UNINTERPRETED CALCULI AND «SOUND» LOGICAL SYSTEMS *

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In his *Introduction to Mathematical Logic* (Vol. I), Church states that «it is desirable or practically necessary for purposes of logic to employ a specially devised language, a *formalized language,* «[[I] p. 2] and that «to adopt a particular formalized language thus involves adopting a particular theory or system of logical analysis» [[I] p. 3]. Then later, he makes the standard distinction between 1) the «purely formal part of the language», i.e., the «uninterpreted calculus or *logistic system*», which is set up «in abstraction from all considerations of meaning» [[I] p. 48] and 2) the formalized language itself, which does not come into being until an *interpretation* has been provided for the logistic system [[I] p. 54]. So far so good.

In this paper I wish to argue 1) that Church's subsequent analysis of «soundness» rules out certain formalized systems of logic which meet all syntactical, semantical and pragmatic requirements, and 2) that his terminology in describing the so-called purely syntactical elements suggests unnecessary restrictions upon the notion of an uninterpreted calculus. The argument applies not only to Church but to prevailing contemporary practice, and brings into question the prevailing view of a system of logic as an axiomatized system, or language, yielding a set of universally true statements.

I

With respect to the first point, I shall show how it is possible to construct systems which meet all formal requirements laid down for the usual purposes of logic as standard systems, but which, by Church's definition of «soundness» would turn out to be «unsound» interpretations or languages, and thus are to be «rejected.»

I shall call the systems to be considered «antilogistic systems.» These are defined as follows: «S is an antilogistic system» =df «1) S contains the kinds of syntactical elements and rules which are characteristic of a logistic system [cf. Church; [1], pp. 47-54]; 2) S

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employs primitive symbols, *with the same customary interpretations,* which appear in some established, consistent, formalized language or deductive system which yields only truths as derived formulas; 3) S yields only derived formulas which are false or inconsistent, instead of true formulas.»

The following example of an antilogistic system, P_{Ba}, is based on a refinement, P_B, of the formulation of the propositional calculus which appeared in *Principia Mathematica* [cf. [1] p. 157]. The propositional calculus, P_B, includes, (among others) the following syntactical elements: 1) the primitive constants '$\vee$' and '—', 2) the abbreviations, D1:

$$(S \supset S') = \text{df } (—S \vee S'), \text{ and D2: } (S \,.\, S') = \text{df } —(S \vee —S'),$$

3) the four primitive formulas,

A1. $(—(p \vee p) \vee p)$
A2. $(—q \vee (p \vee q))$
A3. $(—(p \vee q) \vee (q \vee p))$
A4. $(—(—q \vee r) \vee (—(p \vee q) \vee (p \vee r)))$

(in which I have replaced occurrences of '$\supset$' by unabbreviated expressions), and 4) the rule of transformation, «If $\vdash S$ and $\vdash (—S \vee S')$, then $\vdash S'$.".

From the system P_B we construct the purely formal part of the antilogistic analogue, P_{Ba}, simply by replacing '$\vee$' with '.' in the four axioms and in the rule of inference, interchanging '$\vee$' and '.' in Definition 2, and replacing '$\vee$' by '.' as a primitive symbol. Thus in P_{Ba} we have the *primitive logical constants* '.' and '—', the *abbreviation*:

D2a. $(S \vee S') = \text{df } —(S \,.\, —S'),$

the four *primitive formulae*:

A1a. $(—((p \,.\, p) \,.\, p)$
A2a. $(—q \,.\, (p \,.\, q))$
A3a. $(—(p \,.\, q) \,.\, (p \,.\, q))$
A4a. $(—(—q \,.\, r) \,.\, (—(p \,.\, q) \,.\, (p \,.\, r)))$

and the «*rule of transformation*:

If $\vdash S$ and $\vdash (—S \,.\, S')$, then $\vdash S'$

All other elements and rules of the two systems remain the same.

