whistling. We may also claim that the act of walking while whistling is a subact of the act of walking, that is $(A \cap B) \subset A$. It is seen that we may interpret the algebra of subsets of U with the usual set operations of union, intersection, and set theoretical complement as an algebra of acts. These operations enable us to combine acts in some formal manner.

We now consider a feature common to our system which enables us to speak of the performance of any act or action. Associated with the universe of actions U is a certain collection Φ of subsets of U called the *performance universe* of U. We shall impose the condition that each member of U is contained in at least one member of Φ. Any member of Φ will be called a *performance set*. If x is an element of U and if ψ is any performance set which contains x we shall say that ψ is a *performance neighborhood* or more briefly a *p—nbhd* of x. If ψ is a p—nbhd of x we shall say that x is performed with respect to ψ. If A is an act and if ψ is a performance set then if A and ψ have a non null intersection, that is if $(A \cap \psi) \neq O$ where O is the empty set, then we shall say that A is performed with respect to ψ.

When making statements in the subjunctive mood it might be said that one is talking about a hypothetical universe and a hypothetical set of events each member of which occurs in this universe. Each performance set ψ in Φ serves as such a universe.

We now illustrate the use of Φ in making counterfactual statements about acts. Suppose the following is asserted: 'If I had studied for my math exam the instructor would certainly have passed me'. We wish to reformulate this statement in terms of our mathematical model, in particular making use of the family Φ. If we think of A as the act of my studying for my exam and B the act of the instructor passing me then we wish to assert that whenever A is performed, B is performed. Applying Φ we assert that for each ψ in Φ, whenever A is performed with respect to ψ then so is B. Thus we have that for each ψ in Φ, whenever $(A \cap \psi) \neq O$, $(B \cap \psi) \neq O$.

We now examine briefly one final feature of our system, a feature we use in order to introduce the notion of moral permission and obligation. Associated with the universe of actions U is a certain non empty subset π. We shall call π the collection of all *permissable instances*. We shall say that an act A is *permitted* if it intersects π that is if $(A \cap \pi) \neq O$.

I. *Activity Domains.*

Having outlined the essential features of the type of system we shall study we now consider a formal development.

1.1 Definition. Let U be a set, π a subset of U and Φ a collection of subsets of U such that

1. $\pi \neq O$

and 2. $\bigcup\{\psi \varepsilon \Phi\} = U$.

Then the triplet $M = (U, \Phi, \pi)$ will be called an *activity domain.*

1.2 Definition. Let $M = (U, \Phi, \pi)$ be an activity domain. An act A in U is *permitted* if $(A \cap \pi) \neq O$.

1.3 Definition. For any activity domain M an act A is *forbidden* if A is not permitted.

In the following sequence of propositions A and B are acts in a fixed activity domain. We state the propositions without proof.

1.4 Theorem. The act $A \cup B$ is permitted if and only if A is permitted or B is permitted.

1.5 Theorem The act $A \cup B$ is forbidden if and only if both A and B are forbidden.

1.6 Theorem. If the act $A \cap B$ is permitted then both A and B are permitted.

1.7 Theorem. If A is forbidden or if B is forbidden then $A \cap B$ is forbidden.

II. *Negative Acts.*

We wish to formulate a definition of the negative of an act in terms of activity domains. We shall consider it as a set operator on the family of subsets of U where U is the universe of actions of any activity domain. Although the negative operator we shall define enjoys some of the properties of the set theoretical complement, it does not usually coincide with the complement operator. In the following argument we trace one of the consequences of defining the negative of an act as its set theoretical complement.

In accordance with the usual definition of obligation in terms of permission we shall say that an act is obligatory if its negative is not permitted. If we define the negative of an act as its complement in U then it would follow that an act A is obligatory if and only if $(-A \cap \pi) = O$, where $-A$ denotes the complement of A in U. Hence it would follow that A is obligatory if and only if $\pi \subset A$. Thus each obligatory act would contain the collection of all permissable instances. As a consequence for example, if x is a permissable instance of walking and if A is the obligatory act of paying taxes then x is an instance of A. This is needless to say quite an undesirable state of affairs.

We now proceed to define an operation on the collection of acts which we call the *negative operation*. We wish to describe the negative of an act A as the largest act such that whenever it is performed, A is not performed.

In the following development we consider an arbitrary activity domain M, and allow upper case letters to denote arbitrary acts of M.

