DECISION ALGORITHMS FOR SOME FUNCTIONAL CALCULI WITH MODALITY

M. J. POLIFERNO

1. *Introduction*. 1.1. The decision algorithms described in this paper (¹) constitute adaptations of methods due to Quine ([6]) (²) and Anderson ([1]) to systems including both quantification and modality. In particular, a monadic, first order functional calculus MQ based on von Wright's calculus M (von Wright [9], p. 85) is formulated and a decision algorithm constructed for it. With the help of the decision algorithm, MQ is shown to be consistent and non-trivial (in various senses; see 5.3 and 5.4) and an interpretation of MQ is discussed in section 6. In section 5 it is shown that suitable modifications of the algorithm for MQ yield algorithms for S4Q (defined in 5.6) and S5Q (defined in 5.12) and for a subclass of the well formed formulas of the Barcan system $S4Q^1$ (defined in 5.10).

1.2. The decision algorithms described in Anderson [1] and Quine [6] are special cases of the ones presented here. Although the techniques developed in Anderson [1] are used here, the results in that paper are not presupposed. On the other hand, results in Quine [6] are presupposed, though the techniques developed there are not used (directly).

2. *The calculus MQ*. 2.1. The alphabet of MQ consists of $\aleph_0$ propositional variables, $\aleph_0$ functional variables, $\aleph_0$ individual variables, four operators, and two parentheses. p, q, r, p_1, q_1, r_1, p_2, ... are metavariables ranging over propositional variables; f, g, h, f_1, g_1, h_1, f_2, ... are metavariables ranging over functional variables; and x, y, z, x_1, y_1, z_1, x_2, ... are metavariables ranging over individual variables. $\sim$, . , $\Diamond$, and E are metaconstants denoting the four operators; and (and) are metaconstants denoting the parentheses.

2.2. A formula of MQ is any finite sequence of members of the alphabet. The symbols A, B, C, A_1, B_1, C_1, A_2, ... are metavariables ranging over formulas.

(¹) This paper is a condensation of a dissertation presented for the degree of Doctor of Philosophy at Yale University. I am indebted to Professor Alan Ross Anderson for helpful criticisms and suggestions.

(²) Numbers in brackets refer to the bibliography.

2.3. The class of well formed formulas (wffs) of MQ is recursively characterized in the usual way.

2.4. Bound and free occurrences of individual variables in wffs are defined in the usual way, as are $\vee$, $\supset$, $\equiv$, $\Box$, and (x).

2.5. Axioms of MQ. If A, B, and C are wffs, then each of the following is an axiom of MQ.

A1. $A \supset A \,.\, A$

A2. $A \,.\, B \supset A$

A3. $(A \supset B) \supset (\sim (B \,.\, C) \supset \sim (C \,.\, A))$

A4. $A \supset \Diamond A$

A5. $\Diamond (A \vee B) \supset \Diamond A \vee \Diamond B$

A6. $\sim \Diamond \sim (A \supset B) \supset (\Diamond A \supset \Diamond B)$

A7. $B \supset (Ex)\, A$, where B results from the substitution of y for all free occurrences of x in A, provided no free occurrence of x in A is in a wf'd part of A of the form (Ey)C.

A8. $(Ex)\, A \supset A$, where x is not free in A.

A9. $\sim (Ex) \sim (A \supset B) \supset ((Ex)\, A \supset (Ex)\, B)$

A10. $(Ex) \sim \Diamond A \supset \sim \Diamond (Ex)\, A$

2.6. Rules of inference of MQ. (Write «A e MQ» for «A is a theorem of MQ.»)

R0. If A is an axiom of MQ, then A e MQ.

R1. (Modus ponens) If A e MQ and $A \supset B$ e MQ, then B e MQ.

R2. (Necessitation) If A e MQ, then $\sim \Diamond \sim A$ e MQ.

R3. (Generalization) If A e MQ, then $\sim (Ex) \sim A$ e MQ.

2.7. A1, A2, and A3 are Rosser's axioms for the propositional calculus ([8], pp. 55-56). It is clear that the propositional calculus P and the monadic, first order functional calculus PQ are subsystems of MQ. Note also that MQ is an extension of von Wright's calculus M to include quantification. (See Anderson's formulation of M ([1], p. 212).)

2.8. The rule of intersubstitutability of material equivalents holds for MQ. (The proof given in Church [3], pp. 189-190, can easily be extended to apply to MQ because of R2 and A6.)

2.9. The following are theorems of MQ, and the proofs are straightforward:

(1) $\sim \Diamond (Ex)\, A \equiv (Ex) \sim \Diamond A$

(2) $\Diamond (Ex)\, A \equiv (Ex) \Diamond A$

(3) (Ex) $\sim \Diamond$ A $\equiv \sim$ (Ex) $\Diamond$ A
(4) (Ex) $\Diamond$ A $\equiv \sim$ (Ex) $\sim \Diamond$ A
(5) (Ex) $\sim \Diamond$ A . (Ex) B $\equiv$ (Ex) ($\sim \Diamond$ A . B)
(6) (Ex) $\Diamond$ A . (Ex) B $\equiv$ (Ex) ($\Diamond$ A . B)

2.10. Definitions. *Open* and *closed* wffs, and the *closure* $\overline{\mathrm{A}}$ of a wff A are defined in the usual way. A wff of the form (Ex) A is called a *quantification*, and a wff of the form $\Diamond$A is called an *atom*.

3. The calculus $\overline{\mathrm{MQ}}$. 3.1. The decision algorithm for MQ will be obtained via a system $\overline{\mathrm{MQ}}$ which is like MQ except that all its theorems are closed.

The alphabet of $\overline{\mathrm{MQ}}$ is that of MQ, and the wffs are those of MQ. The axioms of $\overline{\mathrm{MQ}}$ are the closures of the axioms of MQ (label them $\overline{\mathrm{A1}}$ - $\overline{\mathrm{A10}}$) and $\overline{\mathrm{A11}}$: $\overline{\text{(Ex)} \Diamond \text{A} \supset \Diamond \text{(Ex) A}}$. The rules of inference are modus ponens ($\overline{\mathrm{R1}}$), necessitation ($\overline{\mathrm{R2}}$), and $\overline{\mathrm{R0}}$. $\overline{\mathrm{MQ}}$ is called *the closure of MQ*.

