A BINARY PRIMITIVE IN DEONTIC LOGIC

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1.- *Introduction*

In this note I intend to show that deontic logic can be based on the logic of a relationship B that might be identified with some concept of *betterness*. A modified three-valued decision procedure can be proved to be effective for this logic. The relations of our system to the calculi proposed by S. Halldén in his study *On the Logic of 'Better'* are obviously of particular interest and will receive some treatment. The non-trivial differences between our logics of 'better' are, I suggest, simply due to the fact that Hallden's notion of betterness is different from mine which is framed with an eye on deontic logic.

2.- *The calculus BD*

By the class of well-formed formulae of the propositional calculus (wffs of PC) we understand, as usual, the least class *a* such that (i) every propositional variable (p, q, r, ...) is in *a*, and (ii) if α and β are in *a*, then $N\alpha$, $K\alpha\beta$, $A\alpha\beta$, $C\alpha\beta$ and $E\alpha\beta$ (Polish notation) are in *a*.

By the class of *wffs of the calculus BD* we shall then understand the least class *b* such that (i) if α and β are wffs of PC, then $B\alpha\beta$, $S\alpha\beta$, $O\alpha$, $F\alpha$, $I\alpha$, and $P\alpha$ are in *b*, and (ii) if α and β are in *b*, then $N\alpha$, $K\alpha\beta$ etc. are in *b*.

BD contains the usual definitions of A, C, and E in terms of N and K. Besides, it contains the following definitions:

D1. Spq =def NABpqBqp
D2. Op =def BpNp
D3. Fp =def BNpp
D4. Ip =def SpNp
D5. Pp =def NBNpp

Bpq is read as «p is deontically better than q» or «p has a higher deontic value than q», Spq as «p and q are of equal deontic value», Op as «it is obligatory that p» or as «it ought to be the case that p», Fp as «it is forbidden that p», Ip as «it is indifferent that p» or, more idiomatically, «it is indifferent whether p or non-p», and Pp as «it is permitted (right) that p».

We propose the following set of postulates for BD:

P1. CBpNpNBNpp (*)
P2. CBCpqNCpqCBpNpBqNq
P3. EBpqAAKBpNpSqNqKBpNpBNqqKSpNpBNqq

And these rules of inference are laid down:

R1. *Detachment.* If α and $C\alpha\beta$ are theses of BD, then β is a thesis of BD.

R2. *Substitution.* If α is a thesis of BD and β is like α except for containing a wff of PC at each place where α has a certain variable, then β is a thesis of BD.

R3. *Necessitation.* If α is a thesis of PC, then $B\alpha N\alpha$ is a thesis of BD.

R4. *Replacement.* If α is a thesis of PC and β is the result of replacing every variable in α by a wff of BD, then β is a thesis of BD.

R5. *Double negation.* A double negation occurring at any place in a wff of BD may be dropped.

3.- *A decision procedure for BD*

The decision problem of the system just presented can be effectively solved; and we now proceed to the description of a method by which any wff of BD can be tested for validity in a mechanical manner.

We first introduce the following tables:

K	1	$\frac{1}{2}$	0	N
1	1	$\frac{1}{2}$	0	0
$\frac{1}{2}$	$\frac{1}{2}$	?	0	$\frac{1}{2}$
0	0	0	0	1

B	1	$\frac{1}{2}$	0
1	0	1	1
$\frac{1}{2}$	0	0	1
0	0	0	0

For the moment, just note that $K\frac{1}{2}\frac{1}{2}$ has not been uniquely evaluated. Derived tables for the connectives A, C, E, S, O etc. are of course easily constructed. The following interpretation of 1, $\frac{1}{2}$ and 0 might now be suggested:

(1) If 1, $\frac{1}{2}$, or 0 are assigned to a wff α of PC, they are taken as the «deontic values» obligatoriness, indifference, and prohibitedness respectively. In this case ,K11 = 1, *e.g.*, means that a conjunction of two obligatory facts (actions) is itself obligatory.

(2) If 1 or 0 are assigned to a wff β of BD ($\frac{1}{2}$ is readily seen to be

* This postulate is redundant.

incapable of being assigned to such a formula), they are taken as the values truth and falsity respectively. *E.g.*, $B10 = 1$ means that if p is obligatory and q forbidden, then it is true that p is better than q, $K11 = 1$ now means that a conjunction of two true propositions is itself true, and so on.