2.1 Definition. Let A be an act. Then we define the negative of A, A^n as:

$A^n = \{x\text{: for each } \psi \text{ in } \Phi \text{ if } x \varepsilon \psi \text{ then } (A \cap \psi) = O\}$.

2.2 Theorem. For any act A, $A^n \subset -A$.

Proof: Let $x \ \varepsilon \ A^n$. Choose a p—nbhd ψ of x. This can be done by 1.1-2. Then $(A \cap \psi) = O$ since $x \varepsilon A^n$. Hence $x \varepsilon \psi \subset -A$.

As an aid in deriving certain properties of the operator n we introduce the following definition.

2.3 Definition. Let N(A) designate the union of all performance sets which intersect A or,

$N(A) = \cup\{\psi \varepsilon \Phi : (A \cap \psi) \neq O\}$.

2.4 Theorem. For any act A, $A^n = -N(A)$.
Proof: $x \varepsilon A^n$ if and only if p—nbhd of x intersects A.
No p—nbhd of x intersects A if and only if $x \not\varepsilon N(A)$, that is, if and only if $x \varepsilon -N(A)$.

2.5 Corollary. For any acts A and B, if $A \subset B$, then $B^n \subset A^n$.
Proof: It is easily seen that if $A \subset B$, then $N(A) \subset N(B)$, and hence $-N(B) \subset -N(A)$. Thus the corollary follows from 2.4.

2.6 Theorem. For any act A, $A \subset A^{nn}$.
Proof: Suppose $x \varepsilon A$. Then every p—nbhd of x intersects A. Hence no p—nbhd of x can intersect A^n. Hence by definition x must lie in the negative of A^n that is, A^{nn}.

2.7 Theorem. For any act A, $A^n = A^{nnn}$.
Proof: By 2.6, $A^n \subset (A^n)^{nn} = A^{nnn}$. Also by 2.6, $A \subset A^{nn}$ and hence by 2.5, $(A^{nn})^n =$
$A^{nnn} \subset A^n$. Thus we obtain the equality.

2.8 Theorem. 1. $(A^n U B^n) \subset (A \cap B)^n$.
2. $(AUB)^n = (A^n \cap B^n)$.

Proof of 1: $(A \cap B) \subset A$ and hence by 2.5, $A^n \subset (A \cap B)^n$. Similarly $B^n \subset (A \cap B)^n$. Thus $(A^n U B^n) \subset (A \cap B)^n$.
Proof of 2: $A \subset (AUB)$. Thus $(AUB)^n \subset A^n$. Similarly $(AUB)^n \subset B^n$. Hence $(AUB)^n \subset (A^n \cap B^n)$. Now suppose that $x \varepsilon (A^n \cap B^n)$. Let ψ be a p—nbhd of x. Then since x lies in both A^n and B^n, $(A \cap \psi) = O$ and $(B \cap \Psi) = O$. Therefore $(AUB) \cap \Psi = O$. We conclude that $x \varepsilon (AUB)^n$. Thus $(A^n \cap B^n) \subset (AUB)^n$ and we establish the equality.

2.9 Theorem. 1. $U^n = O$.
2. $O^n = U$.

Proof of 1. Let $x \varepsilon U$. Let ψ be any p—nbhd of U. Then $(\psi \cap U) \neq O$. Hence $x \not\varepsilon U^n$. We conclude that $U^n = O$.
Proof of 2: From 1 it follows that $O^n = U^{nn}$. By 2.6, $U \subset U^{nn}$. We conclude that $O^n = U$.

It is to be observed that theorems 2.5 - 2.9 describe properties of of the operator n which are weaker or at most as strong as the corresponding properties of the set theoretical complement. It will be shown in the appendix that the crucial property which distinguishes the negative operator from the operator of set theoretical complement is the fact that the negative of the negative of an act is not always the original act.

We now investigate the notions of permission and obligation applying our definition of the negative operator.

2.10 Definition. An act A is ***obligatory*** if A^n is not permitted.

2.11 Theorem. For any act A if—A is forbidden then A is obligatory. The proof is a simple application of 2.2.

2.12 Theorem. For any acts A and B if $A \cap B$ is obligatory then both A and B are obligatory.
Proof: If $A \cap B$ is obligatory then $((A \cap B)^n \cap \pi) = O$. Hence by 2.8 we have that $((A^n \cup B^n) \cap \pi) = O$. Hence $(A^n \cap \pi) = O$ and $(B^n \cap \pi) = O$. We conclude that both A and B are obligatory.