3.2. Metatheorem.

(a) If A e MQ, then $\overline{\mathrm{A}}$ e $\overline{\mathrm{MQ}}$.

(b) If $\overline{\mathrm{C}}$ e $\overline{\mathrm{MQ}}$, then C e MQ.

3.3. The proof is straightforward. (The contrapositive of $\overline{\mathrm{A11}}$ is used in the proof of (a), in showing that if A is a consequence of B (e MQ) by necessitation, where $\overline{\mathrm{B}}$ e $\overline{\mathrm{MQ}}$, then $\overline{\mathrm{A}}$ e $\overline{\mathrm{MQ}}$.)

3.4. It is a corollary of 3.2 that any decision algorithm for $\overline{\mathrm{MQ}}$ yields automatically an algorithm for MQ, since A e MQ iff $\overline{\mathrm{A}}$ e $\overline{\mathrm{MQ}}$.

3.5. Definition.

(a) If A is a functional variable, then Ax is *completely open*.

(b) If A and B are completely open, so are $\sim$A, $\Diamond$A, and A . B.

3.6. Definitions. Let A be a wff. A is *uniform* iff at most one individual variable occurs in A, *null uniform* iff no individual variable occurs in A, and *uniform in x* iff A is uniform and x occurs in A. A is a *basic quantification* iff it consists of (Ex) followed by a completely open wff uniform in x and $\Diamond$-free, and a *basic wff* iff every wf'd part of A of the form (Ex) B is a basic quantification. A is *normal* iff A is closed, basic, and uniform.

3.7. Note that the definition of *uniform* in 3.6 differs from that in Quine [6] in that in the latter *uniform* is defined for completely open wffs only. The definition of *basic quantification* in 3.6 differs from that in [6] in the $\Diamond$-free requirement.

3.8. Metatheorem. Let A be a closed wff of $\overline{MQ}$ in primitive notation. Then there exists an effective procedure for obtaining a normal wff B such that $B \equiv A$ e $\overline{MQ}$.

3.9. Proof. Apply the following replacement rules as often as possible to A. Replace a wf'd part of the form:

Q1.	$\sim \sim C$	by	C
Q2.	$(Ex) \sim (C_1 . C_2)$	by	$\sim (\sim (Ex) \sim C_1 . \sim (Ex) \sim C_2)$
Q3.	$(Ex) (\sim (C_1 . C_2) . D)$	by	$\sim (\sim (Ex) (\sim C_1 . D) . \sim (Ex) (\sim C_2 . D))$
Q4.	$(Ex) (D . \sim (C_1 . C_2))$	by	$\sim (\sim (Ex) (D . \sim C_1) . \sim (Ex) (D . \sim C_2))$
Q5.	$(Ex) (D_1 . \sim (C_1 . C_2) . D_2)$	by	$\sim (\sim (Ex) (D_1 . \sim C_1 . D_2) . \sim (Ex) (D_1 . \sim C_2 . D_2))$
Q6.	$(Ex) C$	by	C, if x is not free in C
Q7.	$(Ex) (C . D)$	by	$C . (Ex) D$, if x is not free in C
Q8.	$(Ex) (D . C)$	by	$(Ex) D . C$, if x is not free in C
Q9.	$(Ex) (D_1 . C . D_2)$	by	$(Ex) (D_1 . D_2) . C$, if x is not free in C
Q10.	$(Ex) \sim \Diamond C$	by	$\sim \Diamond (Ex) C$
Q11.	$(Ex) (\sim \Diamond C . D)$	by	$(Ex) \sim \Diamond C . (Ex) D$
Q12.	$(Ex) (D . \sim \Diamond C)$	by	$(Ex) D . (Ex) \sim \Diamond C$
Q13.	$(Ex) (D_1 . \sim \Diamond C . D_2)$	by	$(Ex) (D_1 . D_2) . (Ex) \sim \Diamond C$
Q14.	$(Ex) \Diamond C$	by	$\Diamond (Ex) C$
Q15.	$(Ex) (\Diamond C . D)$	by	$(Ex) \Diamond C . (Ex) D$
Q16.	$(Ex) (D . \Diamond C)$	by	$(Ex) D . (Ex) \Diamond C$
Q17.	$(Ex) (D_1 . \Diamond C . D_2)$	by	$(Ex) (D_1 . D_2) . (Ex) \Diamond C$

Note that the procedure does not go on indefinitely. For instance, each of the rules Q2-Q5 removes a part of the form $\sim (C_1 . C_2)$ from the scope of an occurrence of E, and eventually all such parts will be removed. Similarly, the rules Q10-Q17 remove parts of the form $\Diamond$ C from scopes of occurrences of E, etc.

Call the result A^1. By 2.7, 2.8, 2.9, and 3.2 $A^1 \equiv A$ e $\overline{MQ}$. Moreover, A^1 is basic because if $(Ex)A_1$ is any wf'd part of A^1, then A_1 is completely open and uniform in x by Q6-Q9 and A_1 is $\Diamond$-free by Q10-Q17.

Q18. Let y be an individual variable that does not occur in A^1. For each individual variable x that occurs in A^1 replace x by y in all its occurrences in A^1.

Call the resulting formula B. Then $A^1 \equiv B$ e $\overline{MQ}$ because all individual variables in A^1 are bound to quantifiers with non-overlapping scopes. Hence, $A \equiv B$ e $\overline{MQ}$. Moreover, B is normal.

3.10. Definition. Let A be a closed wff of $\overline{MQ}$. Then A* is a *normal form of A* iff A* can be derived from A by the method of 3.9.