The interpretation just proposed seems natural enough. From the point of view of the decision problem of BD, however, we can dispense with interpretations of 1, $\frac{1}{2}$ and 0 altogether, so we need not worry about the oddity that our «ambiguous» interpretation might be thought to involve.

We can now present the decision procedure for BD by giving the following instructions for testing validity.

I1. Apply the tables given above and their derivatives.

I2. As for the cases not covered by these tables, we may have $K\frac{1}{2}\frac{1}{2} = \frac{1}{2}$ *or* $K\frac{1}{2}\frac{1}{2} = 0$, $A\frac{1}{2}\frac{1}{2} = \frac{1}{2}$ *or* $A\frac{1}{2}\frac{1}{2} = 1$, $C\frac{1}{2}\frac{1}{2} = \frac{1}{2}$ *or* $C\frac{1}{2}\frac{1}{2} = 1$, and $E\frac{1}{2}\frac{1}{2} = \frac{1}{2}$ *or* $E\frac{1}{2}\frac{1}{2} = 1$ *or* $E\frac{1}{2}\frac{1}{2} = 0$. Now, in evaluating any such case, consider *all* these possibilities for $K\frac{1}{2}\frac{1}{2}$, $A\frac{1}{2}\frac{1}{2}$ etc.

I3. Application of I1 and I2 to any formula of BD results in a truth-table yielding 1 or 0 for each row of the table. Now, keep only the rows that yield 0. (If there are none, the formula is valid and the test is already at an end.) Then determine whether each of these rows meets at least one of the following conditions (i)-(iii).

(i) The formula contains a wff α of PC or a class $/ \alpha_1 \ldots \alpha_n /$ $(n \geqslant 2)$ of wffs of PC, and, further, a wff β of PC or a class $/ \beta_1 \ldots \beta_m /$ $(m \geqslant 2)$ of wffs of PC, which are such that *either*

(a) $C\alpha\beta$ is a tautology of PC, and the row assigns 1 or $\frac{1}{2}$ to α but 0 to β, or 1 to α but $\frac{1}{2}$ to β, *or*

(b) $CK\alpha_1 \ldots \alpha_n\beta$ is a tautology of PC, and the row assigns 1 to each of $\alpha_1 \ldots \alpha_n$ but $\frac{1}{2}$ or 0 to β, or $\frac{1}{2}$ to *exactly one* of $\alpha_1 \ldots \alpha_n$ and 1 to the others but 0 to β, *or*

(c) $C\alpha A\beta_1 \ldots \beta_m$ is a tautology of PC, and the row assigns 1 or $\frac{1}{2}$ to α but 0 to each of $\beta_1 \ldots \beta_m$, or 1 to α but $\frac{1}{2}$ to *exactly one* of $\beta_1 \ldots \beta_m$ and O to the rest, *or*

(d) $CK\alpha_1 \ldots \alpha_n A\beta_1 \ldots \beta_m$ is a tautology of PC, and the row assigns 1 to each of $\alpha_1 \ldots \alpha_n$ but 0 to each of $\beta_1 \ldots \beta_m$ or $\frac{1}{2}$ to *exactly one* of $\beta_1 \ldots \beta_m$ and O to the rest, or it assigns $\frac{1}{2}$ to *exactly one* of $\alpha_1 \ldots \alpha_n$ and 1 to the rest but 0 to each of $\beta_1 \ldots \beta_m$.

(ii) The formula contains a class $/ \alpha_1 \ldots \alpha_n /$ $(n \geqslant 2)$ of wffs of PC such that *either*

(a) $A\alpha_1 \ldots \alpha_n$ is a tautology of PC, and the row assigns 0 to each of $\alpha_1 \ldots \alpha_n$, or $\frac{1}{2}$ to *exactly one* of them and 0 to the others, *or*

(b) $K\alpha_1 \ldots \alpha_n$ is a contradiction of PC, and the row assigns 1 to each of $\alpha_1 \ldots \alpha_n$, or $\frac{1}{2}$ to *exactly one* of them and 1 to the rest.

(iii) The formula contains a wff α of PC such that *either*

(a) α is a tautology of PC, and the row assigns 0 or $\frac{1}{2}$ to it, *or*

(b) α is a contradiction of PC, and the row assigns 1 or $\frac{1}{2}$ to it.

The following *rule of elimination* can now be enunciated: RE. A row assigning 0 to a formula of BD may be dropped, if and only if, it meets at least one of the conditions (i)--(iii) just stated.