Contrary to the results of most deontic logics, the converse of 2.12 is not a theorem. This, in the author's opinion, is as it should be. If A and B are both obligatory this does not necessarily imply that an act must be performed each instance of which is an instance of both A and B. For example, if both sleeping and attending classes are obligatory acts one would not wish to say that the act of sleeping while attending class is obligatory.

In setting up an axiomatic system for a deontic logic one is always certain to ensure that all obligatory acts are permitted. Unfortunately this is not a theorem in the general theory of activity domains. The problem arises therefore to obtain some additional structure for activity domains to correct this situation.

2.13 Definition. Let M be an activity domain. We shall say that M satisfies axiom M_1 if there exists an action x in π such that each p neighborhood of x lies in π.

2.14 Lemma. An activity domain satisfies axiom M_1 if and only if $(-\pi)^n \neq O$.
Proof: Let M be any activity domain. Then $x \in (-\pi)^n$ if and only if every p—nbhd of x lies in π. Hence M satisfied axiom M_1 if and only if there exists an element x in U such that $x \in (-\pi)^n$.

2.15 Theorem. Let M be an activity domain. A necessary and sufficient condition that all obligatory acts are permitted is that M satisfies axiom M_1.
Proof: Suppose that M satisfies axiom M_1. Let A be any forbidden act. We show that A is not obligatory. Since A is forbidden, $A \subset -\pi$. Hence by 2.5, $(-\pi)^n \subset A^n$. By 2.2 $(-\pi)^n \subset -(-\pi) = \pi$. By 2.14 $(-\pi)^n \neq O$. Hence there exists an element x in $(-\pi)^n$ such that $x \in A^n$. Also $x \in \pi$. Therefore A^n is permitted. Hence A is not obligatory.

We now suppose that all obligatory acts are permitted. Then the act $-\pi$ is not permitted and hence is not obligatory. Thus $((-\pi)^n \cap \pi) \neq O$. We conclude that $(-\pi)^n \neq O$.

2.16 Corollary. Let M be an activity domain. A necessary and sufficient condition for M to satisfy axiom M_1 is that for any act A either A is permitted or its negative is permitted.

In all deontic logics if an act is forbidden its negative is obligatory. This follows from the definition of obligation and the fact that the negative of the negative of an act is the original act. Again this is not a theorem in the general theory of activity domains. As before we obtain certain additional conditions under which this property can be established.

2.17 Definition. Let M be an activity domain. We shall say that M satisfies axiom M_2 if for each element x in π there exists a p—nbhd ψ of x and an element y in ψ such that for any p—nbhd ψ' of y, $\psi' \subset \pi$.

2.18 Lemma. Let M be an activity domain. Then M satisfies axiom M_2 if and only if $(-\pi)^{nn} = -\pi$.

Proof: Suppose that M satisfies axiom M_2. Let x be any instance of π. Then there exists a p—nbhd ψ of x and an element y in ψ satisfying the conditions of 2.17. Thus for any p—nbhd ψ' of y, $(\psi' \cap -\pi) = O$. We conclude that $y \varepsilon (-\pi)^n$. Therefore a p—nbhd of x, ψ, intersects $(-\pi)^n$. We conclude that x is not a member of $(-\pi)^{nn}$. But x was chosen as an arbitrary member of π. Thus $(-\pi)^{nn} \subset -\pi$. Applying 2.6 we obtain the equality.

Conversely suppose that $(-\pi)^{nn} = -\pi$. Let $x \varepsilon \pi$. Then x is not a member of $(-\pi)^{nn}$. Hence there exists a p—nbhd ψ of x such that $(\psi \cap (-\pi)^n) \neq O$. Choose an element y in this intersection. Then it is clear that every p—nbhd of y is contained in π. Hence M satisfies axiom M_2.

2.19 Theorem. Let M be an activity domain. A necessary and sufficient condition that for each act A if A is forbidden then the negative of A is obligatory is that M satisfies axiom M_2.

Proof: Suppose that M satisfies axiom M_2. Let A be any forbidden act. Then $A \subset -\pi$. Hence $A^{nn} \subset (-\pi)^{nn} = -\pi$. Hence A^{nn} is forbidden and we conclude that A^n is obligatory.