3.11. Definitions. Propositional variables, closed atoms, and basic quantifications are called *constituents*. (Note that every constituent is a closed wff.) If A is a functional variable and x an individual variable, then Ax is called a *component*.

3.12. Definitions. Let A be a completely open wff uniform in x and ◇-free. Let $A_1, \ldots, A_k$ be the k distinct components of A. Let $p_1, \ldots, p_k$ be k distinct propositional variables. Substitute p_i for A_i throughout A. The result $A^{(P)}$ of this substitution is called a *medadic analogue of A*, and A is a *medadic tautology* iff $A^{(P)}$ e P.

3.13. If A is a medadic tautology, then A e MQ ($\overline{A}$ e $\overline{MQ}$). Moreover, there is an effective procedure for determining whether or not $A^{(P)}$ e P.

3.14. Definitions. Let A be a normal wff. Then A is a truth function of one or more of its k distinct constituents $A_1, \ldots, A_k$, and a *truth table* for A ($\mathcal{T}$(A)) can be constructed in the usual way. If the value of A in Row (i) (the ith row) of $\mathcal{T}$ (A) is T (F), then Row (i) will be called a T (an F) - row of $\mathcal{T}$(A). Note that $\mathcal{T}$(A) is also a table for $A_1, \ldots, A_k$.

3.15. Definition.

(a) If A is a propositional variable and B a functional variable, then A and Bx are *wffs of degree 0*.

(b) If A and B are wffs of degrees m and n, respectively, then ~ A and (Ex) A are *wffs of degree m*, ◇ A is a *wff of degree m+1*, and A.B is a *wff of degree max* $\{m, n\}$.

(This definition is adapted from Anderson [1], p. 203.)

Note that every wff is a wff of some degree, and conversely.

4. *A decision algorithm for* $\overline{MQ}$. 4.1. Definition. A is a *tautology of degree n of* $\overline{MQ}$ (*n-tautology of* $\overline{MQ}$) iff

(1) A is a normal wff of degree n of $\overline{MQ}$,

and (2) every F-row of $\mathcal{T}$(A) satisfies at least one of the following six conditions:

T1. Some constituent (of A) of the form ◇ B has F while B has T.

T2. Some constituent of the form $\diamond$ B has T, some constituents of the forms $\diamond D_1, \ldots, \diamond D_h$ ($h \geqslant 1$) all have F, while $B \supset D_1 v \ldots v D_h$ is a tautology of degree $n_1 < n$.

T3. Some constituent of the form $\diamond$ B has T while $\sim$ B is a tautology of degree $n_1 < n$.

T4. Some constituents of the forms (Ex) $D_1, \ldots,$ (Ex) D_h ($h \geqslant 1$) all have F while $D_1 v \ldots v D_h$ is a medadic tautology.

T5. Some constituent of the form (Ex) B has T, some constituents of the forms (Ex) $D_1, \ldots,$ (Ex) D_h ($h \geqslant 1$) all have F, while $B \supset D_1 v \ldots v D_h$ is a medadic tautology.

T6. Some constituent of the form (Ex) B has T while $\sim$ B is a medadic tautology.

(This definition is adapted from Anderson [1], p. 212, and Quine [6], p. 6.)

4.2. Metatheorem. Every tautology (i.e., n-tautology for some n) of $\overline{MQ}$ is a theorem of $\overline{MQ}$.

4.3. Proof. By 3.2(a) it is sufficient to show that every tautology A of $\overline{MQ}$ is a theorem of MQ. This is done by mathematical induction on the degree of A.

4.4. Observe that if $\mathcal{T}$(A) has no F-rows, then A e MQ. Proof: Let $A_1, \ldots, A_k$ be those k distinct constituents of A that are not of the form $\diamond$ B, and let $B_1, \ldots, B_l$ be those l distinct constituents of A that are of the form $\diamond$ B. Let $p_1, \ldots, p_k, q_1, \ldots, q_l$ be k+l distinct propositional variables not occurring in A. Substitute l_i for A_i throughout A. Then substitute q_j for B_j throughout the result A^1, substituting first for those constituents of highest degree, then for those of next highest degree, etc. Call the result A^{11}. A^{11} is a wff of P. Since $\mathcal{T}_{\overline{MQ}}$(A) (table for A in $\overline{MQ}$) is also a table for A^{11} in P, $\mathcal{T}_P(A^{11})$ has no F-rows. Hence A^{11} e P and A e MQ.

4.5. Let A be a tautology of degree 0. Then each F-row of $\mathcal{T}$(A) satisfies T4, T5, or T6. Hence, by Quine's decision algorithm for PQ ([6]), A e PQ. Thus A e MQ. (Alternatively, a proof similar to that of 3.5 in Anderson [1] can be given, and appeal to Quine's result avoided.)

4.6. Let A be a tautology of degree $n \geqslant 1$. Suppose that for every $n_1 < n$ if C is a tautology of degree n_1, then C e MQ. To show A e MQ. The proof is like that of 3.5 in Anderson [1], which can easily be adapted to M and extended to take into account T4-T6 of 4.1 of the present paper.

4.7. Definition. Let A be a wff of $\overline{MQ}$. Let $A_1, \ldots, A_k$ be those distinct wf'd parts of A of the form $\diamond$ C. Let $p_1, \ldots, p_k$ be k distinct propositional variables not occurring in A. Substitute p_i for A_i throughout A, substituting first for those wf'd parts of highest degree, etc. The result $\mathring{A}$ of this substitution is called *an associate of A*. (Note that $\mathring{A}$ is a wff of $\overline{PQ}$.)

4.8. Lemma. If $A \in \overline{MQ}$, and A has as an associate a theorem of $\overline{PQ}$, then A has a normal form A^* which has as an associate a theorem of $\overline{PQ}$ and which is, hence, a tautology of $\overline{MQ}$.