The pure logistic systems of P_B and P_{Ba}, are syntactically equivalent; the only difference between them is that the marks '.' and '$\vee$' interchange roles (save in the inessential definition, D1, of '$\supset$'). This syntactical equivalence permits us to ascribe P_{Ba} all of the purely syntactical properties which have been established for P_B, including effectiveness, consistency (in its three syntactical senses [cf. [1], p. 108], and completeness (in its three syntactical senses [cf [1], p. 109]. The example above is not an isolated system. Given any one of the axiomatizations of the PM type propositional calculus, an antilogistic analogue could be constructed with all the syntactical properties of the initial PM-type system, by the use of known principles of duality. The antilogistic analogues would have the same number of primitive formulae, and for each axiom in turn the same number and groupings of variables and occurrences of variables; they would differ only in the interchange of logical constants in the unnabbreviated primitive formulae, in some abbreviations and rules of transformation, and in choice of the primitives. The construction of antilogistic systems can easily be extended to quantification theory and higher levels of logic, and presumably to non-mathematical deductive systems as well. All of these systems would meet, thus far, the first requirement specified above for antilogistic systems: i.e., the requirements Church sets down [[1], pp. 47-54] for logistic systems.

The second requirement of antilogistic systems stipulates that we shall employ primitive symbols with their *customary interpetations.* Thus our primitive constants '.' and '—', as well as abbreviations involving these, like '$\vee$' and '$\supset$', retain the standard truth-functional definitions related to the English words «and», «not», «or» and «if... then.» Taken *by themselves,* there can be no more or less objection to these semantical definitions than might be brought against them in any standard system of logic. But the result of retaining these standard interpretations in connection with P_{Ba} is that each of the primitive formulae, and every expression reached through application of rules of transformation become logical falsehoods on this interpretation. Thus what we have is a system which produces inconsistent statement forms, rather than a method of producing logically true statement forms (1). By virtue of these interpretations, and principles

(1) Antilogistic systems, though related in use to Aristotle's «demonstration *per impossibile*» [2] and Ladd-Franklin's method of the antilogism [3], as well as certain methods of Skolem [4], and Hintikka and Beth [5], in testing for inconsistency, are *not* the same. When such systems are formalized, they yield only truths, e.g., of the sort «'(p.-p)' is inconsistent»; whereas

of duality, then, the third requirement for antilogistic systems is met.

It is at this point that Church, by his definitions, labels the «formalized language» P_{Ba}, unsound, and calls for its rejection. For Church says,

> «... we call an interpretation of a logistic system *sound* if, under it, all the axioms either denote truth or have always the value truth, and if further the same thing holds of the conclusion of any immediate inference if it holds of the premisses. In the contrary case we call the interpetation *unsound*. A formalized language is called sound or unsound according as the interpretation by which it is obtained from a logistic system is sound or unsound. And an unsound interpretation or an unsound language is to be rejected.» [[1] p. 55]

But why does Church require that, in addition to the semantic correlations between logical constants and English usage provided by truth-functional definitions, the *primitive and derived formulae* of the interpreted formalized language must be true in all possible cases? The answer appears to lie in the presupposition, widely shared, that a system of logic is necessarily a system for deducing truths from truths. As we shall see, this presupposition is suggested implicitly in the terminology he uses to define logistic systems.

Before considering the latter point, however, let us consider whether antilogistic systems, as we defined them, should necessarily be rejected, as Church would affirm. Presumably Church's injunction to reject «unsound» languages means, in some sense, that the latter should not be taken seriously by logicians and thus should not appear in treatises or textbooks on logic, except, perhaps, as curious examples of what should not be done. But it is easily shown that antilogistic systems have precisely the same utility for the traditional tasks of logic as do the more usual axiomatizations. If so, an admonition to eliminate them from serious considerations would appear unwarranted.

our system yields logically false forms. The closest approximation to antilogistic systems was presented by Hirano in 1934 [6], who took P_B, supplemented by quantification theory, and altered the *interpretation* of 'V' and '(x)' so that they were read «and» and «There is some x...», thus making all axioms and theorems falsehoods on interpretation. This system meets the first and third conditions laid down for antilogistic systems, but fails to meet the second. Our antilogistic systems differ from Hirano's in expository and pragmatic advantages, rather than in theoretical significance, however.