If and only if every row assigning 0 to a formula of BD can be dropped in virtue of RE[1], the formula is valid in BD.

Such is our decision procedure for BD. We are not here going to prove that passing the test affords a condition that is both necessary and sufficient for theoremhood in BD. I have done so elesewhere[2] for a closely similar test and a deontic logic equally strong as the one involved in BD. The metalogical argument required for BD will contain no new points of particular interest, so it might well be omitted in this paper.

4.- *Deontic logic in BD*

It is easily shown that the system BD contains a subsystem of deontic logic which is derivable as follows. Let us refer to any wff of BD that involves no occurrence of the connectives B or S as a *deontic wff*. Application of the definitions D2 and D5 to the postulates P1 and P2 yields the theorems

T1. COpPp
T2. COCpqCOpOq

Let R1′, R2′ and R4′ be special cases of respectively R1, R2, and R4, where α, β, $C\alpha\beta$ are supposed to be deontic wffs. Let further R3′ be like R3 except for containing $O\alpha$ where R3 has $B\alpha N\alpha$.

Obviously, we have as a subsystem of BD a calculus taking T1 and T2 (plus the equivalence EPpNONp, provable by D2, D5, R4 and R5) as its postulates and R1′ — R4′ as its rules of inference. Moreover, this calculus appears adequate enough for deontic logic. From the syn-

(1) The rule RE is in certain respects similar to a rule of elimination of which QUINE makes use in «On the Logic of Quantification», *Journal of Symbolic Logic 10,* (1945), pp. 1-12.

(2) See «Postulate sets and decision procedures for some systems of deontic logic», forthcoming in *Theoria.*

tactical point of view it is but a weakened version of Feys's system T of modal logic (von Wright's M) (3).

5.- *BD as a logic of 'better'*

In this section we shall concern ourselves with those wffs of BD that are not deontic, *i.e.* such wffs of BD as contain no occurrences of the connectives O, F, I or P. As was promised in the introduction, we are going to compare the present logic of 'better' with some systems proposed by Halldén for a similar relationship (4). It is to be kept in mind, however, that our respective systems are designed to capture different intuitive notions of betterness.

First, there are some trivial differences that can be passed over quickly.

(i) Certain expressions that are recognized as well formed by Halldén's two theories A and B are not so according to BD, *viz.*, (a) wffs of PC, and (b) expressions arising from a «mixture» of wffs of PC and wffs of BD, *e.g.* CBpqp.

(ii) Disregarding the groups just mentioned, we note that every non-deontic wff of BD is a wff of Halldén's theory A but not of his theory B. Theory A admits, besides, expressions where the connectives B and S are iterated, *e.g.* BBpqSqp. In the theory B, the connectives B and S are allowed only to connect propositional variables and their negations (single or iterated); thus, our BD-axiom P2 is ill-formed in theory B.

(iii) Halldén's systems use all PC-theorems as postulates, whereas BD has the rule R4 much to the same effect.

As for the rules of inference of theories A and B, let it be observed that both of them contain a rule of detachment, one of substitution, and one allowing for interchangeability of equivalents within the respective systems. We just note here that an analogous rule of interchangeability of equivalents is derivable in BD, which thus contains suitable counterparts to the principles of inference for A and B.

Let us then pass to consideration of the postulates of the systems A (5) and B (6).

(3) See *e.g.* B. Sobocinski, «Note on a modal system of Feys von Wright», *Journal of Computing Systems 1*, (1953), pp. 171-178.

(4) *On the Logic of «Better»*, Uppsala 1957.

(5) *Ibid.*, ch. ii, sect. 6-8.

(6) *Ibid.*, ch. iv, sect. 17.

A1. CBpqNBqp	B1. = A1
A2. CKBpqBqrBpr	B2 = A2
A3. Spp	B3 = A3
A4. CSpqSqp	B4 = A4
A5. CKSpqSqrSpr	B5 = A6
A6. CKBpqSqrBpr	B6. CBpqBNqNp
A7. EBpqBKpNqKqNp	B7. AABpqSpqBqp
A8. ESpqSKpNqKqNp	

Observe that none of these systems involves any definition like D1 in BD; thus, the connectives B and S are both taken as primitives.

The following result is perhaps worth stating.

If α is a wff both of B and BD, then α is a thesis of B only if α is a thesis of BD.

It is readily verified by the decision method of BD that the B-postulates B1—B7 are theses of BD. And we just noted above (though without giving the detailed proof) that we can derive in BD analogues of the rules of inference for B that apply to wffs of BD.