4.9. Proof. It will be convenient to consider the second part of the assertion first. Suppose A^* has as an associate a theorem $\mathring{A}^*$ of $\overline{PQ}$. Then by Quine's decision algorithm for PQ ([6]), $\mathring{A}^*$ is a tautology of $\overline{PQ}$; that is, every F-row of $\mathcal{T}_{\overline{PQ}}(\mathring{A}^*)$ satisfies T4, T5, or T6. Therefore every F-row of $\mathcal{T}_{\overline{MQ}}(A^*)$ satisfies T4, T5, or T6. Thus A^* is a tautology of $\overline{MQ}$.

4.10. Consider now the first part of the assertion. If A is $\diamond$-free, the result is obvious. Assume A contains at least one occurrence of $\diamond$. Let $A_1, \ldots, A_k$ ($k \geqslant 1$) be the k distinct wf'd parts of A of the form $\diamond$ C to which propositional variables are assigned per 4.7. (Then no A_i occurs solely as a part of an A_j.) Apply Q1-Q9 and Q18 to A as often as possible. Call the result A^1. A^1 has as an associate a theorem of $\overline{PQ}$. Suppose A^1 is not normal. Apply to A^1 any one of Q10-Q17 which is appropriate, and (λ): do this in such a way that if Q_i ($10 \leqslant i \leqslant 17$) is applied to a part B it is so applied to every occurrence of B in A. Call the result A^{11}. Consideration of the following eight cases will show that A^{11} has as an associate a theorem of $\overline{PQ}$.

4.11. Case 1. Suppose Q10 is applied to B in A^1 to get A^{11}. Then B is of the form $(Ex) \sim \diamond B_1$ and A^1 is of the form $D_1(Ex) \sim \diamond B_1 D_2$, where D_1 and D_2 are formulas (finite sequences of primitive symbols) of $\overline{MQ}$. Assume (without loss of generality, because of (λ)) that D_1 and D_2 have no occurrence of B.

(1) $D_1(Ex) \sim \diamond B_1 D_2$ has as an associate a theorem of $\overline{PQ}$.
Let p be a propositional variable not occurring in (1). Then

(2) $D_1(Ex) \sim p D_2$ has as an associate a theorem of $\overline{PQ}$. Therefore

(3) $D_1 \sim p D_2$ has as an associate a theorem of $\overline{PQ}$. Therefore

(4) $D_1 \sim \diamond (Ex) B_1 D_2$ has as an associate a theorem of $\overline{PQ}$. Therefore

(5) A^{11} has as an associate a theorem of $\overline{PQ}$.

4.12. Case 2. Suppose Q11 is applied to B in A^1 to get A^{11}. Then B is of the form $(Ex)(\sim\Diamond B_1.B_2)$ and A^1 is of the form $D_1(Ex)(\sim\Diamond B_1.B_2)D_2$.

(1) $D_1(Ex)(\sim\Diamond B_1.B_2)D_2$ has as an associate a theorem of $\overline{PQ}$. Let p be a propositional variable not occurring in (1). Then

(2) $D_1(Ex)(\sim p.B_2)D_2$ has as an associate a theorem of $\overline{PQ}$. Therefore

(3) $D_1\sim p.(Ex)B_2D_2$ has as an associate a theorem of $\overline{PQ}$. Therefore

(4) $D_1\sim\sim(Ex)\sim\Diamond B_1.(Ex)B_2D_2$ has as an associate a theorem of $\overline{PQ}$. Therefore

(5) $D_1(Ex)\sim\Diamond B_1.(Ex)B_2D_2$ has as an associate a theorem of $\overline{PQ}$. Therefore

(6) A^{11} has as an associate a theorem of $\overline{PQ}$.

4.13. The remaining cases are similar.

4.14 If A^{11} is not normal, the argument can be repeated, yielding a wff A^{111} which has as an associate a theorem of $\overline{PQ}$. Eventually a normal form A^* will appear which has as an associate a theorem of $\overline{PQ}$.

This completes the proof of the lemma.

4.15. Metatheorem. If A is an axiom of $\overline{MQ}$, then A has a normal form A^* which is a tautology of $\overline{MQ}$.

4.16. Proof. $\overline{A1}$, $\overline{A2}$, $\overline{A3}$, $\overline{A7}$, $\overline{A8}$, and $\overline{A9}$ are associates of theorems of $\overline{PQ}$, so by 4.8 they have normal forms which are tautologies of $\overline{MQ}$.

4.17. Consider $\overline{A6}$: $\overline{\sim(\sim\Diamond(A.\sim B).\Diamond A.\sim\Diamond B)}$. Let D_n be $\sim(Ex_n)\ldots(Ex_1)(\sim\Diamond(A.\sim B).\Diamond A.\sim\Diamond B)$, where $\sim\Diamond(A.\sim B).\Diamond A.\sim\Diamond B$ contains x_1, …, x_n free but no other free individual variables ($n\geqslant 0$). Any instance of $\overline{A6}$ is of the form D_n for some $n\geqslant 0$. Let D^1_n be $\sim(\sim\Diamond(Ex_n)\ldots(Ex_1)(A.\sim B_1).\Diamond(Ex_n)\ldots(Ex_1)A.\sim\Diamond(Ex_n)\ldots(Ex_1)B)$. Then $D_n \equiv D^1_n$ e $\overline{MQ}$ (by successive applications of some of Q10-Q17). Let λ_1 be a normal form for $(Ex_n)\ldots(Ex_1)(A.\sim B)$, λ_2 for $(Ex_n)\ldots(Ex_1)A$, and λ_3 for $(Ex_n)\ldots(Ex_1)B$. Then $\sim(\sim\Diamond\lambda_1.\Diamond\lambda_2.\sim\Diamond\lambda_3)$ is a normal form for D_n. $\overline{A \supset B \vee A.\sim B}$ e $\overline{MQ}$. Therefore $(Ex_n)\ldots(Ex_1)A \supset (Ex_n)\ldots(Ex_1)B \vee (Ex_n)\ldots(Ex_1)(A.\sim B)$ e $\overline{MQ}$. Therefore $\lambda_2 \supset \lambda_3 \vee \lambda_1$ e $\overline{MQ}$ and has as an associate a theorem of $\overline{PQ}$. Hence $\lambda_2 \supset \lambda_3 \vee \lambda_1$ is a tautology of $\overline{MQ}$. Therefore $\sim(\sim\Diamond\lambda_1.\Diamond\lambda_2.\sim\Diamond\lambda_3)$ is too (Every F-row of its table satisfies T2).

