RELEVANCE [1]

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The late C.I. Lewis defined what Moore was later to call «entailment» as follows: A entails B if and only if it is not logically possible for A to be true and B false. [2] It follows from this definition that an impossible proposition entails any proposition; that a necessary proposition is entailed by any proposition; that any two impossible propositions entail each other; and that any two necessary propositions entail each other. Lewis was the first to notice these odd or paradoxical consequences and was himself tempted to belief that they showed his definition to be inadequate. One common diagnosis of the flaw in Lewis' analysis is that it allows for cases where entailment holds between irrelevant propositions. Doubtless this is so, and intuitively, one might easily think that it is irrelevance that accounts for the radical falsehood of Lewis' paradoxes. This assessment, however, was rejected by Lewis. He constructed for the paradoxes what purport to be independent deductive proofs which he confidently challenged his critics to fault.

There arises, then, a pair of related questions whose point is sharpened somewhat by the relative inattention accorded them by philosophers of logic:

(i) What exactly does it mean to say that two or more propositions are irrelevant the one to the other ?

(ii) Is it, after all, true that the irrelevance of any two propositions (for some sense of «irrelevance» appropriate to the present context) implies that neither can stand to the other in the relation of entailment ? [3]

Whatever our preanalytic intuitions about irrelevance and about its importance for the truth or falsity of the paradoxes, Lewis' proofs endow our questions with a rather special urgency. The very fact that we have been confronted with what, for all the world, is impeccable deductive justification of the paradoxes requires us to press

(1) Read originally at the eighth annual congress of the *Canadian Philosophical Association* in Halifax, June 13, 1964.

(2) See, e.g., LEWIS and C. H. LANGFORD, *Symbolic Logic*, New York (1932), p. 124 and Ch. VIII.

(3) As we shall see, this formulation of the question embodies a mistaken assumption — that irrelevance is symmetrical. See Section V below.

our demands to know in what sense do the paradoxes violate the relevance condition for entailment-statements and for what reason that condition is properly thought to be a condition.

II

Here we must pause to dispose of one tempting and facile answer to (i), which in turn, trivially, answers (ii). On this interpretation of propositional irrelevance, to say that A is irrelevant to B is to say only that neither the truth (nor the falsity) of A is evidence, deductive or inductive, for the truth (or the falsity) of B. For this sense of irrelevance it cannot be argued without circularity that the irrelevance of A to B is the reason why A does not entail B. If part of what it *means* to say that A is irrelevant to B is that A does not entail B, then it is not possible to decide that A *is* irrelevant to B without begging the question whether A *entails* B. Thus if this is the interpretation of irrelevance upon which the objection to the paradoxes relies, the objection fails.

This raises a third, and I think, the crucial question:

(iii) Is it possible adequately to precise the concept of irrelevance, and so to answer question (i), without begging or illicitly prejudging question (ii), while at the same time providing information in terms of which (ii) is susceptible of a clear answer?

That is, is it possible to define irrelevance under the following conditions:

first: the paradoxes are clearly shown to exemplify irrelevance so defined;

second: the decision whether A is irrelevant to B is possible without having first to determine whether A entails B.

and

third: the information yielded by the definition is sufficient to show that from the fact that A is irrelevant to B, it follows that A does not entail B.

III

I have already remarked the relative uninterest of modern philosophers in the formalization of the relevance condition. One welcome exception is to be found in the work of Nuel D. Belnap, Jr. [4]

Belnap lists thirteen preanalytic requirements which he thinks any viable analysis of entailment must satisfy. Of the thirteen only three are of importance for our immediate purposes:

C I If 'from A_1 ,... A_n to infer B' is a rule of inference, then the inference from A_1 ,..., A_n to B is valid; i.e., any conjunction of premisses *entails* all derivations yielded therefrom by the application of the rules of inference.

C II A theory of entailment ought to be only as strong as is consistent with its fundamental aims. E.g., it ought not rule out any *non-controversial* entailment-statement.

and

C III *The principle of relevance*. If A and B have no propositional variables in common, then 'A entails B' is rejected as a theorem of the system.