We may also note that the converse of the above result does not hold, *viz.*, that for any wff both of B and BD every thesis of BD is a thesis of B as well. By Halldén's infinite matrix M, which he proves to be characteristic of B, we verify that the BD-postulate P3 is no thesis of B (7). (Assign, say, the value T_3 to p and T_2 to q, and P3 gets the final value F_{-1} that is not designated according to Matrix M.) A weaker version of P3 is however provable in B, *viz.*,

P3′. CAAKBpNpSqNqKBpNpBNqqKSpNpBNqqBpq

Let it also be observed that P1 is derivable in B by simple substitution of Np for q in B1, and further, that suitable special cases of the BD-rules R1, R2, R4 and R5, applicable to wffs of B and BD, are derivable in B in an obvious way. Finally, the BD-postulate P2 is no wff of B, and the rule R3 cannot give rise to any thesis of B — since no thesis of PC consists just of a single variable or of negations, single or iterated, of single variables — so they need not concern us. Hence, it would hold that for any wff of B and BD every thesis of BD is a thesis of B, unless the converse of P3′ above were unprovable in B.

We then pass to consideration of Halldén's theory A. Let it first be noted that the postulates A7 and A8 are not theses of BD, which can be seen as follows. Assign the value $\frac{1}{2}$ both to p and q in A7 and A8, and consider in the case of KpNq the possibility $K\frac{1}{2}\frac{1}{2}=\frac{1}{2}$ and in that

(7) *Ibid.*, ch. iv, sect. 20, 21.

of KqNp the possibility $K\frac{1}{2}\frac{1}{2}=0$ (cf. instruction I2 above). This results in rows which assign 0 both to A7 and A8 and cannot be dropped in virtue of our rule RE. The following weaker formulae are however derivable in BD and verified by our method:

A7′. CBpqBKpNqKqNp
A8′. CSKpNqKqNpSpq

The remaining A-postulates A1—A6 are also theses of BD.

Certain wffs of both A and BD should be considered that are discussed by Halldén and are all rejected by him as theses of system A (8). He also proves that none of these formulae is derivable in A (9).

H1. CBpKqrKBpqBpr
H2. CKBpqBprBpKqr
H3. CBKpqrKBprBqr
H4. CKBprBqrBKpqr
H5. BApNpKpNp
H6. SApNpKpNp
H7. BKpNpApNp
H10 = B7

Of these formulae H1, H4, H6 and H7 are unprovable in BD as well. The following value-assignment falsifies H1: $p=\frac{1}{2}$, $q=\frac{1}{2}$, $r=\frac{1}{2}$, $Kqr=K\frac{1}{2}\frac{1}{2}=0$. As for H4, we have $p=\frac{1}{2}$, $q=\frac{1}{2}$, $r=0$, $Kpq=K\frac{1}{2}\frac{1}{2}=0$, falsifying that formula. On the other hand, we easily verify that H2, H3, H5 and H10 are theses of BD. We thus see that unlike theory A the system BD admits — in virtue of H2 and H3 — that the connective B is to a certain extent distributive with respect to conjunction. Some further formulae of a similar distributive character might deserve consideration in this connexion (10), *e.g.* the following:

H1a. CBpAqrABpqBpr
H1b. CBpAqrKBpqBpr
H2a. CABpqBprBpAqr
H2b. CKBpqBprBpAqr
H3a. CBApqrABprBqr
H3b. CBApqrKBprBqr
H4a. CABprBqrBApqr
H4b. CKBprBqrBApqr

(8) *Ibid.*, ch. iii, sect. 11, 12, 14.

(9) *Ibid.*, ch. iii, sect. 16.

(10) See H. N. Castañeda's review of Halldén, *Philosophy and Phenomenological Research 19*, (1959), p. 266.

Let it be noted that the pairs H1a-b and H4a-b are theorems of BD whereas the remaining pairs are not.

H5 is of course a thesis of BD in virtue of the rule R3. A noteworthy difference between the systems A and BD is indeed that this rule R3 is not derivable in A. If it where, H5 would be a thesis of A. But Halldén has strictly proved that neither H5, nor any wff of A of the type Bαβ, is a theorem of A ([9]).

As regards H10, we mention that the unprovability of this formula in theory A constitutes one of the chief differences of this system from the calculi B and BD.

Let me conclude this comparison by conjecturing that none of the BD-postulates P2 or P3 is deducible in theory A. I have not proved this, however.

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