4.18. The arguments for the other axiom schemata are similar.

4.19. Definition. Row(i) of $\mathcal{T}$(A) is *satisfactory* if it satisfies at least one of T1-T6 and *unsatisfactory* otherwise.

Thus A is a tautology of $\overline{\text{MQ}}$ iff every F-row of $\mathcal{T}$(A) is satisfactory.

4.20. Lemma. If $\mathcal{T}$(A) has an unsatisfactory F-row, say Row(i), then $\mathcal{T}$(~(B.~A)) has a row, say Row(j), in which A has F and which is also unsatisfactory.

4.21. Proof. (The proof given here is adapted from that of the same lemma in Anderson [1].) If every constituent of B is also a constituent of A, then $\mathcal{T}$(A) is the same as $\mathcal{T}$(~(B.~A)); hence j=i.

4.22. Suppose B has constituents $C_1, \ldots, C_k$ ($k \geqslant 1$) which are not also constituents of A, and suppose they are arranged in order of increasing degree, so that

$$\deg(C_1) \leqslant \deg(C_2) \leqslant \ldots \leqslant \deg(C_k)$$

Consider the following sequence of normal formulas:

$$G_1 = A$$
$$G_{h+1} = G_h \,.\sim (C_h \,.\sim C_h),\ h = 1, 2, \ldots, k$$

G_{h+1} has exactly one more constituent, namely C_h, than G_h, so that $\mathcal{T}(G_{h+1})$ has exactly one more column than and twice as many rows as $\mathcal{T}(G_h)$. Moreover, $\mathcal{T}(G_{k+1}) = \mathcal{T}$(~(B.~A)). Hence it will be sufficient to show that if $\mathcal{T}(G_h)$ has an unsatisfactory F-row, then $\mathcal{T}(G_{h+1})$ has an unsatisfactory row in which G_h has F. The contrapositive is established as follows.

4.23. Sublemma. If C.~(D.~D) is a tautology of degree n which has exactly one more constituent, namely D, than C, then C is a tautology.

4.24. Proof. Mathematical induction on n.

Suppose n = 0. If D is a propositional variable, then the sublemma (for n = 0) is immediate. Suppose D is of the form (Ex)N. Assume C.~((Ex)N.~(Ex)N) is a tautology of degree 0 and C is not a tautology. Then $\mathcal{T}$(C) has an unsatisfactory F-row, say Row(i). Let Row(i) of $\mathcal{T}$(C) assign values $V_1, \ldots, V_k$ (each of which is T or F) to the k constituents $C_1, \ldots, C_k$ of C. Then in $\mathcal{T}$(C.~((Ex)N.~(Ex)N)) there will be a row (Row(i_T)) which assigns the same values to the constituents $C_1, \ldots, C_k$ of C but assigns T to (Ex)N; similarly, there is a Row(i_F) of $\mathcal{T}$(C.~((Ex)N.~(Ex)N)) which assigns $V_1, \ldots, V_k$ to the constituents $C_1, \ldots, C_k$ but assigns F to (Ex)N. Thus $\mathscr{T}$(C) looks in part as follows:

	C_1	...	C_k
Row(i):	V_1	...	V_k

and $\mathscr{T}$(C.~((Ex)N.~(Ex)N)) looks in part as follows:

	C_1	...	C_k	(Ex)N
Row(i_T):	V_1	...	V_k	T
Row(i_F):	V_1	...	V_k	F

Row(i_T) and Row(i_F) are both satisfactory. It will be shown that, consequently, Row(i) is satisfactory. There are three cases to consider, according as Row(i_T) satisfies T4, T5, or T6.

4.25. Case 1. Row(i_T) satisfies T4. Then in Row(i_T) some constituents (Ex)$D_1, \ldots,$ (Ex)D_h ($h \geqslant 1$) all have F while $D_1 v \ldots v D_h$ is a medadic tautology. Then none of (Ex)$D_1, \ldots,$ (Ex)D_h is (Ex)N because (Ex)N has T in Row(i_T). Hence (Ex)$D_1, \ldots,$ (Ex)D_h are constituents of C, so that Row(i) satisfies T4 and is, therefore, satisfactory.

4.26. Case 2. Row(i_T) satisfies T5. Then some constituent (Ex)B has T in Row(i_T), some constituents (Ex)$D_1, \ldots,$ (Ex)D_h ($h \geqslant 1$) all have F in Row(i_T), while $B \supset D_1 v \ldots v D_h$ is a medadic tautology. If (Ex)B is not (Ex)N, then it is clear that Row(i) satisfies T5. Suppose (Ex)B is (Ex)N. Then Row(i_T) looks in part as follows (where the dashes represent a succession of F's):

	(Ex)D_1	...	(Ex)D_h	(Ex)N
Row(i_T):	F	- - -	F	T

where

(α) $$N \supset D_1 v \ldots v D_h$$

is a medadic tautology. There are now three subcases, according as Row(i_F) satisfies T4, T5, or T6.