Condition III defines irrelevance: if A and B fail to share a propositional variable, then A is irrelevant to B.

Here, then, we have a specification which satisfies only the first two our of own conditions governing the formalization of irrelevance. The paradoxes, do, to be sure, exemplify this sense of irrelevance, and it is possible to decide whether A is in this sense irrelevant to B, without first having to determine whether A entails B. But it does not follow from Belnap's definition alone, that if A is irrelevant to B, then A does not entail B; and so our third condition is not satisfied. Only if one accepts Belnap's C III, can one use the fact of irrelevance in Belnap's sense to rule out the paradoxes. But to accept C III is to beg all the important questions; and moreover, I think it is possible to show that C III is inconsistent with Belnap's other conditions, C I and C II.

Given Belnap's C III, we must deny the following entailment-statements:

(1) «The book is blue» entails «The book is coloured».

(2) «Proposition A is true» entails «Proposition A is truth-valued.»

(3) «The figure is square» entails «The figure is rectangular.»

In general, C III rules out all entailment-statements which hold ei-

(4) *A Formal Analysis of Entailment*, Technical Report No. 7, Office of Naval Research Contract No. S A R / Nonr - 609 (16), New Haven (1960; and «Entailment and Relevance», *Journal of Symbolic Logic*, 25 (1960), pp. 144-6.

ther in virtue of a determinate-determinable relation, or in virtue of a so-called genus-species relation. This surely violates C II: one gets rid of the paradoxes only at the expense of denying other entailment-statements which it would not occur to us to question.

But more disturbing is the fact that each step of Lewis' *proofs* (5) of the paradoxes satisfies the relevance-condition; and each would seem to be sanctioned by a sound rule of inference. If so, then, by C I, we must agree that the paradoxes are true (albeit truths easily overlooked), a fact which, by C III, we must deny.

The embarrassing strength of C III shows itself in yet another respect. It throws out of court the results of simultaneous substitutions into formulae of the system. One is debarred from moving from

(i) $p \supset (p \vee q)$

to

(ii) $r \supset (r \vee s)$, simultaneously substituting r for p, and s for q.

However, it does not rule out substitutions which, *piecemeal*, achieve the same result.

(iii) $p \supset (p \vee q)$

yields

(iv) $r \supset (r \vee q)$, (substituting r for p),

which, in turn, yields

(v) $r \supset (r \vee s)$, (substituting s for q).

Now, given that substitution is a rule of inference, one must say, by C I, that (iii) entails (iv), and that (iv) entails (v); and given the transitivity of entailment, one is forced to agree that (iii) entails (v); but, by C III, one cannot agree to this since (iii) and (v) share no propositional variables.

The only escape is either to reject one or more of C I through C III, or to reject at least one of the rules of inference invoked by Lewis' proofs, and to reject substitution as a rule of inference. This is not the place to argue one's options' (6) but I think many will agree that the least desperate expedient is to dispense with the relevance-condition, C III, at least as Belnap has formulated it. Suffice it to say that the application of C III yields consequences no less counter-intuitive than the very paradoxes it was designed to rule out.

(5) Lewis' independent proofs are, of course, well known. They may be found in *Symbolic Logic*, pp. 248-252.

IV

However, there is, I think, another sense of irrelevance which may apply to the paradoxes and which helps sustain the impression that they must be false. Irrelevance in this sense is a matter of the logical bearing of premisses of deductive arguments upon their conclusions. Consider the following argument.

Given: (1) E ,
(2) A ,
(3) $A \supset B$,
and (4) $B \supset C$,
to prove that C v D.

	(7) C v D	(2) and (3), modus ponens
Proof:	(5) B	(4) and (5), modus ponens
	(6) C	(6), addition.
		Q.E.D.