Now suppose that if any act is forbidden its negative is obligatory. In particular $-\pi$ is forbidden and hence $(-\pi)^n$ is obligatory. Hence $(-\pi)^{nn} \subset -\pi$. Thus we obtain the equality, $(-\pi)^{nn} = -\pi$. By 2.18 we conclude that M satisfies axiom M_2.

2.20 Theorem. Every activity domain which satisfies axiom M_2 also satisfies axiom M_1.

Proof: Let M be any activity domain which satisfies axiom M_2. Then $(-\pi)^{nn} = -\pi$. Since $\pi \neq O$, $-\pi \neq U$. Hence $(-\pi)^{nn} \neq U$. Therefore by 2.9 $(-\pi)^{n} \neq O$. Hence by lemma 2.14 M satisfies axiom M_1.

2.21 Corollary. Let M be an activity domain. If for each forbidden act A, the negative of A is obligatory, then all obligatory acts are permitted.

III. *A Causal Relation*

In section II we used the performance universe Φ of an activity domain M to define the negative of an act and to establish theorems about negative acts. In this section we use Φ to define a causal relation between acts.

3.1 Definition. Let M be an activity domain. We shall say that an act A in M *causes* an act B in M if whenever A is performed B is performed, i.e. for each performance set ψ in Φ if $(A \cap \psi) \neq O$ then $(B \cap \psi) \neq O$. For the sake of convenience we shall write 'AcB' in place of the statement 'A causes B'.

In the following discussion we shall consider an arbitrary activity domain M and shall allow the upper case letters A and B to denote arbitrary acts of M.

3.2 Definition. Let A be any act. We define the *cause of A* denoted by A^c as the set:

$$A^c = \{x\text{: for each } \psi \text{ in } \Phi \text{ if } x \varepsilon \psi \text{ then } (A \cap \psi) \neq O\}.$$

3.3 Theorem. Let A and B be acts. Then AcB if and only if $A \subset B^c$.
Proof. Suppose A causes B. Let $x \varepsilon A$. Let ψ be any p—nbhd of x. Then $(A \cap \psi) \neq O$. Hence $(B \cap \psi) \neq O$. We conclude that $x \varepsilon B^c$. Hence $A \subset B^c$.

Now suppose that $A \subset B^c$. Let ψ be any performance set. Suppose $(A \cap \psi) \neq O$. Then we can choose an action x in $A \cap \psi$. Since $x \varepsilon A \subset B^c$ it follows that $(B \cap \psi) \neq O$. We conclude that A causes B.

3.4 Theorem. 1. AcA.
2. $A \subset A^c$.

3.5 Theorem. For any acts A, B, and C, if AcB and BcC then AcC.
Proof: Suppose that AcB and BcC. Let $\psi \varepsilon \Phi$. Then if $(A \cap \psi) \neq O$, $(B \cap \psi) \neq O$. Also if $(B \cap \psi) \neq O$, $(C \cap \psi) \neq O$. Hence if $(A \cap \psi) \neq O$, $(C \cap \psi) \neq O$. Hence AcC.

3.6 Theorem. For any acts A and B, if $A \subset B$, $A^c \subset B^c$.

3.7 Theorem. For any act A, $A^c \subset A^{nn}$.
Proof: Suppose that there exists an action x in A^c such that $x \notin A^{nn}$. Then there exists a p—nbhd ψ of x such that $(A^n \cap \psi) \neq O$, and $(A \cap \psi) \neq O$. Choose an element y in $A^n \cap \psi$. Then although $y \varepsilon A^n$ and ψ is a p—nbhd of y, $(A \cap \psi) \neq O$, a contradiction.

3.8 Theorem. For any act A, $A^{cc} = A^c$.
Proof: By 3.4, $A^c \subset A^{cc}$. Let $x \varepsilon A^{cc}$. Consider any p—nbhd ψ of x. Then $(A^c \cap \psi) \neq O$. Hence there exists an action y in A^c such that ψ is a p—nbhd of y. Since $y \varepsilon A^c$, $(A \cap \psi) \neq O$. We conclude that each p—nbhd of x intersects A. Hence $x \varepsilon A^c$. Thus $A^{cc} \subset A^c$.

The question might arise as to the number of distinct sets which can be obtained by the use of the operators n and c. The following theorem answers this question.

3.9 Theorem. For any act A, $A^n = A^{nc} = {}^{cn}$.
Proof: By 3.4 and 3.7 we have that $A \subset A^c \subset A^{nn}$. Hence by 2.5 and 2.7, $A^n = A^{nnn} \subset A^{cn} \subset A^n$. Hence $A^n = A^{cn}$.