Subcase 1. Row(i_F) satisfies T4. Then Row(i_F) looks in part as follows:

	(Ex)$D^1{}_1$	...	(Ex)$D^1{}_a$
Row(i_F):	F	- - -	F

where

$$D^1{}_1 v \ldots v D^1{}_a$$

is a medadic tautology or

$$D^1_1 \vee \ldots \vee D^1_a \vee N$$

is a medadic tautology. If $D^1_1 \vee \ldots \vee D^1_a$ is a medadic tautology, then it is clear that Row(i) is satisfactory. Suppose $D^1_1 \vee \ldots \vee D^1_a \vee N$ is a medadic tautology. Then, since Row(i_T) and Row(i_F) differ only in the value assigned to (Ex)N, Row(i) must look in part as follows:

	$(Ex)D^1_1$	...	$(Ex)D^1_a$	$(Ex)D_1$	...	$(Ex)D_h$
Row(i):	F	---	F	F	---	F

Consider the formula

$$D^1_1 \vee \ldots \vee D^1_a \vee D_1 \vee \ldots \vee D_h$$

Since

(α) $$N \supset D_1 \vee \ldots \vee D_h$$

is a medadic tautology and

$$D^1_1 \vee \ldots \vee D^1_a \vee N$$

is a medadic tautology, it follows that

$$D^1_1 \vee \ldots \vee D^1_a \vee D_1 \vee \ldots \vee D_h$$

is a medadic tautology. Hence Row(i) satisfies T4 and is, therefore, satisfactory.

The remaining subcases are similar.

4.27. Case 3. The argument is similar to that of 4.26.

4.28. Suppose $n \geqslant 1$. Assume that for every $n_1 < n$ if $C.\sim(D.\sim D)$ is a tautology of degree n_1 which has exactly one more constituent, namely D, than C, then C is a tautology. If D is a propositional variable, then the sublemma (for n) is immediate. This leaves two cases, namely, D of the form (Ex)N and D of the form $\Diamond$N. Suppose D is of the form (Ex)N. Assume $C.\sim((Ex)N.\sim(Ex)N)$ is a tautology of degree n and C is not a tautology. Then $\mathcal{T}(C)$ has an unsatisfactory F-row, say Row(i). Let Row(i) of $\mathcal{T}(C)$ assign values $V_1, \ldots, V_k$ to the k constituents $C_1, \ldots, C_k$ of C. Then consider Row(i_T) and Row(i_F) of $\mathcal{T}(C.\sim((Ex)N.\sim(Ex)N))$ as before (4.24). Row(i_T) and Row(i_F) are both satisfactory. It is to be shown that, consequently, Row(i) is satisfactory. There are six cases to consider, according as Row(i_T) satisfies T1-T6. It is clear, however, that if Row(i_T) satisfies T1, T2, or T3, then Row(i) satisfies T1, T2, or T3, respectively; while if Row(i_T) satisfies T4, T5, or T6, then by the same arguments as for $n = 0$ Row(i) is satisfactory.

4.29. Suppose D is of the form $\Diamond N$. Assume $C.\sim(\Diamond N.\sim\Diamond N)$ is a tautology of degree n and C is not a tautology. Then $\mathscr{T}(C)$ has an unsatisfactory F-row, say Row(i). Let Row(i) of $\mathscr{T}(C)$ assign values $V_1, \ldots, V_k$ to the k constituents $C_1, \ldots, C_k$ of C. Then consider Row(i_T) and Row(i_F) of $\mathscr{T}(C.\sim(\Diamond N.\sim\Diamond N))$ as before (4.24). They are both satisfactory. It is to be shown that, consequently, Row(i) is satisfactory. There are six cases to consider, according as Row(i) satisfies T1-T6. It is clear, however, that if Row(i_T) satisfies T4, T5, or T6, then Row(i) satisfies T4, T5, or T6, respectively. Thus there are only three cases to examine in detail. These are like those in the proof of 3.19 of Anderson [1], which can easily be adapted to M and extended to take into account T4-T6.

4.30. Metatheorem. If A is a theorem of $\overline{MQ}$, then A has a normal form A* which is a tautology of $\overline{MQ}$.

4.31. Proof. If A is an axiom of $\overline{MQ}$, then 4.15 yields the result.

4.32. Suppose A is a consequence of B ($\in \overline{MQ}$) by $\overline{R2}$, so that A is $\sim\Diamond\sim B$, where B has a normal form B* which is a tautology of degree n. To show that A has a normal form A* which is a tautology of degree n+1. $A^* = \sim\Diamond\sim B^*$ is a normal form of A. Let the constituents of B* be $B_1, \ldots, B_k$ $(k \geqslant 1)$. Then the constituents of A* are $B_1, \ldots, B_k, \Diamond\sim B^*$. Then if Row($i_F$) is an F-row of $\mathscr{T}(A^*)$ it must assign T to $\Diamond\sim B^*$. Then Row(i_F) satisfies T3.

4.33. Suppose A is a consequence of B ($\in \overline{MQ}$) and $B \supset A$ ($\in \overline{MQ}$) by $\overline{R1}$, where B and $B \supset A$ have normal forms B* and $(B \supset A)^*$ which are tautologies. To show A has a normal form A* which is a tautology. $(B \supset A)^* = (\sim(B.\sim A))^* = \sim(B^*.\sim A^*) = B^* \supset A^*$. B* and $B^* \supset A^*$ are tautologies. To show that A* is a tautology. (The argument uses 4.20 and is like that in Anderson [1], p. 208.) Suppose B* and $\sim(B^*.\sim A^*)$ are tautologies but A* is not. Then there is an unsatisfactory F-row in $\mathscr{T}(A^*)$, say Row(i). By 4.20 $\mathscr{T}(\sim(B^*.\sim A^*))$ has an unsatisfactory row in which A* has F, say Row(j). Consider the value of B* in Row(j). If B* has F in Row(j), then $\mathscr{T}(B^*)$ has an unsatisfactory F-row. This contradicts 4.20. If B* has T in Row(j), then, since A* has F in Row(j), $\sim(B^*.\sim A^*)$ has F in Row(j), so that $\mathscr{T}(\sim(B^*.\sim A^*))$ has an unsatisfactory F-row. This yields a contradiction. Hence A* is a tautology.

This completes the proof of 4.30.