Now, no use was made of premiss (1). The truth of E was not required to show that (C v D). That premiss could have been dropped without affecting the outcome of the proof. E is not a necessary ingredient in the proof of (C v D), and in that sense E is irrelevant to (C v D). Let us denominate this sense of irrelevance «irrelevance_2», to distinguish it from Belnap's conception of it. Next, we observe that E does not entail (C v D). And some will suppose that the fact that E is irrelevant_2 is the *reason* why E does not entail C v D. If true, this is an important fact, for it seems that we could in general determine whether any two propositions were irrelevant_2 without begging the question against the possibility of their being related by entailment, and after having done so, use the fact that the one is irrelevant_2 to the other as a *reason* for saying that the one does not entail the other.

But this is an illusion. The fact, that, in our example, E does not entail (C v D) is unrelated to the fact E is irrelevant_2 to (C v D). This can be shown by devising a proof exactly like the one above save that each occurrence of «C» is replaced by «E». Thus,

Given: (1) E ,
(2) A ,
(3) $A \supset B$,
and (4) $B \supset E$,

([6]) Belnap elects to repudiate disjunctive syllogism. See *A Formal Analysis*, p. 33 ff. However, the denial of disjunctive syllogism reduces to absurdity; this I hope to show in a fortcoming publication.

to prove that E v D.

Proof:	(5) B	(2), (3), modus ponens
	(6) E	(4), (5), modus ponens
	(7) E v D	(6), addition.
		Q. E. D.

Here the situation is exactly as before — at least as far as irrelevance$_2$ goes. The first premiss, E, is irrelevant in the sense that it could be dropped without affecting the outcome of the proof. It is not necessary to use it to derive the conclusion. It is therefore irrelevant$_2$ to the conclusion. But it does not follow from its irrelevance$_2$ that it does not entail the conclusion. E *does* entail (E v D); and E *is* irrelevant$_2$ to (E v D). Thus, in general, the irrelevance$_2$ of A to B is after all no guarantee that A does not entail B. And so it would appear that, once again, our third condition is not satisfied: from the fact that one proposition is irrelevant$_2$ to another it does not follow that the one does not entail the latter.

More than that, it is hardly likely that anyone should wish to impute irrelevance$_2$ to the proofs of the paradoxes. What premisses is one to think of as dispensable? Which is to be dropped without prejudicing the proof in which it occurs? This sense of irrelevance simply does not apply to Lewis' proofs, and we have in this fact a second reason for rejecting the objection from irrelevance$_2$.

V

Let us, finally, define irrelevance$_3$. Let us say that A is irrelevant$_3$ to B, if, and only if, the truth-value of B remains constant independently of whether A or its contradictory is true. It should be noted that irrelevance$_3$ is not symmetrical, and that, even if from the fact that A is irrelevant$_3$ to B it follows that A does not entail B, it does not follow that B does not entail A.

Now I think there is no doubt that any statement B is irrelevant$_3$ to any statement of the form $p \,.\! \sim p$, and that any statement A is irrelevant$_3$ to any statement of the form $q \vee \sim q$. That is, the first paradox embodies irrelevance$_3$ from consequent to antecedent, and the second paradox embodies irrelevance$_3$ from antededent to consequent. At the same time, it seems to me clear enough that the third paradox exemplifies irrelevance$_3$ both from antecedent to consequent and from consequent to antecedent as well; and that the same holds for the fourth paradox. But all I wish here to defend and to examine is the irrelevance$_3$ of the first two paradoxes. So I am arguing that our first condition is at least partially satisfied. And so, it would seem, is, our second condition. For surely one can discover that the first two paradoxes exhibit irrelevance$_3$ without first having to decide whether $(p \,.\sim p)$ entails q and whether p entails $(q \vee \sim q)$. It remains to see whether from the fact that they are irrelevant$_3$ it follows that they are not true.