Again by 3.4 and 3.7 we have that $A^n \subset (A^n)^c = A^{nc} \subset (A^n)^{nn} = A^n$. Thus $A^{nc} = A^n$.

3.10 Theorem. For any acts A and B if AcB then $B^n c A^n$.
Proof: If AcB then $A \subset B^c$ by 3.3. Hence by 3.5 and 3.9, $B^n = B^{cn} \subset A^n = A^{nc}$. Hence by 3.3, $B^n c A^n$.

3.11 Theorem. For any acts A and B, $(A \cap B)^c \subset (A^c \cap B^c) \subset (A^c \cup B^c) \subset (A \cup B)^c$.
Proof: $(A \cap B) \subset A$ and hence $(A \cap B)^c \subset A^c$. Similarly $(A \cap B)^c \subset B^c$.
Proof: $(A \cap B) \supset A$ and hence $(A \cap B)^c \subset A^c$. Similarly $(A \cap B)^c \subset B^c$.

Thus $(A \cap B)^c \subset (A^c \cap B^c)$. By a similar argument we can establish that $(A^c \cup B^c) \subset (A \cup B)^c$.

We now deduce some theorems relating to the relation of cause and to the notions of obligation and permission.

3.12 Theorem For any act A, the following are equivalent:

1. A is obligatory.
2. A^{nn} is obligatory.
3. A^c is obligatory.

Proof: A is obligatory if and only if $(A^n \cap \pi) = O$. Since $A^n = A^{nnn}$ we have that A is obligatory if and only if $(A^{nnn} \cap \pi) = O$, and this is true if and only if A^{nn} is obligatory. Hence 1 and 2 are equivalent.

By a similar argument and the fact that $A^{cn} = A^n$ it follows that 1 and 3 are equivalent.

3.13 Theorem. For any acts A and B, if AcB and A is obligatory, then B is obligatory.
Proof: Suppose AcB. Then by 3.3 $A \subset B^c$. Hence 2.5 and 3.9, $B^n = B^{cn} \subset A^n$. If A is obligatory then $(A^n \cap \pi) = O$. Hence $(B^n \cap \pi) = O$. We conclude that B is obligatory.

If an act A causes an act B and if A is permitted it does not necessarily follow that B is permitted. We study the conditions required for this to be valid.

3.14 Definition. Let M be an activity domain. Then we shall say that M satisfies axiom M_3 if for each $x \varepsilon \pi$, x has a p—nbhd ψ such that $\psi \subset \pi$.

3.15 Lemma. M satisfies axiom M_3 if and only if $(-\pi)^c = -\pi$.
Proof: Suppose M satisfies axiom M_3. Let $x \varepsilon (-\pi)^c$. Then each p—nbhd of x intersects $-\pi$. Hence by the statement of M_3 $x \varepsilon -\pi$. We conclude that $(-\pi)^c \subset -\pi$. By 3.4, $-\pi \subset (-\pi)^c$, and hence we obtain the equality.

Now suppose that $-\pi = (-\pi)^c$. Let $x \varepsilon \pi$. Then $x \not\varepsilon -\pi$. Hence $x \not\varepsilon (-\pi)^c$. Hence there exists a p—nbhd ψ of x such that ψ does not intersect $-\pi$ i.e. such that $\psi \subset \pi$. Hence M satisfies axiom M_3.

3.16 Theorem. A necessary and sufficient condition for an activity domain M to satisfy axiom M_3 is that for any two acts A and B of M if AcB and if A is permitted then B is permitted.
Proof: Suppose that M satisfies axiom M_3. Let A and B any two acts such that AcB. We suppose that B is forbidden and prove that A is forbidden. If B is forbidden then $B \subset -\pi$. Hence $B^c \subset (-\pi)^c = -\pi$. By 3.3, $A \subset B^c$. Therefore $A \subset -\pi$. Hence A is forbidden.

Now suppose that the condition holds. We wish to prove that $(-\pi)^c = -\pi$. Let $x \varepsilon (-\pi)^c$. It is sufficient to show that $x \varepsilon -\pi$. Since $x \varepsilon (-\pi)^c$, $\{x\} \subset (-\pi)^c$ and hence by 3.3 we have that $\{x\} c (-\pi)$. Therefore since $-\pi$ is not permitted we conclude by hypothesis that $\{x\}$ is not permitted. Hence $\{x\} \subset -\pi$ or $x \varepsilon -\pi$.