4.34. Metatheorem. If A has a normal form A* which is a tautology, then every normal form of A is a tautology.

4.35. Proof. Suppose A has a normal form A* which is a tautology. Then A is a theorem. Let A^*_1 be a normal form of A other than A*. Then A^*_1 is a theorem. Therefore A^*_1 has a normal form A^{**}_1 which is a tautology. But A^*_1 is already normal. If A^{**}_1 is A^*_1, 4.34 is proved. Suppose A^{**}_1 is not A^*_1. Then A^{**}_1 must be obtained from A^*_1 by some of Q1-Q18 (because of the definition of normal form (3.10)). But Q10-Q17 are certainly not used in getting A^{**}_1 from A^*_1 because A^*_1 is already normal, and Q1-Q9, Q18 preserve tautologyhood, by Quine's decision algorithm for PQ ([6]).

4.36. Metatheorem. Let C be a closed wff of $\overline{MQ}$. Let C* be any normal form of C. Then C e $\overline{MQ}$ iff C* is a tautology of $\overline{MQ}$.

4.37. Proof. If C* is a tautology of $\overline{MQ}$, then by 4.2 C* e $\overline{MQ}$ and hence C e $\overline{MQ}$. If C e $\overline{MQ}$, then by 4.30 and 4.34 C* is a tautology of $\overline{MQ}$.

5. *Further results.* 5.1. MQ is consistent.

5.2. Proof. By the decision algorithm it is evident that for no p, ·p e MQ.

5.3. Note that for no p, $\Diamond p \supset p$ e MQ and $\Diamond p \supset \Box p$ e MQ, and it is not the case that $(Ex)fx \supset (x)fx$ e MQ. Note also that, although $(x)\Box fx \equiv \Box(x)fx$ e MQ and $(Ex)\Box fx \equiv \Box(x)fx$ e MQ, it is not the case that $\Box(Ex)fx \supset (Ex)\Box fx$ e MQ.

5.4. Metatheorem. In MQ, $\Diamond$ cannot be defined in terms of the other primitive symbols.

5.5. Proof. Suppose it is possible to define $\Diamond$. Then there is a formula D containing no occurrence of $\Diamond$ such that $\Diamond p \equiv D$ e MQ. Let D* be a normal form of D. Then $\Diamond p \equiv D^*$ is a tautology of $\overline{MQ}$. Let $D_1, \ldots, D_k$ be the k distinct constituents of D*. Each D_i $(1 \leqslant i \leqslant k)$ is either a propositional variable or a quantification. $\mathcal{T}(\Diamond p \equiv D^*)$ has 4×2^k F-rows, 4 rows for each of the 2^k sets of values $V^{(i)}_1, \ldots, V^{(i)}_k$ for $D_1, \ldots, D_k$. Call these rows Row (F_{11}), Row (F_{12}), Row (F_{13}), Row (F_{14}), Row (F_{21}), ..., Row (F_{2^k4}). Consider Row (F_{i1}), ..., Row (F_{i4}) $(1 \leqslant i \leqslant 2^k)$.

	P	$\Diamond$P	D_1	...	D_k	D*
Row (F_{i1}):	T	T	$V^{(i)}_1$	...	$V^{(i)}_k$	F
Row (F_{i2}):	F	T	$V^{(i)}_1$	...	$V^{(i)}_k$	F
Row (F_{i3}):	T	F	$V^{(i)}_1$	...	$V^{(i)}_k$	T
Row (F_{i4}):	F	F	$V^{(i)}_1$	...	$V^{(i)}_k$	T

Each of these rows must be satisfactory. Row (F_{i3}) satisfies T1. Since no D_i is an atom, none of the other three can satisfy T1, T2, or T3.

Then they must satisfy T4, T5, or T6 by virtue of $D_1, \ldots, D_k$. Therefore Row (F_{i3}) also satisfies T4, T5, or T6 by virtue of $D_1, \ldots, D_k$. Thus each F-row of $\mathcal{T}(\Diamond P \equiv D^*)$ is satisfactory by virtue of $D_1, \ldots, D_k$ only. Hence the same is true of $\mathcal{T}(p \equiv D^*)$. Thus $p \equiv D^*$ e MQ. Then $\Diamond p \equiv p$ e MQ. This contradicts 5.3. Thus $\Diamond$ is independent.

5.6. The calculus S4Q. If the formula $\Diamond\Diamond A \supset \Diamond A$ is added as an axiom to M, the resulting system is S4. (S4 is described in Lewis and Langford [4], p. 501, p. 493, and pp. 125-126.) Accordingly, S4Q is the system which results from MQ if $\Diamond\Diamond A \supset \Diamond A$ is added to MQ as an axiom, and $\overline{S4Q}$ is the closure of S4Q.

5.7. Definition. A is a *tautology of degree n of $\overline{S4Q}$ (n - tautology of $\overline{S4Q}$)* iff

(1) A is a normal wff of degree n of $\overline{S4Q}$,

and (2) every F-row of $\mathcal{T}(A)$ satisfies at least one of the conditions T1-T6, or

T2[1]. Some constituent of the form $\Diamond B$ of degree $n_1 \leqslant n$ has T, some constituents of the form $\Diamond D_1, \ldots, \Diamond D_h$, $\Diamond C_1, \ldots, \Diamond C_m$ all have F ($h \geqslant 0$, $m \geqslant 0$, $h+m \geqslant 1$), while $B \supset D_1 v \ldots v D_h v\ C_1 v \ldots v\ C_m$ is an (n_1—1)-tautology of $\overline{S4Q}$. (Anderson [1], *Correction*[3])

5.8. Metatheorem. Let C be a closed wff of $\overline{S4Q}$. Let C* be any normal form of C. Then C e $\overline{S4Q}$ iff C* is a tautology of $\overline{S4Q}$.

5.9. Proof. The proof is analogous to that of 4.36.

5.10. The Barcan system S4Q[1]. If in S4Q A10 is replaced by A10[1]: $\Diamond(Ex)A \supset (Ex)\Diamond A$, the system S4Q[1] which results is the monadic part of the one described in Barcan [2], pp. 1-2 and p. 15. S4Q[1] is a subsystem of S4Q; that is every theorem of S4Q[1] is a theorem of S4Q. On the other hand, every normal theorem of S4Q is a theorem of S4Q[1] because such a theorem A is a tautology of $\overline{S4Q}$, and inspection of the proof of 4.2 shows that A has a proof which does not require A10. This proves the following.