Given that p is irrelevant$_3$ to $(q \vee \sim q)$, it surely does not follow that p does not entail $(q \vee \sim q)$. It *might* follow that p is not a *necessary* condition of $(q \vee \sim q)$, but this is consistent with p's being a sufficient condition of $(q \vee \sim q)$. Were it the case that irrelevance$_3$ held in reverse, i.e. between consequent and antecedent, things would be different. But since it is not the case that $(q \vee \sim q)$ is irrelevant$_3$ to p, this is of merely academic interest. (7) So I think we may conclude, in the case of the second paradox, that irrelevance$_3$ is compatible with its truth.

But what of the first paradox? Surely it is possible to *prove* its falsity from the fact it embodies irrelevance$_3$.

Granted that (1) q is irrelevant$_3$ to $(p \,.\sim p)$, it certainly seems to follow that

(2) it is not the case that q is a necessary condition of $(p \,.\sim p)$.
And this implies that

(3) it is not the case that $(p \,.\sim p)$ is a *sufficient* condition of q.
But what is this but to acknowledge that $(p \,.\sim p)$ does not entail q? No one will deny that the first paradox can be discovered to embody irrelevance$_3$ without first deciding whether $(p \,.\sim p)$ entails q; so it would seem absolutely essential to fault our proof if one is to continue to believe that the paradox is true.

Can the argument be faulted? Certainly the move from (2) to (3) is absolutely in order; and the inference from (1) to (2) appears perfectly sound.

(7) Suppose p is some false proposition. Then if $(q \vee \sim q)$ is true, p surely remains false; but if the *negation* of $(q \vee \sim q)$ were true, then p, along with every proposition would be true. So $(q \vee \sim q)$ is not irrelevant to p. But surely I have begged the question. I have assumed the very thing in question, namely, that if the negation of $(q \vee \sim q)$ were true then every proposition would be true, and this is just Lewis' first paradox. It can be shown, however, that *if* that paradox is false it is *not* because it contains irrelevance$_3$. If. then, one rejects the paradoxes simply because of irrelevance$_3$, I have not begged the question in denying that $(q \vee \sim q)$ is irrelevant to p.

However, there is a situation in which even though the falsity of A does not alter the truth-value of B, it is nonetheless true that A is a necessary condition of B. This obtains when it is the case both that A and that $\sim$A are necessary conditions of B. If both A and $\sim$A are necessary conditions of B, it remains true to say that the truth-value of B is unaltered by the falsity of A; but it is false, by hypothesis, to infer that, for this reason, A is not necessary to B. It is of some importance that just such a situation is said to obtain when B is a proposition of the form $(p \,.\! \sim p)$; it is just this that Lewis' first paradox purports to reveal. For what that paradox shows is that if $(p \,.\! \sim p)$ then every proposition whatever is a necessary condition of its truth.

The principle we used to take us from the irrelevance$_3$ of q to $(p \,.\! \sim p)$ to the claim that q is not therefore a necessary condition of $(p \,.\! \sim p)$ is illegitimately strong. It should have read *not*

(a) If A is irrelevant$_3$ to B, then $\sim$ (A is a necessary condition of B)

but rather,

(b) If A is irrelevant$_3$ to B, then ($\sim$ (A is a necessary condition of B) if, and only if $\sim$($\sim$A is a necessary condition of B)).

In our argument, principle (b) is not strong enough to carry us from step (1) to step (2), and it is there that the argument fails.

What this shows is that not even the irrelevance$_3$ from consequent to antecedent is sufficient to falsify the entailment from antecedent to consequent. If the objection from irrelevance to Lewis' paradoxes is to prevail it must be for a sense of «irrelevance» that we have not yet managed to formulate.

One parting shot: perhaps principle (b) above commits the mistake of supposing that each of two mutually contradictory propositions, A and $\sim$ A, could be necessary for the truth of a proposition B. If this is so, then my attack on the argument from irrelevance$_3$ fails. But that argument is nonetheless emasculated. It can no longer be used to falsify the first paradox; for in its restored form it already presupposes the falsity of the paradox by presupposing the impossibility of both A and $\sim$ A being necessary conditions of B.

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