The principle traditional functions served by formal logic are two: to test arguments for validity, and to test statements for logical truth, logical falsehood or consistency. The theorems of P_B (or *Principia Mathematica*) constitute an array of formulae which, by virtue of the interpretation, are schemata of logically true propositions. To *use Principia Mathematica* to test an argument in ordinary language for validity one i) schematizes each separate statement in the argument, using the symbolism of P_B, ii) forms a conditional having the conjunction of the premisses for an antecedent and the conclusion as a consequent, again in the symbolism of P_B, iii) determines whether the propositional form just constructed occurs (can be derived) among the axioms and theorems of P_B; if this propositional form does so occur the argument is pronounced *valid*. To determine the logical truth of a *statement*, or a conjunction of statements, one follows essentially the same process, omitting step ii). To determine the logical falsehood of a *statement*, or the inconsistency of a set of *statements*, one schematizes the statement or set of statements in the symbolism of P_B, denies the result, and determines whether the resulting denial is among the axioms and theorems of P_B.

Analogous methods are available for antilogistic systems. Instead of beginning with the notion of logical truth as something to be attained, let us begin with the notion of inconsistency as something to be avoided. A logically 'valid' argument is then defined as one such that the acceptance of the premisses and the denial of the conclusion would be inconsistent; a logically true statement is defined simply as the denial of an inconsistent statement. To *test* an argument for validity in P_{Ba} one i) schematizes each statement in the argument as before, in the symbolism of the antilogistic system, ii) conjoins the premisses with the denial of the conclusion (as contrasted with forming a conditional), iii) determines whether the propositional form just constructed occurs (can be «derived») among the primitive and derived formulae of the antilogistic system; if it does so occur, the argument is *valid*. To determine the logical falsehood of a statement or group of statements, one follows essentially the same process, omitting step ii), and determines directly whether the given form is found among the primitive or «derived» formulae of the system. To determine logical truth, one tests in like manner the denial of a propositional form. Other subsidiary tasks of logic, like determining relationships of contrariety, contradiction, independence, etc., can be handled with equal ease in antilogistic systems.

In general then, an antilogistic system of logic, can serve the same functions as the more common variety, though it does it by systematic development of a list of inconsistent forms, rather than a systematic development of logically true schemata. There is no good reason to reject such systems, either on syntactical, semantical or pragmatic grounds.

II

It is unlikely that most contemporary logicians ,including Church and Carnap, would deny that they more or less pre-suppose that «logistic systems», or «uninterpreted calculi» are intended to be given interpretations which will make the primitive and derived formulae come out true in all possible cases. Thus Church gives as a reason for his definition of soundness that «... it is intended that the proof of a theorem shall always justify its assertion», and Carnap writes:

> «One who constructs a syntactical system usually has in mind from the outset some interpretation of this system... While his intended interpretation can receive no explicit indication in the syntactical rules — since these rules must be strictly formal — the author's intention respecting interpretation naturally affects his choice of the formation and transformation rules of the syntactical system. E.g. he chooses primitive signs in such a way that certain concepts can be expressed. He chooses sentential formulas in such a way that their counterparts in the intended interpretation can appear as meaningful declarative sentences. His choice of primitive sentences must meet the requirement that these primitive sentences come out as true sentences in the interpetation. And his rules of inference must be such that if the sentence S_j is directly derivable from a sentence S_i (or from S_{i1} and S_{i2} where S_i is $S_{i1} . S_{i2}$), then $S_i \supset S_j$ turns out to be a true sentence under the customary interpretation of '$\supset$'. These last requirements ensure that all provable sentences also come out true.» [[7] p. 101]

Although Carnap's first four sentences suggest that a purely syntactical system places no restriction on the interpetation, his last three sentences *do* place a restriction on such calculi, through requiring that the primitive and derivable sentences must come out true on interpretation, and it is not clear whether this is a restriction on the uninterpreted system, or on the interpretations which may be given it. In either case, it raises a question as to whether the purely formal parts of systems are as free from relationship with particular interpretations as is initially suggested.