5.11. Metatheorem. If C is a normal formula of S4Q[1], then C e S4Q[1] iff C e S4Q.

([3]) The following was brought to my attention by Professor Anderson and has been taken into account in 5.7. Clause II of the *Correction* does not entail clause II of the original paper. Hence, instead of replacing clause II of the original paper by II of the *Correction*, one must add II of the *Correction* to II of the original paper, so that the procedure for S4 requires four conditions instead of three. Then the argument goes through in the same way, except that there are more cases to consider.

completely normal form of C iff C^*_c can be derived from C by the
5.12. The calculus S5Q. If the formula $\Diamond{\sim}\Diamond A \supset {\sim}\Diamond A$ is added as an axiom to M, the resulting system is S5 (Lewis and Langford [4], p.501). Accordingly, S5Q is the system which results from MQ if $\Diamond{\sim}\Diamond A \supset {\sim}\Diamond A$ is added to MQ as an axiom, and $\overline{S5Q}$ is the closure of S5Q.

5.13. It is known that every wff of S5 can be reduced (effectively) to a wff of degree at most 1. (See Parry [5], p. 151, footnote 19, and references there given.)

5.14. Metatheorem. Let C be a closed wff of $\overline{S5Q}$. Let C* be any normal form of C. Then there exists an effective procedure for obtaining a wff B of $\overline{S5Q}$ such that $B \equiv A \; e \; \overline{S5Q}$ and B is of degree at most 1.

5.15. Proof. Apply 5.13 to C*, treating quantifications in C* as though they were propositional variables.

5.16. Definition. Let C be a closed wff of $\overline{S5Q}$. Then C^*_c is a *completely normal form of* C iff C^*_c can be derived from C by the method of 5.15.

5.17. Metatheorem. Let C be a closed wff of $\overline{S5Q}$. Let C^*_c be any completely normal form of C. Then $C \; e \; \overline{S5Q}$ iff $C^*_c \; e \; \overline{MQ}$.

5.18. Proof. The proof is obvious.

6. *An interpretation of MQ*. The distinguishing feature of MQ (and of S4Q and S5Q) is A10: $(Ex){\sim}\Diamond A \supset {\sim}\Diamond(Ex)A$. This axiom plays a crucial role in the algorithm for reducing formulas to normal form, inasmuch as it permits the removal of occurrences of $\Diamond$ from scopes of occurrences of E. The fact that all wffs of MQ can be reduced to normal form makes it possible to apply Quine's interpretation of modal logic on the pre-quantificational level to MQ as well. ([7]. «$\Box$p» may be interpreted as «p is logically true», or «'p is true' is analytic.»)

Note that, by A10, $(Ex)\Box fx \supset \Box(x)fx \; e \; MQ$. In fact, $(Ex)\Box fx \equiv \Box(x)fx \; e \; MQ$ and $(x)\Box fx \equiv \Box(x)fx \; e \; MQ$ (but it is not the case that $\Box(Ex)fx \supset (Ex)\Box fx \; e \; MQ$). This suggests the following extension of Quine's interpretation to cases where $\Diamond$'s do occur in scopes of occurrences of E. Interpret «$\Box$f» as «f is necessarily a universal property.» (Then «$\Box$f» is not interpreted as a property.) This amounts to reading «$\Box$fx» as if it were «$\Box$(x)fx.»

More generally, interpret «$\Box A$,» where A is any wff, as if it were «$\Box \bar{A}$,» and give the latter Quine's interpretation.

Supporting Metatheorem. $D_1 \Box A D_2$ e MQ iff $D_1 \Box \bar{A} D_2$ e MQ, where D_1 and D_2 are finite sequences of primitive symbols of MQ.

Proof. It is sufficient to show that $\overline{D_1 \sim \Diamond \sim A D_2} \equiv \overline{D_1 \sim \Diamond \sim \bar{A} D_2}$ e MQ. Let the free individual variables of A be $x_1, \ldots, x_n$. Then in MQ: $\overline{D_1 \sim \Diamond \sim A D_2} \equiv \overline{D'_1 \sim \Diamond (Ex_n) \ldots (Ex_1) \sim A D'_2} \equiv \overline{D'_1 \sim \Diamond \sim \bar{A} D'_2} \equiv \overline{D'_1 \sim \Diamond (Ex_n) \ldots (Ex_1) \sim \bar{A} D'_2} \equiv \overline{D_1 \sim \Diamond \sim \bar{A} D_2}$.

Trinity College, Department of Mathematics
Hartford 6, Connecticut.

M. J. POLIFERNO

BIBLIOGRAPHY

[1] A. R. ANDERSON, «Improved decision procedures for Lewis' calculus S4 and von Wright's calculus M», *The journal of symbolic logic*, vol. 19 (1954), pp. 201-214. (See «Correction to a paper on modal logic», *ibid.*, vol. 20 (1955), p. 150)

[2] Ruth C. BARCAN (Ruth (Barcan) MARCUS), «A functional calculus of first order based on strict implication», *The journal of symbolic logic*, vol. 11 (1946), pp. 1-16.

[3] Alonzo CHURCH, *Introduction to mathematical logic*, vol. 1, Princeton, 1956.

[4] C. I. LEWIS and C. H. LANGFORD, *Symbolic logic*, New York, 1932.

[5] W. T. PARRY, «Modalities in the *Survey* system of strict implication», *The journal of symbolic logic*, vol. 4 (1939), pp. 137-154.

[6] W. V. QUINE, «On the logic of quantification», *The journal of symbolic logic*, vol. 10 (1945), pp. 1-24.

[7] W. V. QUINE, «The problem of interpreting modal logic», *The journal of symbolic logic*, vol. 12 (1947), pp. 43-48.

[8] J. B. ROSSER, *Logic for mathematicians*, New York, 1953.

[9] G. H. VON WRIGHT, *An essay in modal logic*, Amsterdam, 1951.