3.17 Theorem. Let M be an activity domain which satisfies axiom M_2. Then M also satisfies axiom M_3.
Proof: If M satisfies axiom M_3 then $(-\pi)^{nn} = -\pi$. But $-\pi \subset (-\pi)^c \subset (-\pi)^{nn} = -\pi$. We conclude that $(-\pi)^c = -\pi$. Hence M satisfies axiom M_3.

3.18 Corollary. Let M be an activity domain such that for each forbidden act A the negative of A is obligatory. Then for any two acts A and B, if A is permitted and A causes B then B is permitted.

IV. *Appendix.*

A. *Negative operators of order two.*

In this section we show that if a set operator enjoys certain of the reasonable properties of the negative operator and also the additional property that the operator is of order two then the operator necessarily coincides with the operation of set theoretical complement. Actually we prove the following:

Theorem. Let S be any set. Let * be a set operator on the family of subsets of S such that * satisfies the following:

1. For any subsets A and B of S, if $A \subset B$ then $B^* \subset A^*$.
2. For any subset A of S, $A^* \subset -A$.
3. For any subset A of S, $A^{**} = A$.

Then for any subset A of S, $A^* = -A$.

It is to be noted that the operation n satisfies the first two of the above conditions. The third condition asserts that the operator * is of order two.

We prove the above theorem as a corollary to the following sequence of lemmas.

4.1 Lemma. $O^* = S$.
Proof: $O \subset S^*$. Hence $S = S^{**} \subset O^*$. Thus $S = O^*$.

4.2 Lemma. For any $A \subset S$, $(-A)^* \subset A$.
Proof: $A^* \subset -A$ and hence $(-A)^* \subset A^{**} = A$.

4.3 Lemma. For any element x in S, $\{x\} = (-\{x\})^*$.
Proof: By 4.2, $(-\{x\})^* \subset \{x\}$. Also $(-\{x\})^* \neq O$, for otherwise we would have that $-\{x\} = (-\{x\})^{**} = O^* = S$, a contradiction. The only non empty subset of $\{Lx\}$ is $\{Lx\}$ itself. Hence $(-dxd))^* = dxd$.

4.4 Lemma. For any $A \subset S$, $A \subset (-A)^*$, and therefore $A = (-A)^*$.
Proof: Let $x \varepsilon A$. Then $-A \subset -\{x\}$. Hence $(-\{x\})^* \subset (-A)^*$. Thus we have that $x \varepsilon \{x\} = (-\{x\})^* \subset (-A)^*$. We conclude that $A \subset (-A)^*$. By 4.2 we obtain the equality.

4.5 Corollary. For any subset A of S, $-A = A^*$.
Proof: By 4.4, $(-A)^* = A$. Hence $(-A)^{**} = -A = A^*$.

It is interesting to note that we can omit hypothesis 1 from the theorem ($A \subset B$ implies that $B^* \subset A^*$) provided that S is finite.

B. *The operation of complement and the performance universe.*

We now proceed to characterize those activity domains for which the negative operator coincides with the operation of the set theoretical complement. We prove the following:

4.6 Theorem. For any activity domain $M = (U, \Phi, \pi)$, the operator n coincides with the set complement if and only if $\Phi = \{\{x\} : x \varepsilon U\}$.

Proof: Suppose that $\Phi = \{\{x\} : x \varepsilon U\}$. Let $A \subset U$. Choose any element x in $-A$. Then x has only one p—nbhd, namely $\{x\}$. Also $\{x\} \cap A = O$. We conclude that $x \varepsilon A^n$. Hence $-A \subset A^n$. But $A^n \subset -A$. Therefore $A^n = -A$.

Now suppose that n coincides with the set theoretical complement. Let $x \varepsilon U$. Choose any element y in U distinct from x. Let ψ be any p—nbhd of x. Then we have that $x \varepsilon -\{y\} = \{y\}^n$. Hence $\psi \cap \{y\} = O$. We conclude that $y \not\varepsilon \psi$. Thus if ψ is any p—nbhd of x and if $y \neq x$, $y \not\varepsilon \psi$. We conclude that $\psi = \{x\}$. Each element in U possesses at least one p—nbhd and hence for each x in U, x possesses one and only one p—nbhd, $\{x\}$. Hence $\Phi = \{\{x\} : x \varepsilon U\}$.

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