In fact, both in Church and Carnap, (among others), the terminology used to describe the elements of «logistic» or «syntactical» systems, is prejudicial. Thus Church, after mentioning primitive symbols, formulae, well-formed formulae and rules of formation as elements in his «logistic» systems, describes additional elements in the following words:

> «...certain among the well-formed formulas are laid down as axioms. And finally (primitive) *rules of inference* (or *rules of procedure*) are laid down, rules according to which, from appropriate well-formed formulas as *premisses,* a well-formed formula is *immediately inferred* as *conclusion.* (So long as we are dealing only with a logistic system that remains uninterpreted, the terms *premiss, immediately infer, conclusion* have only such meaning as is conferred upon them by rules of inference themselves.)
>
> «A finite sequence of one or more well-formed formulas is called a *proof* if each of the well-formed formulas in the sequence either is an axiom or is immediately inferred from preceding well-formed formulas in the sequence by means of one of the rules of inference. A proof is called a proof *of* the last well-formed formula in the sequence, and the *theorems* of the logistic system are those well-formed formulas for which proofs exist.» [[1] pp. 49-50. Italics are Church's.]

Carnap, like Church, employs the terms «proof», «inference» and «premiss», in describing elements of a «syntactical» system; though he does not use «axiom» (but «primitive sentence») or «theorem».

Now all of these expressions («axiom», «rule of inference», «proof», «premiss», «conclusion», «theorem») take on familiar and ordinary additional meaning when the purely formal system is interpreted in the usual way as a deductive system of logically true propositional forms. But if the interpeted system is an antilogistic system (as in P_{Ba}above), and we have followed Church's (or Canap's) terminology in describing the purely syntactical parts, then the attempt to carry this terminology over into the interpreted system yields strange and unnatural results. For the word «proof» clearly carries with it, in ordinary language, the suggestion that what is proved is true, and *shown* true on the basis of previously established truths. The interpreted «theorems» of the antilogistic system, however, are not only false, but logically inconsistent. Each of the «premisses» is likewise logically false. But how can one «prove» any conclusion from «premisses» which are intentionally false and inconsistent ? In the usual sense of «proofs» an antilogistic system contains no proofs at all, and is not intended to. It becomes apparent on examination that the transformation sequences in an antilogistic system are not the kind one would call «inferences» at all; in the case of the rule, «*From* $\vdash S$ *and* $\vdash (—S . S')$ *infer* $\vdash S'$», the two premisses could not possibly both be true, and it is not intended that the conclusion be true. This rule does not conform to any usual concepts of *deduction* at all; it is simply a tool or instrument which works, in an antilogistic system, to get a list of logical falsehoods from other logical falsehoods. To view formulas, knowingly interpreted as inconsistencies, as «axioms» and «theorems», under these conditions, event though this refers merely to the sign system apart from the interpretation, would be, at least, a perverse employment of common words.

In conclusion, 1) it would be arbitrary to reject antilogistic systems from logic ,since no strictly formal arguments have been brought against them, and pragmatic arguments in their favor are quite as strong as those for ordinary deductive systems of formal logic; 2) if we are to make the study of uninterpreted calculi a field of investigation, we should use terms to describe such calculi which will not prejudge or presuppose the type of interpretation which can or should be made. If there are reasons for preferring interpretations which yield only truths for primitive and derived formulae, these reasons can be presented and discussed, but this will not be part of the theory of *uninterpeted* calculi; and 3) definitions of soundness and principles of acceptability respecting interpretations should be broad enough to permit and include antilogistic systems (and perhaps others). The requirement that «sound» interpetations must be deductive systems yielding only true theorems from true premisse, is then, an unnecessary restriction; and a system of logic is not necessarily a system of true statements.

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